

THE UPPER DOMINATION RAMSEY NUMBER $u(4, 4)$

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Abstract

The upper domination Ramsey number $u(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with color red and blue, $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$, where B and R is the subgraph of K_p induced by blue and red edges, respectively; $\Gamma(G)$ is the maximum cardinality of a minimal dominating set of a graph G . In this paper, we show that $u(4, 4) \leq 15$.

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1. INTRODUCTION

Our notation comes from [6] and [7]. Let $G = (V(G), E(G))$ be a graph with a vertex set $V(G)$ of order $p = |V(G)|$ and an edge set $E(G)$. If v is a vertex in $V(G)$, then the open neighborhood of v is $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is $N_G[v] = \{v\} \cup N_G(v)$. The external private neighborhood of v relative to $S \subseteq V(G)$ is $epn(v, S) =$

$N(v) - N[S - \{v\}]$. The open neighborhood of a set S of vertices is $N_G(S) = \bigcup_{v \in S} N_G(v)$, and the closed neighborhood is $N_G[S] = N_G(v) \cup S$.

A set $S \subseteq V(G)$ is a *dominating set* in G if each vertex v of G belongs to S or is adjacent to some vertex in S . A set $S \subseteq V(G)$ is an *irredundant set* if for each $s \in S$ there is a vertex w in G such that $N_G[w] \cap S = \{s\}$. A set $S \subseteq V(G)$ is *independent* in G if no two vertices of S are adjacent in G . If S is an irredundant set in G and $v \in S$, then the set $N[v] - N[S - \{v\}]$ is nonempty and is called the set of *private neighbors of v* in G (relative to S), denoted by $pn_G(v, S)$ or simply by $pn(v, S)$. The *upper domination number* of G , denoted by $\Gamma(G)$, is the maximum cardinality of a minimal dominating set of G . The *upper irredundance number* of G , denoted by $IR(G)$, is the maximum cardinality of an irredundant set of G . The *independence number* of G , denoted by $\beta(G)$, is the maximum cardinality among all independent sets of vertices of G . A minimal dominating set of cardinality $\Gamma(G)$ is called a $\Gamma(G)$ -set. Similarly, an irredundant set of cardinality $IR(G)$ is called an $IR(G)$ -set.

It is apparent that irredundance is a hereditary property.

Remark 1. Any independent set is also irredundant.

Remark 2. Every minimal dominating set is an irredundant set. Consequently, we have $\Gamma(G) \leq IR(G)$ for every graph G .

Remark 3 [5]. A set $D \subseteq V(G)$ is a minimal dominating set if and only if it is dominating and irredundant, and therefore, if $\Gamma(G) < IR(G)$, then no IR -set is dominating.

Remark 4. Every maximum independent set is also a dominating set, thus we have $\beta(G) \leq \Gamma(G)$ for every graph G .

Hence the parameters $\beta(G), \Gamma(G), IR(G)$ are related by the following inequalities which were observed by Cockayne and Hedetniemi [3].

Theorem 1 [3]. *For every graph G , $\beta(G) \leq \Gamma(G) \leq IR(G)$.*

Let $G_1, G_2, \dots, G_t$ be an arbitrary t -edge coloring of K_n , where for each $i \in \{1, 2, \dots, t\}$, G_i is the spanning subgraph of K_n whose edges are colored with color i . The classical *Ramsey number* $r(n_1, n_2, \dots, n_t)$ is the smallest value of n such that for every t -edge coloring $G_1, G_2, \dots, G_t$ of K_n ,

there is an $i \in \{1, 2, \dots, t\}$ for which $\beta(\overline{G_i}) \geq n_i$, where $\overline{G}$ is the complement of G . The *irredundant Ramsey number* denoted by $s(n_1, n_2, \dots, n_t)$, is the smallest n such that for every t -edge coloring $G_1, G_2, \dots, G_t$ of K_n , there is at least one $i \in \{1, 2, \dots, t\}$ for which $IR(\overline{G_i}) \geq n_i$. The irredundant Ramsey numbers exist by Ramsey's theorem, and by Remark 1 $s(n_1, n_2, \dots, n_t) \leq r(n_1, n_2, \dots, n_t)$ for all n_i , where $i = 1, 2, \dots, t$. The *upper domination Ramsey number* $u(n_1, n_2, \dots, n_t)$ is defined as the smallest n such that for every t -edge coloring $G_1, G_2, \dots, G_t$ of K_n , there is at least one $i \in \{1, 2, \dots, t\}$ for which $\Gamma(\overline{G_i}) \geq n_i$.

