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ON PARTITIONS OF HEREDITARY PROPERTIES OF GRAPHS

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Abstract

In this paper a concept $\mathcal{Q}$-Ramsey Class of graphs is introduced, where $\mathcal{Q}$ is a class of bipartite graphs. It is a generalization of well-known concept of Ramsey Class of graphs. Some $\mathcal{Q}$-Ramsey Classes of graphs are presented (Theorem 1 and 2). We proved that $\mathcal{T}_2$, the class of all outerplanar graphs, is not $\mathcal{D}_1$-Ramsey Class (Theorem 3). This results leads us to the concept of acyclic reducible bounds for a hereditary property $\mathcal{P}$. For $\mathcal{T}_2$ we found two bounds (Theorem 4). An improvement, in some sense, of that in Theorem 5 is given.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For a graph $G = (V, E)$ and $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by vertices of U.

A *k-colouring* of a graph G is a mapping f from the set of vertices of G to the set of k colours such that adjacent vertices receive distinct colours. An *acyclic k-colouring* of a graph G is a k-colouring of G satisfying the subgraph induced by every pair of colour classes has no cycle. The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$.

Similarly, for a class $\mathcal{P}$ of graphs, the *acyclic chromatic number of* $\mathcal{P}$, denoted by $\chi_a(\mathcal{P})$, is defined as the maximum $\chi_a(G)$ over all graphs $G \in \mathcal{P}$.

This number has been studied extensively over past thirty years. Several authors have been able to determine $\chi_a(\mathcal{P})$ for some classes $\mathcal{P}$ of graphs such as graphs of maximum degree 3, considered by Grűnbaum in [10] and of maximum degree 4, studied by Burstein in [7]. The acyclic chromatic number of planar graphs was found by Borodin in 1979, see [3] for details. Planar graphs with "large" girth, outerplanar and 1-planar graphs also were considered, see for instance [4, 5], etc.

In nineties Sopena *at al.*, have begun their studies on acyclic colourings of graphs with respect to hereditary properties of graphs. Namely, they have considered outerplanar, planar graphs and graphs with bounded degree, see [1, 2]. To precise this notion, we need some definitions. We follow [6].

Let $\mathcal{I}$ denote the class of all finite simple graphs. A *property of graphs* is any nonempty class of graphs from $\mathcal{I}$, which is closed under isomorphisms. A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed under subgraphs, i.e., if $H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if for each graph G all of whose components have the property $\mathcal{P}$ it follows that $G \in \mathcal{P}$, too. By $\mathbb{L}^a$ we denote the set of all additive hereditary properties of graphs. We list some additive hereditary properties:

$$\begin{aligned} \mathcal{O} &= \{G \in \mathcal{I} : E(G) = \emptyset\}, \\ \mathcal{O}^k &= \{G \in \mathcal{I} : \chi(G) \leq k\}, \\ \mathcal{T}_2 &= \text{the class of all outerplanar graphs}, \\ \mathcal{D}_1 &= \text{the class of all acyclic graphs}. \end{aligned}$$

A hereditary property $\mathcal{P}$ can be uniquely determined by the set of *minimal forbidden subgraphs* which can be defined as follows:

$$\boldsymbol{F}(\mathcal{P}) = \{G \in \mathcal{I} : \ G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\}.$$

Let $\mathcal{F}$ be a family of graphs, $Forb(\mathcal{F})$ is defined to be the property of all graphs having no subgraph isomorphic to any graph of $\mathcal{F}$. Thus, $\mathcal{P} = Forb(\boldsymbol{F}(\mathcal{P}))$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ be hereditary properties of graphs. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-*colouring* of a graph G is a mapping f from the set of vertices of G to a set of k colours such that for every colour i, the subgraph induced by the i-coloured vertices has property $\mathcal{P}_i$.

Suppose $\mathcal{F}$ is a nonempty family of connected bipartite graphs, each with at least 2 vertices.

