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ON ARBITRARILY VERTEX DECOMPOSABLE UNICYCLIC GRAPHS WITH DOMINATING CYCLE

Sylwia Cichacz and Irmina A. Zioło

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. A. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: cichacz@agh.edu.pl
e-mail: ziolo@agh.edu.pl

Abstract

A graph G of order n is called arbitrarily vertex decomposable if for each sequence $(n_1, \ldots, n_k)$ of positive integers such that $\sum_{i=1}^{k} n_i = n$, there exists a partition $(V_1, \ldots, V_k)$ of vertex set of G such that for every $i \in \{1, \ldots, k\}$ the set V_i induces a connected subgraph of G on n_i vertices. We consider arbitrarily vertex decomposable unicyclic graphs with dominating cycle. We also characterize all such graphs with at most four hanging vertices such that exactly two of them have a common neighbour.

Keywords: arbitrarily vertex decomposable graph, dominating cycle.

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1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $|V(G)| = n$. A sequence $\tau = (n_1, \ldots, n_k)$ of positive integers is called *admissible for G* if $n_1 + \ldots + n_k = n$. We shall write $((n_1)^{s_1}, \ldots, (n_l)^{s_l})$ for the sequence $(\underbrace{n_1, \ldots, n_1}_{s_1}, \ldots, \underbrace{n_l, \ldots, n_l}_{s_l})$. If $\tau = (n_1, \ldots, n_k)$ is an admissible sequence for the graph G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set $V(G)$ such that for each $i \in \{1, \ldots, k\}$ the subgraph $G[V_i]$ induced by V_i is a connected graph on n_i vertices, then τ is called *G-realizable* or *realizable* *in* G and the sequence $(V_1, \ldots, V_k)$ is said to be a *G-realization of* τ or a *realization of* τ *in* G. Each set V_i will be called a τ*-part* of a realization of τ in G. A graph G is called *arbitrarily vertex decomposable* (*avd* for short) if each admissible sequence for G is realizable in G.

Arbitrarily vertex decomposable graphs have been investigated in several papers ([1] – [5] for example). The problem originated from some applications to computer networks ([1]). It is obvious that every traceable graph is avd since every path is avd.

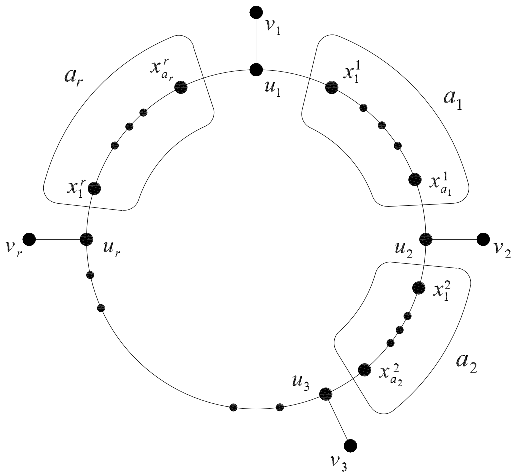

Figure 1. $Sun(a_1, \ldots, a_r)$

A *sun with* r *single rays* is a graph of order $n \geq 2r$ with r hanging vertices $v_1,\ldots,v_r$ whose deletion yields a cycle C_{n-r}, and each vertex u_i adjacent to v_i is of degree three. Each hanging edge u_iv_i is called a *single ray*. If the sequence of vertices u_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices, each of degree two, between u_i and u_{i+1}, $i = 1, \ldots, r$ (the indices taken modulo r), then this sun is denoted by $Sun(a_1, \ldots, a_r)$ and is unique up to isomorphism (Figure 1).

For every $i \in \{1, \ldots, r\}$, the single ray u_iv_i can be replaced by a multiple ray in the following way. After removing the vertex v_i we add vertices $v_i^1,\ldots,v_i^j$ and edges $u_iv_i^1,\ldots,u_iv_i^j$ and obtain the ray $\{u_iv_i^1, \ldots, u_iv_i^j\}$ of multiplicity $j \geq 1$. Note that for every sun the unique cycle is dominating. By $Sun'(a_1, \ldots, a_r)$ we will denote a sun with one double ray $\{u_1v_1^1, u_1v_1^2\}$ and $r-1$ single rays $u_2v_2,\ldots,u_rv_r$.