In the case $t = 2$, $r(m, n)$ is the smallest integer p such that for every 2-coloring of the edges of K_p with colors red (R) and blue (B), $\beta(B) \geq m$ or $\beta(R) \geq n$. Similarly, the irredundant Ramsey number $s(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with colors red (R) and blue (B) satisfies $IR(B) \geq m$ or $IR(R) \geq n$. Finally, the upper domination Ramsey number $u(m, n)$ is the smallest integer p such that every 2-coloring of the edges of K_p with colors red (R) and blue (B) satisfies $\Gamma(B) \geq m$ or $\Gamma(R) \geq n$.

It follows from Theorem 1 that for all m, n ,

$$s(m, n) \leq u(m, n) \leq r(m, n),$$

and for the purpose of our proof of the main result, let us recall the following results.

Theorem 2 [2]. $s(4, 4) = 13$.

Theorem 3 [4]. $r(3, 4) = 9$.

Theorem 4 [4]. $r(4, 4) = 18$.

2. MAIN RESULT

First we state the following

Lemma 5. *Let (R, B) be a 2-edge coloring of K_n such that $\Gamma(B) \leq 3$, $IR(B) \geq 4$ and $\beta(R) \leq 3$. Then there exists an irredundant set X of B such that $|X| = 4$ and $epn(x, X) \neq \emptyset$ for each $x \in X$.*

Proof. Let Y be an IR -set of B and X the subset of Y such that $epn(x, Y) \neq \emptyset$ for all $x \in X$. Suppose firstly that $|X| = 3$; say $X = \{x_1, x_2, x_3\}$ and

let $X' = \{x'_1, x'_2, x'_3\}$, where $x'_i \in \text{epn}(x_i, Y)$, $i = 1, 2, 3$. Note that each x'_i is joined by red edges to all vertices in $Y - \{x_i\}$. Since $|Y| \geq 4$, there is a vertex $w \in Y - X$ such that $\text{pn}(w, Y) = \{w\}$; hence w is joined by red edges to the vertices in $X \cup X'$. Furthermore, by Remark 3 there is also a vertex $v \in V(B) - N[Y]$; so v is joined by red edges to all vertices in Y . But $\beta(B) < 3$ and so, to avoid a red K_4 , the above-mentioned red edges force all edges between vertices in $X' \cup \{v\}$ to be blue. But this is a blue K_4 , contradicting $\beta(R) \leq 3$. The case $|X| \leq 2$ is easy and omitted. ■

Now we are ready to prove the following theorem.

Theorem 6. $u(4, 4) \leq 15$.

Proof. Let (R, B) be a 2-edge coloring of K_{15} and suppose that $\Gamma(R) \leq 3$ and $\Gamma(B) \leq 3$. By Theorem 1, $\beta(R) \leq 3$ and $\beta(B) \leq 3$. By Theorem 2, $s(4, 4) = 13$ and therefore, without loss of generality, we may assume that $IR(B) \geq 4$. We only consider the case $IR(B) = 4$; the case $IR(B) \geq 5$ is similar but simpler, and thus omitted. Then, by Lemma 5, there exists an IR -set X of B in which $\text{epn}(x, X) \neq \emptyset$ for each $x \in X$. Let $V(K_{15}) = \{0, 1, \dots, 9, x, y, z, w, t\}$, $X = \{0, 2, 4, 6\}$ and $Y = \{1, 3, 5, 7\}$, where for each $i \in Y$, $i \in \text{epn}_B(i - 1, X)$. Thus there is a blue matching consisting of the edges $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$ and $\{6, 7\}$, and each vertex $i \in X$ is joined to all vertices $j \in Y - \{i + 1\}$ by red edges, according to the private neighbor property. Since $\Gamma(B) < IR(B)$, Remark 3 applied to the irredundant sets X and Y implies that there are vertices u and v joined by red edges to the vertices in X and Y , respectively. If $u = v$, then $X \cup \{u\}$ is irredundant in B and $IR(B) \geq 5$, which is not the case. Hence we may assume that $u \neq v$; say $u = 9$ and $v = 8$. Similarly, we may assume that $\{8, 9\}$ is red, otherwise $X' = X \cup \{8\}$ is irredundant in B (where $9 \in \text{epn}(8, X')$).