A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-colouring of a graph G is said to be *$\mathcal{F}$-free* if for every two distinct colours i and j, the subgraph induced by all the edges linking an i-coloured vertex and a j-coloured vertex does not contain a subgraph isomorphic to any graph F in $\mathcal{F}$. These $\mathcal{F}$-free $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$-colourings are a natural generalization of acyclic colourings if $\mathcal{F} = \{C_{2p} : p \geq 2\}$, star-forest colourings if $\mathcal{F} = \{P_4\}$, and so on.

We assume that $\mathcal{F}$ is a minimal set of forbidden subgraphs for a property $\mathcal{Q}$, i.e., $\mathcal{F} = \boldsymbol{F}(\mathcal{Q})$.

A property $\mathcal{R} = \mathcal{P}_1 \underset{\mathcal{Q}}{\circ} \mathcal{P}_2 \underset{\mathcal{Q}}{\circ} \cdots \underset{\mathcal{Q}}{\circ} \mathcal{P}_n$ is defined as the set of all graphs having an $\boldsymbol{F}(\mathcal{Q})$-free $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-colouring.

If $\mathcal{Q} = \mathcal{D}_1$ then we use the notation $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2 \odot \cdots \odot \mathcal{P}_n$.

A partition of $V(G)$ generated by an $\mathcal{F}$-free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring of G is called an *$\mathcal{F}$-free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-partition.* An $\mathcal{F}$-free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colouring, where $\mathcal{P}_i = \mathcal{I}$ for $i = 1, ..., k$, will be called briefly an *$\mathcal{F}$-free colouring.* If $\mathcal{F}$ consists of a single graph F, then it will be called an F-free colouring (partition) for short.

For definitions and notations not presented here, we refer to [6, 9].

2. Ramsey Classes of Graphs

A hereditary property $\mathcal{P}$ is called a *$\mathcal{Q}$-Ramsey Class* if for every $G \in \mathcal{P}$ there is an $H \in \mathcal{P}$ such that $H \underset{\mathcal{Q}}{\to} G$, i.e., for every $\boldsymbol{F}(\mathcal{Q})$-free bicolouring of H there is a monochromatic subgraph isomorphic to G.

It is easy to see that if $\mathcal{P}$ is a $\mathcal{Q}$-Ramsey Class and $\mathcal{Q}' \subseteq \mathcal{Q}$, then $\mathcal{P}$ is a $\mathcal{Q}'$-Ramsey Class.

Proposition 1. *Let $k \geq 2$. Then $\mathcal{O}^k$ is a $\mathcal{D}_1$-Ramsey Class.*

Proof. Let $G \in \mathcal{O}^k$ and $\alpha(G) = \alpha$. It is easy to see that the graph $H = K_{k \times (\alpha+2)}$, the complete k-partite balanced (i.e., each colour class has the same cardinality $\alpha + 2$) graph satisfies the requirements of Theorem. ■

Theorem 1. *Let F be a connected bipartite graph and $\mathcal{Q}$ =Forb(F). Then $\mathcal{O}^k$ is a $\mathcal{Q}$-Ramsey Class for $k \geq 2$.*

Proof. Let F be a given connected bipartite graph and let F be a subgraph of $K_{r,s}$ with $r \leq s$, and let $G \in \mathcal{O}^k$. Consider a graph $H = K_{k \times n}$, where

$n \geq s + \alpha(G) + 1$. Suppose that $\{X, Y\}$ is an F-free bipartition of H. It implies that one of sets X, Y, say X, has at most s elements in each colour class V_i of H. Similarly, each colour class V_i of H has at least $\alpha(G)$ elements in Y, thus G is a subgraph of $H[Y]$. ■

Let k be a positive integer. A *k-clique* is a complete graph of order k. A *k-tree* is a graph defined inductively as follows: A k-clique is a k-tree. If G is a k-tree, and K is a subgraph of G isomorphic to a k-clique, then a graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a *partial k-tree*. The *tree-width* of a graph G is zero if G is edgeless; otherwise it is a smallest integer k such that G is a partial k-tree, and will be denoted by $t_w(G)$. Nontrivial forests have tree-width 1, while every graph has some tree-width.