In [5] the authors characterized all avd suns with at most three single rays. Every sun with one single ray is arbitrarily vertex decomposable since it is traceable.

Theorem 1. *A graph $Sun(a,b)$ is arbitrarily vertex decomposable if and only if at most one of the numbers a and b is odd. Moreover, $Sun(a,b)$ of order n is not avd if and only if $((2)^{\frac{n}{2}})$ is the unique admissible and non-realizable sequence.*

Theorem 2. *A graph $Sun(a,b,c)$ is not arbitrarily vertex decomposable if and only if at least one of the following three conditions is fulfilled:*

(1) *at least two of the numbers a, b, c are odd,*

(2) $a \equiv b \equiv c \equiv 0 (\mathrm{mod}\ 3)$,

(3) $a \equiv b \equiv c \equiv 2 (\mathrm{mod}\ 3)$.

These results have been used to prove Ore-type conditions for a graph to be avd ([6]).

It turned out that for suns with single rays realisations of l-good sequences are interesting. Let $\tau = (n_1, \ldots, n_k)$ be an admissible sequence for a graph G of order n. An element n_i of τ is called *good* if either $n_i = 1$ or n_i is even. For $l \geq 0$, the sequence τ is called *l-good* if τ contains at least $\min(l,k)$ good elements.

Theorem 3. *Every $(r-2)$-good sequence is realizable in a graph $Sun(a_1, \ldots, a_r)$, $r \geq 2$, if and only if at most one of the numbers $a_1, \ldots, a_r$ is odd.*

Let S be a sun such that S has a ray of multiplicity at least 3 or S has at least two double rays. Then the sequences $(2, \ldots, 2)$ for even order or $(1, 2, \ldots, 2)$ for odd order are admissible and not realizable in S. Hence S is not avd. According to the above remark we will consider only suns with one double ray. Section 2 concerns the realization of l-good sequences with one double ray and $r-1$ single rays. In Section 3 we characterize all avd suns with one double and at most two single rays.

Given an admissible sequence $\tau = (n_1, \ldots, n_k)$ for a graph G of order n, we will use the following convention to describe a realization $(V_1, \ldots, V_k)$ of τ in G. We choose an ordering $s = (v_1, \ldots, v_n)$ of the vertex set of G. Then we define the τ-parts according to the sequence s, that is $V_1 = \{v_1, \ldots, v_{n_1}\}$, $V_2 = \{v_{n_1+1}, \ldots, v_{n_1+n_2}\}$ and so on.

2. Realizations of l-Good Sequences

Theorem 4. *Every $(r-2)$-good sequence is realizable in a graph $Sun'(a_1, \ldots, a_r)$, $r \geq 2$, if and only if the following conditions hold:*

(1) *the numbers $a_1,\ldots,a_r$ are even,*

(2) *there exists $j \in \{1, \ldots, r\}$ such that $a_j \not\equiv 2(\mathrm{mod}\ 3)$,*

(3) *$a_1 \not\equiv 2(\mathrm{mod}\ 3)$ or $a_r \not\equiv 2(\mathrm{mod}\ 3)$ or there exists $j \in \{2, \ldots, r-1\}$ such that $a_j \not\equiv 0(\mathrm{mod}\ 3)$.*

Proof. Let n denote the order of the graph $G = Sun'(a_1, \ldots, a_r)$ (Figure 2).