We now make a few observations about the effects that a red edge between two vertices in X (or Y) has on the colors of the other edges between vertices in $X \cup Y \cup \{8, 9\}$. For simplicity, we consider the edge $\{1, 3\}$; similar remarks hold for the other edges. Suppose therefore that $\{1, 3\}$ is red. Then

Observation 1. $\{i, 8\}$ is blue for $i \in \{4, 6\}$, otherwise $\{1, 3, i, 8\}$ induces a red K_4 , thus contradicting $\beta(B) \leq 3$.

Observation 2. $\{4, 6\}$ is blue, otherwise $\{1, 3, 4, 6\}$ induces a red K_4 .

Observation 3. $\{1, 9\}$ and $\{3, 9\}$ are blue, otherwise, if (say) $\{1, 9\}$ is red, then $\{2, 8\}$ (respectively $\{2, 4\}, \{2, 6\}$) is blue to avoid the red K_4 induced by $\{1, 2, 8, 9\}$ (respectively $\{1, 2, 4, 9\}, \{1, 2, 6, 9\}$), thus forming the blue K_4 on $\{2, 4, 6, 8\}$, by Observation 1 and Observation 2. This contradicts $\beta(R) \leq 3$.

Now, if (say) $\{1, 3\}, \{1, 5\}$ and $\{1, 7\}$ are all red, then by Observation 2, $\{2, 4, 6\}$ induces a blue triangle and thus by Observation 1, $\{2, 4, 6, 8\}$ induces a blue K_4 , a contradiction. Therefore

Observation 4. No vertex in X (or Y) is adjacent in R to all other vertices of X (or Y).

Observation 5. The red subgraph induced by X is triangle-free, otherwise any such red triangle forms a red K_4 with vertex 9; similarly, the red subgraph induced by Y is triangle-free.

The remaining part of the proof is divided into two parts.

- Part 1: there is a vertex $v \in Y$ such that v is joined by exactly two red edges to the remaining vertices of Y .
- Part 2: there is no vertex $v \in Y$ such that v is joined by two red edges to the remaining vertices of Y .

Part 1

Without loss of generality, let us suppose that edges $\{1, 3\}, \{1, 5\}$ are red. By Observations 1–5 we have $\{1, 7\}, \{1, 9\}, \{2, 6\}, \{2, 8\}, \{3, 5\}, \{3, 9\}, \{4, 6\}, \{4, 8\}, \{5, 9\}, \{6, 8\}$ and $\{7, 9\}$ are blue, the edge $\{2, 4\}$ is red. To avoid a blue triangle $(3, 5, 7)$ we have that at least one of the edges $\{3, 7\}, \{5, 7\}$ must be red. This forces $\{0, 8\}$ to be blue. Now, we have to consider three cases:

- Case 1: $\{3, 7\}$ and $\{5, 7\}$ are red.
- Case 2: $\{3, 7\}$ is blue, $\{5, 7\}$ is red.
- Case 3: $\{3, 7\}$ is red, $\{5, 7\}$ is blue.

Case 1. In this case, we have that $\{3, 7\}$ and $\{5, 7\}$ are red. By an observation similar to Observation 1, the edges $\{0, 2\}$ and $\{0, 4\}$ are blue. Similarly, $\{0, 6\}$ is red.

Suppose $\{8, x\}$ is blue. If x is joined by red edges to $\{2, 4\}$, then, to avoid a red K_4 , the edges $\{1, x\}, \{7, x\}$ and $\{9, x\}$ are blue, and we obtain a blue K_4 on $\{1, 7, 9, x\}$.

Similarly, if x is joined by red edges to $\{0, 6\}$, then to avoid a red K_4 , the edges $\{3, x\}$, $\{5, x\}$ and $\{9, x\}$ are blue, and we obtain a blue K_4 on $\{3, 5, 9, x\}$.

Suppose $\{2, x\}$ is blue. Then $\{6, x\}$ is red, since otherwise a blue K_4 on $\{2, 6, 8, x\}$ results. Since $\{6, x\}$ is red, $\{0, x\}$ is blue. But then we have a blue K_4 on $\{0, 2, 8, x\}$. Thus $\{2, x\}$ is red, and so $\{4, x\}$ is blue. To avoid a blue K_4 on $\{4, 6, 8, x\}$, $\{6, x\}$ is red. Since $\{6, x\}$ is red, $\{0, x\}$ is blue. But then we have a blue K_4 on $\{0, 4, 8, x\}$.