Let us denote by

$$\mathcal{TW}_k = \{G \in \mathcal{I} : t_w(G) \leq k\}.$$

According to G. Ding, B. Oporowski, D.P. Sanders and D. Vertigan, see [8], we recall the notion of "large" k-trees.

Let k be a positive integer. We will define some classes of k-trees, each with a *level* function λ defined on its vertices. Let the level of a subgraph of a graph with a level function be the maximum level of its vertices. Let $T(k, 0, 0)$ be the k-clique, and each of its vertices have level zero. Let l, r be non-negative integers. We will proceed by induction on l. The k-tree $T(k, l, r)$ and its level function are obtained from k-tree $T = T(k, l-1, r)$ (or $T = T(k, 0, 0)$ if $l = 1$) and its level function by the following: For each k-clique K of T that has level $l - 1$, add r new vertices, join each of them to all vertices of K, and declare the new vertices to be at level l. For a new vertex v added, let $K(v)$ denote this k-clique K of level $l - 1$.

Proposition 2 [8]**.** *The graph $T(k, l, r)$ is a k-tree and every k-tree is a subgraph of $T(k, l, r)$, for some l, r.*

Theorem 2. *Let $k \geq 2$. Then $\mathcal{TW}_k$ is a $\mathcal{D}_1$-Ramsey Class.*

Proof. Let $G \in \mathcal{TW}_k$. By Proposition 2 it follows that $G \subseteq T(k, l, r)$ for some integers l, r. Let $H = T(k, p, s)$, $s, p > 1$. We claim that if p and s are large enough, then in every acyclic bicolouring f with a bipartition $\{U_1, U_2\}$ of $V(H)$, $H[U_1] \supseteq T(k, l, r)$ or $H[U_2] \supseteq T(k, l, r)$.

Firstly, let us observe that if J is any k-clique in H then J has at least $k-1$ monochromatic vertices, say $|V(J) \cap U_1| \geq k-1$, in any acyclic bicolouring f of H.

Secondly, if a k-clique J of level $j < p$ has exactly one vertex in U_2 then there is a monochromatic k-clique J' of the level $j+1$ with $V(J') \subseteq U_1$.

Since we choose p much larger than l therefore, without loss of generality, assume that k-clique K of the level zero is monochromatic, say $V(K) \subseteq U_1$.

Now let x be the vertex of level one in H and $K(x) = K \subseteq U_1$. If $x \in U_2$ then $y \in U_1$ for all vertices $y \neq x$ of level one. Therefore all, except at most one, k-cliques of level one have vertices in U_1. Now, if we consider a vertex x' of level two, with $K(x') \subseteq U_1$, we get that all, except at most one, k-cliques of level two, having common vertices with $K(x')$ have vertices in U_1, and so on. If s is large enough then $T(k,l,r) \subseteq H[U_1]$. ∎

Now we consider the property $\mathcal{T}_2$ which is a proper subclass of $\mathcal{TW}_2$. For $\mathcal{T}_2$ we have the following results.

Theorem 3. *$\mathcal{T}_2$ is not a $\mathcal{D}_1$-Ramsey Class.*

To prove Theorem 3 we need some notations and lemmas.

Let $\mathcal{G} = \{A_l : A_l = T(2,l,1), l > 0, A_0 = T(2,0,0)\}$. It is easy to see that $\mathcal{G}$ is a family of maximal outerplanar graphs with a level function and $A_l \subset A_{l+1}$ for all l.