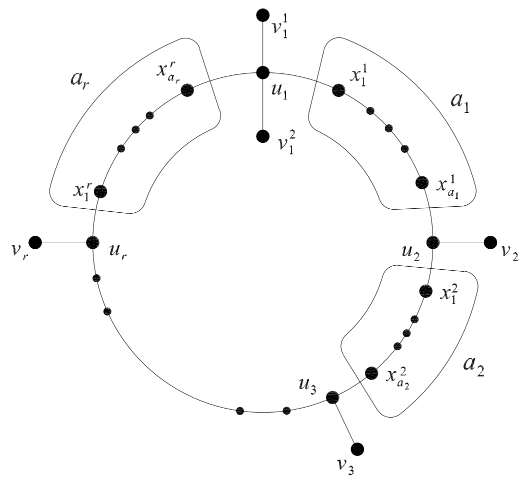

Figure 2. $Sun'(a_1, \ldots, a_r)$

Necessity. If at least one of the numbers $a_1,\ldots,a_r$ is odd then the sequence $(1, (2)^{\frac{n-1}{2}})$ for odd n or the sequence $((2)^{\frac{n}{2}})$ for even n is $(r-2)$-good and not realizable in $Sun'(a_1, \ldots, a_r)$. Sequences of k elements: $(t_1, \ldots, t_{r-2}, (3)^{k-r+2})$ where $t_i \in \{1, 4\}$ for $i = 1, \ldots, r-2$ or $((2)^{r-2}, (3)^{k-r+2})$ are $(r-2)$-good but not realizable when $a_j \equiv 2(\mathrm{mod}\ 3)$ for $j \in \{1, \ldots, r\}$ or when $a_1 \equiv a_r \equiv 2(\mathrm{mod}\ 3)$, $a_j \equiv 0(\mathrm{mod}\ 3)$ for $j \in \{2, \ldots, r-1\}$, respectively.

Sufficiency. By condition (1), the order n is odd. Let $\tau = (n_1, \ldots, n_k)$ be an $(r-2)$-good sequence. Since n is odd, there is an odd number of odd elements in τ. Let $n_1,\ldots,n_{k_1}$ be odd elements for some $k_1 \geq 1$ and

$n_{k_1+1},\ldots,n_k$ be even elements. Let us assume that $n_1 \geq \ldots \geq n_{k_1}$ and $n_{k_1+1} \geq \ldots \geq n_k$. We define sequence $(V_1, \ldots, V_k)$ of τ-parts according to

$$s = (v_1^1, u_1, v_1^2, x_1^1, \ldots, x_{a_1}^1, u_2, v_2, x_1^2, \ldots, x_{a_2}^2, \ldots, u_r, v_r, x_1^r, \ldots, x_{a_r}^r).$$

If all elements of τ are good, that is if every odd element is equal to 1, it is easy to observe that this construction gives a realization of τ in G. Hence we can assume that $n_1 \geq 3$. Suppose that in this case the construction does not give a realization of τ in G. Let i_1 denote the smallest $i \in \{1, \ldots, k\}$ such that the subgraph $G[V_i]$ is disconnected. Since $n_1 \geq 3$, vertices v_1^1, u_1, v_1^2 belong to V_1. It follows that $u_{j_1} \in V_{i_1-1}$ and $v_{j_1} \in V_{i_1}$ for some j_1 such that $2 \leq j_1 \leq r$. Observe that since the number of elements following u_{j_1} in s is odd, the integers n_{i_1-1} and n_{i_1} are odd. Since a_1 is even and n_1 is odd, $i_1 - 1 > 1$. It is clear that $n_{i_1} \geq 3$ and hence $n_{i_1-1} \geq 3$. We consider few possibilities.

Case A. There are at least $r-1$ good elements in τ or $j_1 \in \{3, \ldots, r\}$ or $a_r = 0$.

If the last element n_k is even, then we modify the ordering of elements in τ, obtain $\tau = (n_1, \ldots, n_{i_1-2}, n_k, n_{i_1-1}, \ldots, n_{k-1})$ and define new sequence of τ-parts according to s. If the last element $n_k = n_{k_1}$ is equal to 1, then we modify the ordering of elements in τ, obtain $\tau = (n_1, \ldots, n_{i_1-1}, n_k, n_{i_1}, \ldots, n_{k-1})$ and define new sequence of τ-parts according to s. It is easily seen that in both cases vertices v_1^1, u_1, v_1^2 belong to V_1 and for each $j = 2, \ldots, j_1$ both vertices u_j, v_j belong to connected τ-parts. Then we find the first disconnected subgraph $G[V_{i_2}]$ and repeat the above modification of the sequence τ by moving either its last even element before the element n_{i_2-1} or its last element equal to 1 before the element n_{i_2}. Then we define the sequence of τ-parts according to the modified τ.

The number of necessary modifications is at most $r-1$. Hence if there are at least $r-1$ good elements of τ, then we obtain a realization of τ in G. If $a_r = 0$, then it is easy to observe that $j_1 \leq r-1$. Therefore if $a_r = 0$ or $j_1 \geq 3$, then the number of necessary modifications is not greater than the least possible number $r-2$ of good elements of τ. Hence we finally obtain a realization of τ in G.