Thus vertex 8 is joined by a red edge to every vertex in $\{x, y, z, w, t\}$ and so the red degree of 8 is at least 10. As $r(3, 4) = 9$, we immediately have a red K_4 containing 8 or a blue K_4 amongst the neighbors of 8.

Case 2. In this case, we have that $\{3, 7\}$ is blue and $\{5, 7\}$ is red. Similarly to Observation 1, the edge $\{0, 2\}$ is blue. To avoid a blue K_4 on $\{0, 2, 6, 8\}$, $\{0, 6\}$ is red. If $\{0, 4\}$ is blue, then by using similar methods to those in Case 1, we immediately obtain a contradiction. Thus, edge $\{0, 4\}$ is red.

Next, suppose that vertex 8 has three blue edges incident to vertices $\{x, y, z, w, t\}$. Without loss of generality, let us suppose that edges $\{8, x\}$, $\{8, y\}$ and $\{8, z\}$ are blue.

Now suppose $\{6, x\}$ is blue. Then $\{2, x\}$ and $\{4, x\}$ are red, since otherwise there are two blue K_4 's on $\{2, 6, 8, x\}$ and $\{4, 6, 8, x\}$. But then we have a blue K_4 on $\{1, 7, 9, x\}$. Thus, $\{6, x\}$ is red.

If $\{0, x\}$ is red, we have a blue K_4 on $\{3, 5, 9, x\}$.

Thus there is only one possible method of coloring the edges joining vertices $\{x, y, z\}$ to vertices $\{0, 2, 4, 6\}$: $\{0, x\}$, $\{0, y\}$, $\{0, z\}$, $\{4, x\}$, $\{4, y\}$, $\{4, z\}$ are blue, and $\{2, x\}$, $\{2, y\}$, $\{2, z\}$, $\{6, x\}$, $\{6, y\}$, $\{6, z\}$ are red. But this coloring forces a red K_4 on the set $\{x, y, z, 2\}$, a contradiction.

Thus our assumption that vertex 8 has three blue edges incident to vertices $\{x, y, z, w, t\}$ is incorrect. Similarly, vertex 9 has at most two blue edges to vertices $\{x, y, z, w, t\}$. It is easy to see that there are exactly two blue edges joining vertex 8 (9) to vertices $\{x, y, z, w, t\}$, for otherwise $\deg_R(8) \geq 9$ or $\deg_R(9) \geq 9$, and by the fact $r(3, 4) = 9$ we shall obtain a contradiction. Now, we have to consider three subcases.

Subcase 2.1. In this subcase two blue edges joining vertices 8 and 9 to vertices $\{x, y, z, w, t\}$ have the same end-vertices. Without loss of generality, let us suppose that the end-vertices of these blue edges are x and y .

Suppose $\{6, x\}$ is blue. Then $\{2, x\}$ and $\{4, x\}$ are red, since otherwise there are two blue K_4 's on $\{2, 6, 8, x\}$ and $\{4, 6, 8, x\}$. But then we have a blue K_4 on $\{1, 7, 9, x\}$. Thus $\{6, x\}$ is red.

Suppose $\{0, x\}$ is also colored red. Then $\{3, x\}$ and $\{5, x\}$ are blue, since otherwise two red K_4 's on $\{0, 3, 6, x\}$ and $\{0, 5, 6, x\}$. But then we have a blue K_4 on $\{3, 5, 9, x\}$. Thus $\{0, x\}$ is blue.

Now, suppose $\{1, x\}$ is red. Then $\{3, x\}$ and $\{5, x\}$ are blue, since otherwise there are two red K_4 's on $\{1, 3, 6, x\}$ and $\{1, 5, 6, x\}$. But then we have a blue K_4 on $\{3, 5, 9, x\}$. Thus $\{1, x\}$ is blue.