Lemma 1. *If $G \in \mathcal{T}_2$ then there is an integer $k \geq 0$ such that $G \subseteq A_k$.*

Proof. Clearly, it is enough to consider only maximal outerplanar graphs G with at least 3 vertices. The proof is by induction on the number of vertices of G. If $|V(G)| = 3$ then G is isomorphic to A_1. Assume that for all maximal outerplanar graphs with less than n vertices, $n \geq 3$, the lemma is true. Consider a maximal outerplanar graph G with n vertices. Let $x \in V(G)$ such that $deg_G(x) = 2$ and $G' = G - x$. Applying the induction hypothesis to G' we get $A_k \in \mathcal{G}$ such that $G' \subseteq A_k$. If $G \not\subseteq A_k$ then we construct a graph A_{k+1} from A_k. Since $deg_G x = 2$ it is clear that $G \subseteq A_{k+1}$. ∎

Lemma 2.

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(G_1).$$

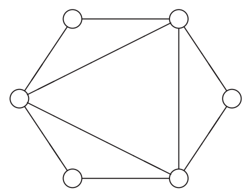

Figure 1. Graph G_1

Proof. From Lemma 1 it follows that it is enough to consider only graphs $A_0, A_1, \ldots, A_n, \ldots$ from the family $\mathcal{G}$. The proof is by induction on n. Obviously, the lemma is true if $n = 0, 1, 2$. We consider the graph A_n and A_{n+1}, for $n \geq 3$, assuming that f is an acyclic colouring of A_n, with a bipartition $\{U_1, U_2\}$ of $V(A_n)$, such that U_1 (the set of red vertices) is independent and $A_n[U_2]$ (the subgraph induced by blue vertices) has the property $\mathcal{R} = Forb(G_1)$. We use f to construct an acyclic colouring f' of A_{n+1}, with a bipartition $\{U_1', U_2'\}$ of $V(A_{n+1})$, such that U_1' (the set of red vertices) is independent and $A_{n+1}[U_2']$ (the subgraph induced by blue vertices) has the property $\mathcal{R}$. First, let $f'(v) = f(v)$ for all vertices in A_{n+1} of level less than $n+1$.

Let x, y be a pair of uncoloured vertices of the level $n+1$ in A_{n+1}. Let us assume, without loss of generality, that x is adjacent to a and b, and y is adjacent to b and c such that they form a triangle in A_n. It is clear that a, b, c have level less than $n+1$ and form a triangle in A_n, and f' has already coloured them.

To colour the vertices of the level $n+1$ we apply the following rules.

Rule 1. If one of the vertices a, b, c is red, then both x and y should be blue.

Rule 2. If a, b, c are blue, then x should be red and y should be blue.

From the construction of the graph A_{n+1} it follows that:

(1) If C is a cycle in A_{n+1} containing x (respectively y), then C contains also either the path (x, b, y) or (x, b, c) (either the path (y, b, x) or (y, b, a) respectively);

(2) If G is a subgraph of A_{n+1} containing x (respectively y), then G contains also a, b, c and y (respectively x).

Colouring rules and (1) implies that the obtained colouring is acyclic.

Similarly, by the rules and (2) we see that blue vertices induce in A_{n+1} a graph with the property $\mathcal{R}$ and red vertices are independent.

It is clear that if we apply these colouring rules to each such a pair of vertices of the level $n+1$, we will obtain a required colouring of A_{k+1}. ■

Now we are ready to prove Theorem 3.

***Proof of Theorem* 3.** We only need to find an outerplanar graph F such that for an arbitrary outerplanar graph H there is an acyclic bipartition $\{U_1, U_2\}$ of $V(H)$, with U_1 being independent and $H[U_2] \not\supseteq F$. Clearly, Lemma 2 yields the graph G_1 (Figure 1), which satisfies the requirements of Theorem. ■

3. Acyclic Reducible Bounds

In this section we give some acyclic reducible bounds for the class of outerplanar graphs. We start with a few definitions.

An additive hereditary property $\mathcal{R}$ is said to be *acyclic reducible* in $\mathbb{L}^a$ if there are nontrivial additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2$ and *acyclic irreducible* in $\mathbb{L}^a$, otherwise.

Obviously, the smallest acyclic reducible property in $\mathbb{L}^a$ is the property $\mathcal{O}^{(2)} = \mathcal{D}_1$.

Theorem 4.