Case B. There are exactly $r-2$ good elements in τ and $j_1 = 2$ and $a_r \geq 2$.

We use the same procedure as in Case A but we define sequence of τ-parts according to the sequence of vertices

$$s^1 = (v_1^2, u_1, v_1^1, x_{a_r}^r, \ldots, x_1^r, u_r, v_r, \ldots, x_{a_2}^2, \ldots, x_1^2, u_2, v_2, x_{a_1}^1, \ldots, x_1^1).$$

If $a_1 = 0$, then we obtain a realization of τ in G. Hence we can assume that $a_1 \geq 2$.

Subcase B.1. $n_1 \geq 5$.
We start our procedure partitioning the set $V(G)$ according to the following sequence of vertices

$$s^2 = (x_{a_r-1}^r, x_{a_r}^r, v_1^1, u_1, v_1^2, x_1^1, \ldots, x_{a_1}^1, u_2, v_2,$$

$$x_1^2, \ldots, x_{a_2}^2, \ldots, u_r, v_r, x_1^r, \ldots, x_{a_r-2}^r).$$

Thus, vertices of the double ray v_1^1, u_1, $v_1^2 \in V_1$ and, since $n_{i_1} \geq 3$, vertices of the first single ray u_2, $v_2 \in V_{i_1}$. Then we proceed in the same way as in Case A. Observe that at each step the first disconnected subgraph corresponds to odd element $n_{i_l} \geq 3$ of the current sequence τ. The element n_{i_l-1} is odd and at least 3, too. Since the number of necessary modifications is at most $r-2$, we obtain a realization of τ in G.

Subcase B.2. $n_1 = 3$.
It follows that $a_1 \equiv 2 (\mathrm{mod}\ 3)$ and, analogously $a_r \equiv 2 (\mathrm{mod}\ 3)$.

B.2.a. $n_{k_1+1} \geq 6$.
We start our procedure with another ordering $\tau = (n_{k_1+1}, n_1, \ldots, n_{k_1}, n_{k_1+2}, \ldots, n_k)$. If $n_{k_1+1} - 4 \geq a_1$ or $n_{k_1+1} \not\equiv 1 (\mathrm{mod}\ 3)$ then we partition the set $V(G)$ according to the sequence

$$s^3 = (x_{a_r}^r, v_1^1, u_1, v_1^2, x_1^1, \ldots, x_{a_1}^1, u_2, v_2, x_1^2, \ldots, x_{a_2}^2, \ldots, u_r, v_r, x_1^r, \ldots, x_{a_r-1}^r).$$

If $n_{k_1+1} - 4 < a_1$ and $n_{k_1+1} \equiv 1 (\mathrm{mod}\ 3)$ then we partition the set $V(G)$ according to the sequence s^2. Thus, vertices of double ray v_1^1, u_1, $v_1^2 \in V_{k_1+1}$.

Since $a_1, \ldots, a_r$ are even, in both cases, there are no $j \in \{2, \ldots, r\}$ such that $v_j \in V_{k_1+1}$ and $u_j \notin V_{k_1+1}$. Observe that in both cases vertices u_2 and v_2 belong to the same τ-part.

Then we proceed again in the same way as in Case A and at each step for the first disconnected subgraph $G[V_{i_l}]$ there are two odd numbers n_{i_l-1},

n_{i_l} of current sequence τ such that $n_{i_l-1} = n_{i_l} = 3$. The number of necessary modifications is at most $r-2$ and we can move $r-3$ good elements. Let us suppose that we have moved $r-3$ good elements and we find the first disconnected subgraph $G[V_{i_{r-2}}]$ with $j_{r-2} = r$. Then the number of elements following u_r in s^3 is equal to 2 modulo 3, the number of elements following u_r in s^2 is equal to 1 modulo 3 and, in both cases, every element of modified τ following $n_{i_{r-2}-1}$ is equal to 3, which is impossible. Hence, in fact, the number of necessary modifications is at most $r-3$ and we obtain a realization of τ in G.