Consequently, to avoid a blue K_4 on $\{1, 7, 9, x\}$, $\{7, x\}$ is red, and to avoid a blue K_4 on $\{0, 2, 8, x\}$, $\{2, x\}$ is red. Then $\{4, x\}$ and $\{5, x\}$ are blue and $\{3, x\}$ is red. Thus there is only one possible method of coloring the edges joining vertices x and y to the vertices of sets X and Y : $\{0, x(y)\}$, $\{1, x(y)\}$, $\{4, x(y)\}$, $\{5, x(y)\}$ are blue, and $\{2, x(y)\}$, $\{3, x(y)\}$, $\{6, x(y)\}$, $\{7, x(y)\}$ are red. But this forces $\{x, y\}$ to be red, and we obtain a red K_4 on vertices $\{3, 6, x, y\}$, a contradiction.

Subcase 2.2. In this subcase vertices 8 and 9 are joined by two blue edges to different vertices among $\{x, y, z, w, t\}$. Assume $\{8, z\}$, $\{8, t\}$, $\{9, x\}$ and $\{9, y\}$ are blue.

To avoid the blue K_4 on $\{3, 5, 9, x\}$, one of the edges $\{3, x\}$ or $\{5, x\}$ is red. Then $\{1, x\}$ is blue, since otherwise there is a red K_4 on either $\{1, 3, 8, x\}$ or $\{1, 5, 8, x\}$. Similarly $\{1, y\}$ is blue.

Next, to avoid the blue K_4 on $\{1, 7, 9, x\}$, edge $\{7, x\}$ is red, and similarly, $\{7, y\}$ is red.

To avoid the blue K_4 on $\{1, 9, x, y\}$ edge $\{x, y\}$ is red. But then $\langle\{7, 8, x, y\}\rangle$ is a red K_4 , a contradiction.

Subcase 2.3. We have to consider the subcase when vertices 8 and 9 are joined by blue edges to exactly one common vertex among $\{x, y, z, w, t\}$. Without loss of generality, assume that $\{8, y\}$, $\{8, z\}$, $\{9, x\}$, $\{9, y\}$ are blue and the remaining edges which join vertices 8 and 9 to $\{x, y, z, w, t\}$ are red. Then we immediately have that $\{w, t\}$ is blue.

Suppose $\{1, x\}$ is red. Then, to avoid two red K_4 's on $\{1, 3, 8, x\}$ and $\{1, 5, 8, x\}$, we obtain that the edges $\{3, x\}$ and $\{5, x\}$ are blue. But then $\langle\{3, 5, 9, x\}\rangle$ is a blue K_4 , a contradiction. We conclude that $\{1, x\}$ is blue, $\{7, x\}$ is red and $\{5, x\}$ is blue.

Suppose $\{2, y\}$ is blue. Then, to avoid a blue K_4 on $\{0, 2, 8, y\}$, $\{0, y\}$ is red. Similarly, to avoid a blue K_4 on $\{2, 6, 8, y\}$, $\{6, y\}$ is red. To avoid a blue

K_4 on $\{3, 5, 9, y\}$, $\{5, y\}$ or $\{3, y\}$ is red. If $\{5, y\}$ is red, then $\{0, 5, 6, y\}$ is a red K_4 . Thus $\{5, y\}$ is blue, and so $\{3, y\}$ is red. But then $\langle\{0, 3, 6, y\}\rangle$ is a red K_4 . Thus $\{2, y\}$ is red.

Suppose $\{4, y\}$ is red. To avoid a red K_4 on $\{1, 2, 4, y\}$ it follows that $\{1, y\}$ is blue. To avoid a red K_4 on $\{2, 4, 7, y\}$, $\{7, y\}$ is blue. But then $\langle\{1, 7, 9, y\}\rangle$ is a blue K_4 , a contradiction. Thus $\{4, y\}$ is blue, and so $\{6, y\}$ is red.

Suppose $\{0, y\}$ is red. To avoid a red K_4 on $\{0, 3, 6, y\}$, $\{3, y\}$ is blue. To avoid a red K_4 on $\{0, 5, 6, y\}$, it follows that $\{5, y\}$ is blue. But then $\langle\{3, 5, 9, y\}\rangle$ is a blue K_4 , a contradiction. Thus $\{0, y\}$ is blue.

Suppose $\{1, y\}$ is red. Then, to avoid a red K_4 on $\{1, 3, 6, y\}$, the edge $\{3, y\}$ is blue. To avoid a red K_4 on $\{1, 5, 6, y\}$, the edge $\{5, y\}$ is blue. But then $\{3, 5, 9, y\}$ is a blue K_4 , which is a contradiction. Thus $\{1, y\}$ is blue.

Suppose $\{5, y\}$ is red. Then, to avoid a red K_4 on $\{2, 5, 7, y\}$, it follows that $\{7, y\}$ is blue. But then we obtain a blue K_4 on $\{1, 7, 9, y\}$. Thus $\{5, y\}$ is blue.

Suppose $\{6, z\}$ is blue. To avoid a blue K_4 on $\{4, 6, 8, z\}$, $\{4, z\}$ is red. To avoid a blue K_4 on $\{2, 6, 8, z\}$, the edge $\{2, z\}$ is red. But then $\langle\{2, 4, 9, z\}\rangle$ is a red K_4 , a contradiction.

Thus $\{6, z\}$ is red. Finally:

- to avoid a red K_4 on $\{0, 6, 9, z\}$, the edge $\{0, z\}$ is blue;
- to avoid a blue K_4 on $\{0, 2, 8, z\}$, the edge $\{2, z\}$ is red;
- to avoid a red K_4 on $\{2, 4, 9, z\}$, the edge $\{4, z\}$ is blue;
- to avoid a blue K_4 on $\{3, 5, 9, y\}$, the edge $\{3, y\}$ is red;
- to avoid a blue K_4 on $\{1, 7, 9, y\}$, the edge $\{7, y\}$ is red;
- to avoid a blue K_4 on $\{3, 5, 9, x\}$, the edge $\{3, x\}$ is red;
- to avoid a blue K_4 on $\{5, 9, x, y\}$, the edge $\{x, y\}$ is red;
- to avoid a blue K_4 on $\{0, 8, y, z\}$, the edge $\{y, z\}$ is red;
- to avoid a red K_4 on $\{2, 7, x, y\}$, the edge $\{2, x\}$ is blue;
- to avoid a red K_4 on $\{2, 7, y, z\}$, the edge $\{7, z\}$ is blue;
- to avoid a red K_4 on $\{3, 6, x, y\}$, the edge $\{6, x\}$ is blue;
- to avoid a red K_4 on $\{3, 6, y, z\}$, the edge $\{3, z\}$ is blue.

Suppose, to the contrary, that $\{w, x\}$ is red. Then $\{3, w\}$ and $\{7, w\}$ are blue, since otherwise $\langle\{3, 8, w, x\}\rangle$ and $\langle\{7, 8, w, x\}\rangle$ are blue K_4 's.

Suppose $\{w, z\}$ is red. If $\{2, w\}$ is red, then $\{2, 9, z, w\}$ is a red K_4 . If $\{6, w\}$ is red, then $\{6, 9, w, z\}$ is a red K_4 . Thus $\{2, w\}$ and $\{6, w\}$ are blue.

If t sends a blue edge to $\{2, 7\}$ and t sends a blue edge to $\{3, 6\}$, we obtain a blue K_4 , and we are done.

Suppose t is joined by red edges to 2 and 7. Then $\{4, t\}$, $\{5, t\}$ and $\{y, t\}$ are blue, since otherwise there are three red K_4 's on $\{2, 4, 7, t\}$, $\{2, 5, 7, t\}$ and $\{2, 7, y, t\}$. But then we obtain a blue K_4 on $\{4, 5, y, t\}$, a contradiction.

Suppose t is joined by red edges to 3 and 6. Then $\{y, t\}$, $\{1, t\}$ and $\{0, t\}$ are blue, since otherwise there are three red K_4 's: $\{0, 3, 6, t\}$, $\{1, 3, 6, t\}$ and $\{3, 6, y, t\}$. But then $\langle\{0, 1, y, t\}\rangle$ is a blue K_4 . Thus $\{w, z\}$ is blue. But then, in both cases, $\{3, 7, w, z\}$ forms a blue K_4 , a contradiction. Consequently, $\{w, x\}$ is blue.

Now, by using the same methods to those for the edge $\{w, x\}$, we prove that $\{x, t\}$ is blue. Suppose $\{x, t\}$ is red. Then $\{3, t\}$ and $\{7, t\}$ are blue, since otherwise, $\langle\{3, 8, x, t\}\rangle$ and $\langle\{7, 8, x, t\}\rangle$ are blue K_4 's.