B.2.b. $n_{k_1+1} \in \{4, 2\}$ or there are no even elements in τ ($k_1 = k$). Then the sequence τ is of the form $((3)^{l_1}, (1)^{l_2}, (4)^{l_3}, (2)^{l_4})$, where $l_1 \geq 1$, l_2, l_3, $l_4 \geq 0$ and $l_2 + l_3 + l_4 = r-2$. We start our procedure of partitioning the set $V(G)$ according to the sequence s.

We proceed in the same way as in Case A and at each step for the first disconnected subgraph $G[V_{i_l}]$, there are two numbers n_{i_l-1}, n_{i_l} of current sequence τ such that $n_{i_l-1} = n_{i_l} = 3$. If during our procedure we need at most $r-2$ modifications then we obtain a realization of τ in G. Hence we may assume that the number of necessary modifications in the procedure is equal to $r-1$. In the first step $j_1 = 2$. We modify the ordering of elements in τ, obtain $\tau = (n_1, \ldots, n_{i_1-2}, n_k, n_{i_1-1}, \ldots, n_{k-1})$ and define new sequence of τ-parts according to s. The next step is for $j_2 = 3$. Let us suppose that $n_k = 2$. Then $a_2 \equiv 0 (\text{mod } 3)$. If either $n_{k_1+1} = 4$ or $n_{k_1} = 1$, then we return to the first step with $j_1 = 2$. We modify the ordering of elements in τ, obtain either $\tau = (n_1, \ldots, n_{i_1-2}, n_{k_1+1}, n_{i_1-1}, \ldots, n_{k_1}, n_{k_1+2}, \ldots, n_k)$ or $\tau = (n_1, \ldots, n_{i_1-1}, n_{k_1}, n_{i_1}, \ldots, n_{k_1-1}, n_{k_1+1}, \ldots, n_k)$, respectively. Thus vertices of rays u_2v_2, u_3v_3 belong to certain connected subgraphs induced by τ-parts of G. Moreover, we will need at most $r-3$ modifications. Hence we obtain a realization of τ in G. We may suppose that $\tau = ((3)^{k-r+2}, (2)^{r-2})$. Since we need $r-1$ modifications, $a_i \equiv 0 (\text{mod } 3)$ for $i \in \{2, \ldots, r-1\}$, contrary to the condition (3). Therefore we may assume that $\tau = ((3)^{k-r+2}, t_1, \ldots, t_{r-2})$, $t_i \in \{1, 4\}$ for $i = 1, \ldots, r-2$. Since we need $r-1$ modifications, $a_i \equiv 2 (\text{mod } 3)$ for $i \in \{1, \ldots, r\}$, contrary to the condition (2). ■

The next corollary follows immediately from the above proof.

Corollary 5. *Every* $(r-1)$*-good sequence is realizable in a graph* $Sun'(a_1, \ldots, a_r)$, $r \geq 2$, *if and only if the numbers* $a_1, \ldots, a_r$ *are even.*

3. Arbitrarily Vertex Decomposable Suns with One Double and at Most Two Single Rays

Observation 6. A graph $Sun'(a)$ is arbitrarily vertex decomposable if and only if the number a is even.

Proof. Let n denote the order of $Sun'(a)$.
Necessity. For odd a the sequence $((2)^{\frac{n}{2}})$ is admissible and non-realizable.
Sufficiency. Let $\tau = (n_1, \ldots, n_k)$ be an admissible sequence for $Sun'(a)$. Since n is odd, there is an odd element n_{i_0} in τ. With another ordering $\tau = (n_{i_0}, n_1, \ldots, n_{i_0-1}, n_{i_0+1}, \ldots, n_k)$ we define the sequence of τ-parts according to $s = (v_1^1, u_1, v_1^2, x_1^1, \ldots, x_a^1)$ and obtain a realization of τ in $Sun'(a)$. ■

The next observation follows immediately from Theorem 4 for $r = 2$ since every admissible sequence is 0-good.

Observation 7. A graph $Sun'(a, b)$ is arbitrarily vertex decomposable if and only if the following conditions hold:

(1) the numbers a, b are even,
(2) $a \not\equiv 2(\text{mod } 3)$ or $b \not\equiv 2(\text{mod } 3)$.

Theorem 8. *A graph $Sun'(a, b, c)$ is arbitrarily vertex decomposable if and only if the following conditions hold:*

(1) *the numbers a, b, c are even,*
(2) $a \not\equiv 2(\text{mod } 3)$ *or* $b \equiv 1(\text{mod } 3)$ *or* $c \not\equiv 2(\text{mod } 3)$,
(3) $[a \not\equiv 2(\text{mod } 3)$ *and* $c \not\equiv 2(\text{mod } 3)]$ *or* $(a + b + c) \not\equiv 2(\text{mod } 3)$.

Proof. Let n denote the order of $G = Sun'(a, b, c)$. If $Sun'(a, b, c)$ is arbitrarily vertex decomposable, Theorem 4 implies (1) and (2). If the condition (3) does not hold, then the sequence $((3)^{\frac{n}{3}})$ is admissible and non-realizable. By Theorem 4 for $r = 3$ it is enough to prove that if conditions (1), (2) and (3) hold, then every admissible sequence without good elements is realizable in G. Let $\tau = (n_1, \ldots, n_k)$ be an admissible sequence of odd elements greater then 1. We assume that $n_1 \geq \ldots \geq n_k \geq 3$. We define the sequence of τ-parts according to

$$s^4 = (v_1^1, u_1, v_1^2, x_1^1, \ldots, x_a^1, u_2, v_2, x_1^2, \ldots, x_b^2, u_3, v_3, x_1^3, \ldots, x_c^3).$$

The induced subgraphs $G[V_i]$ are connected for all i or one of the following two cases occurs.

Case A. There is $i_0 \in \{1, \ldots, k-1\}$ such that $u_2 \in V_{i_0}$ and $v_2 \in V_{i_0+1}$. Then $a \geq 2$, since a is even. If $\tau = ((3)^{\frac{n}{3}})$, then $a \equiv 2 (\bmod\ 3)$ and $n = a+b+c+7 \equiv 0 (\bmod\ 3)$, contrary to (3). Hence we may assume that $n_1 \geq 5$. If $c = 0$, then we define the sequence of τ-parts according to

$$s^5 = (v_3, u_3, v_1^1, u_1, v_1^2, x_1^1, \ldots, x_a^1, u_2, v_2, x_1^2, \ldots, x_b^2)$$

and obtain a realization of τ in G.

Hence we may assume that $c \geq 2$. We define the sequence of τ-parts according to

$$s^6 = (x_c^3, v_1^1, u_1, v_1^2, x_1^1, \ldots, x_a^1, u_2, v_2, x_1^2, \ldots, x_b^2, u_3, v_3, x_1^3, \ldots, x_{c-1}^3).$$

This construction gives a realization of τ, unless there exists an i_1 such that $u_3 \in V_{i_1}$ and $v_3 \in V_{i_1+1}$. In such a case we define the sequence of τ-parts according to

$$s^7 = (x_{c-1}^3, x_c^3, v_1^1, u_1, v_1^2, x_1^1, \ldots, x_a^1, u_2, v_2, x_1^2, \ldots, x_b^2, u_3, v_3, x_1^3, \ldots, x_{c-2}^3).$$

Therefore every induced subgraph $G[V_i]$ is connected for $i \in \{1, \ldots, k\}$.

Case B. The vertices u_2 and v_2 belong to the same τ-part but there is i_0 such that $u_3 \in V_{i_0}$ and $v_3 \in V_{i_0+1}$.

Then $c \geq 2$. If $\tau = ((3)^{\frac{n}{3}})$, then $c \equiv 2 (\bmod\ 3)$ and $n = a + b + c + 7 \equiv 0 (\bmod\ 3)$, contrary to (3). Hence we may assume that $n_1 \geq 5$. We define the sequence of τ-parts according to s^6. This construction gives a realization of τ in G or there exists an i_1 such that $u_2 \in V_{i_1}$ and $v_2 \in V_{i_1+1}$. In the latter case $b \geq 2$, since otherwise the induced subgraphs $G[V_i]$ corresponding to s^6 are all connected. We define the sequence of τ-parts according to s^7. Therefore, since u_1, v_1^1, $v_1^2 \in V_1$, u_2, $v_2 \in V_{i_1+1}$ and u_3, $v_3 \in V_{i_0+1}$, we obtain a realization of τ in G. ■

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