Suppose $\{z, t\}$ is red. If $\{2, t\}$ is red, then $\langle\{2, 9, z, t\}\rangle$ is a red K_4 . If $\{6, t\}$ is red, then $\langle\{6, 9, z, t\}\rangle$ is a red K_4 . Thus $\{2, t\}$ and $\{6, t\}$ are blue.

If $\{2, w\}$, $\{3, w\}$, $\{6, w\}$ and $\{7, w\}$ are blue, then $\{3, 7, w, t\}$, $\{2, 6, w, t\}$ or $\{2, 6, w, x\}$ are a blue K_4 .

Suppose w is joined by red edges to 2 and 7. Then $\{4, w\}$, $\{5, w\}$ and $\{y, w\}$ are blue, since otherwise there are three red K_4 's on $\{2, 4, 7, w\}$, $\{2, 5, 7, w\}$ and $\{2, 7, y, w\}$. But then we obtain a blue K_4 on $\{4, 5, y, w\}$, a contradiction.

Suppose w is joined by red edges to 3 and 6. Then $\{y, w\}$, $\{1, w\}$ and $\{0, w\}$ are blue, since otherwise there are three red K_4 's: $\{0, 3, 6, w\}$, $\{1, 3, 6, w\}$ and $\{3, 6, y, w\}$. But then $\langle\{0, 1, y, w\}\rangle$ is a blue K_4 . Thus $\{z, t\}$ is blue. But then, in both cases, $\{3, 7, t, z\}$ forms a blue K_4 , a contradiction. Hence, $\{x, t\}$ is blue.

Suppose now that $\{z, t\}$ is red. Then $\{2, t\}$ and $\{6, t\}$ are blue, since otherwise $\langle\{2, 9, z, t\}\rangle$ and $\langle\{7, 8, z, t\}\rangle$ are red K_4 's. But then we obtain a blue K_4 on $\{2, 6, t, x\}$. Thus $\{z, t\}$ is blue. Similarly, $\{w, z\}$ is also colored blue. Then, to avoid a blue K_4 on $\{w, t, x, z\}$, it follows $\{x, z\}$ is red.

Now, let us consider a vertex x and all blue edges incident to it. Since $r(3, 4) = 9$, we obtain that x is joined by at most one blue edge to one of vertices 0 and 4.

If $\{0, x\}$ and $\{4, x\}$ are red, then we have a red K_4 on $\{0, 4, 7, x\}$.

First, suppose $\{0, x\}$ is blue. To avoid a red K_4 on vertices $\{1, 5, 8, w\}$ or $\{1, 5, 8, t\}$ we may assume without loss of generality that $\{1, w\}$ is blue. Then $\{5, w\}$ and $\{1, t\}$ are red, and $\{5, t\}$ is blue. To avoid a blue K_4 on vertices $\{0, 1, x, w\}$, $\{0, w\}$ is red. Then $\{6, w\}$ is blue, since otherwise there is a red K_4 on $\{0, 6, 9, w\}$. Similarly, $\{6, t\}$ is red and $\{0, t\}$ is blue. It is easy to see that $\{2, w\}$ and $\{2, t\}$ are red.

Now, consider a vertex z and all blue edges incident to it. Similarly to x , vertex z is joined by exactly one blue edge either to vertex 1 or to 5.

If $\{1, z\}$ is blue and $\{5, z\}$ is red, then, since $\{0, w\}$ is red, we obtain that $\{4, w\}$ is blue and $\{4, t\}$ is red. But then we have a red K_4 on $\{2, 4, 9, t\}$.

If $\{1, z\}$ is red and $\{5, z\}$ is blue, we also easily obtain a contradiction, so $\{0, x\}$ is red. If $\{4, x\}$ is blue, then by using similar arguments we obtain a contradiction, and the proof of this subcase is complete.

Case 3. In this case we have that $\{3, 7\}$ is red and $\{5, 7\}$ is blue. To avoid a red K_4 on $\{0, 3, 4, 7\}$, it follows that $\{0, 4\}$ is blue. To avoid a blue K_4 on $\{0, 4, 6, 8\}$, $\{0, 6\}$ is red. If $\{0, 2\}$ is blue, then by using similar methods to those in Case 1, we obtain a contradiction. Thus edge $\{0, 2\}$ is red and we obtain a coloring isomorphic to that considered in Case 2.

Part 2

Without loss of generality we can assume that $\{1, 3\}$ is red. By Observations 1–5, we obtain that vertices 1 and 3 are joined by blue edges to vertex 9. Edge $\{5, 9\}$ is blue, otherwise a blue K_4 on $\{0, 2, 4, 8\}$ results. Similarly $\{7, 9\}$ is blue, otherwise there is a blue K_4 on $\{0, 2, 6, 8\}$. To avoid a blue K_4 on $\{3, 5, 7, 9\}$, $\{5, 7\}$ is red. So we obtain two red edges, and all the remaining edges of K_5 on $\{0, 2\}$, $\{0, 8\}$, $\{2, 8\}$, $\{4, 6\}$, $\{4, 8\}$, $\{6, 8\}$. When we color the remaining edges of K_5 on $X \cup \{8\}$, we must consider sixteen cases. When $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$, $\{2, 6\}$ are red, we obtain a coloring which is isomorphic to that considered in Part 1, Case 1 above. In the nine following cases:

1. $\{0, 4\}$, $\{0, 6\}$ are blue and $\{2, 4\}$, $\{2, 6\}$ are red,
2. $\{0, 4\}$, $\{2, 4\}$ are blue and $\{0, 6\}$, $\{2, 6\}$ are red,
3. $\{0, 6\}$, $\{2, 6\}$ are blue and $\{0, 4\}$, $\{2, 4\}$ are red,
4. $\{2, 4\}$, $\{2, 6\}$ are blue and $\{0, 4\}$, $\{0, 6\}$ are red,
5. $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$ are blue and $\{2, 6\}$ is red,
6. $\{0, 4\}$, $\{0, 6\}$, $\{2, 6\}$ are blue and $\{2, 4\}$ is red,

7. $\{0, 4\}$, $\{2, 4\}$, $\{2, 6\}$ are blue and $\{0, 6\}$ is red,
8. $\{0, 6\}$, $\{2, 4\}$, $\{2, 6\}$ are blue and $\{0, 4\}$ is red,
9. $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$ and $\{2, 6\}$ are blue,

we immediately obtain a contradiction. In the remaining six cases:

1. $\{0, 4\}$, $\{0, 6\}$, $\{2, 4\}$ are red and $\{2, 6\}$ is blue,
2. $\{0, 4\}$, $\{0, 6\}$, $\{2, 6\}$ are red and $\{2, 4\}$ is blue,
3. $\{0, 4\}$, $\{2, 4\}$, $\{2, 6\}$ are red and $\{0, 6\}$ is blue,
4. $\{0, 6\}$, $\{2, 4\}$, $\{2, 6\}$ are red and $\{0, 4\}$ is blue,
5. $\{0, 6\}$, $\{2, 4\}$ are red and $\{0, 4\}$, $\{2, 6\}$ are blue,
6. $\{0, 4\}$, $\{2, 6\}$ are red and $\{0, 6\}$, $\{2, 4\}$ are blue,

similarly to Case 1, we obtain that vertex 9 is joined by a red edge to every vertex in $\{x, y, z, w, t\}$, so the red degree of 9 is at least 10. This observation completes the proof of Theorem 6. ■

REFERENCES

- [1] R.C. Brewster, E.J. Cockayne and C.M. Mynhardt, *Irredundant Ramsey numbers for graphs*, J. Graph Theory **13** (1989) 283–290.
- [2] E.J. Cockayne, G. Exoo, J.H. Hattingh and C.M. Mynhardt, *The Irredundant Ramsey Number $s(4, 4)$* , Util. Math. **41** (1992) 119–128.
- [3] E.J. Cockayne and S.T. Hedetniemi, *Towards a theory of domination in graphs*, Networks **7** (1977) 247–261.
- [4] R.E. Greenwood and A.M. Gleason, *Combinatorial relations and chromatic graphs*, Canadian J. Math. **7** (1955) 1–7.
- [5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of domination in graphs*, Marcel Dekker, New York, (1998) (Proposition 3.8, p. 72).
- [6] M.A. Henning and O.R. Oellermann, *The upper domination Ramsey number $u(3, 3, 3)$* , Discrete Math. **242** (2002) 103–113.
- [7] M.A. Henning and O.R. Oellermann, *On upper domination Ramsey numbers for graphs*, Discrete Math. **274** (2004) 125–135.

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