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(G_1, G_2),$$

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(G_1, G_3).$$

***Proof*.** We will proof only the first inclusion, the second one can be proved similarly. As in the proof of Lemma 2, we use Lemma 1 to restrict our attention only to graphs $A_0, A_1, \ldots, A_n, \ldots$ from $\mathcal{G}$. The proof is by induction on n. Obviously, it holds for $n = 0, 1, 2$. We consider the graph A_n and A_{n+1}, for $n \geq 3$, assuming that f is an acyclic colouring of A_n, with a bipartition $\{U_1, U_2\}$ of $V(A_n)$, such that U_1 (the set of red vertices) is independent and $A_n[U_2]$ (the subgraph induced by blue vertices) has the property $\mathcal{R} = Forb(G_1, G_2)$. We use f to construct an acyclic colouring f' of A_{n+1}, with a bipartition $\{U_1', U_2'\}$ of $V(A_{n+1})$, such that U_1' (the set of red vertices) is independent and $A_{n+1}[U_2']$ (the subgraph induced by blue vertices) has the property $\mathcal{R}$. First, let $f'(v) = f(v)$ for all vertices in A_{n+1} of level less than $n+1$.

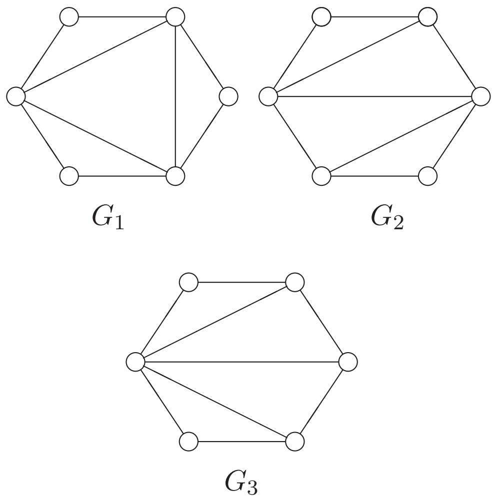

Figure 2. Graphs G_1, G_2 and G_3.

Let x, y be a pair of uncoloured vertices of the level $n+1$ in A_{n+1}. Let us assume that x is adjacent to a and b, and y is adjacent to b and c such that the vertices a, b, c form a triangle in A_n and are coloured.

Now we denote some vertices of A_n. Let $e \neq b$ be a unique vertex adjacent to both a and c, let d be be the vertex different from c adjacent to both a and e (it is a unique vertex in A_n with these properties), and let h be a vertex different from a adjacent to both c and e (it is only one such a vertex in A_n).

It is clear that b has level n. The level of one from $\{a, c\}$ is equal to $n-1$, say c; then the level of a is less than or equal to $n-2$.

To colour the vertices of the level $n+1$ we apply the following rules.

Rule 1. If one of the vertices a, b, c is red, then both x and y should be blue.

Rule 2. If a, b, c are blue, then

(a) if $f'(e) =$ red, then $f'(x) = f'(y) =$ blue.

(b) if $f'(e) =$ blue, then

  (b_1) if $f'(d) =$ red, then $f'(x) =$ red and $f'(y) =$ blue;

  (b_2) if $f'(d) =$ blue, then $f'(x) =$ blue and $f'(y) =$ red.

(Notice that the case $f'(d) = f'(h) =$ blue is impossible, in such case all vertices $\{a, b, c, e, d, h\}$ would be coloured blue and a graph induced by this set in A_n would be isomorphic to G_1.)

From the construction of the graph A_{n+1} it follows that:

(1) If C is a cycle in A_{n+1} containing x (respectively y), then C contains also either the path (x, b, y) or (x, b, c) (either the path (y, b, x) or (y, b, a) respectively);

(2) If F is a subgraph of A_{n+1} isomorphic to G_1, containing x or y, then $V(F) = \{x, y, a, b, c, e\}$;

(3) If F is a subgraph of A_{n+1} isomorphic to G_2, containing x (respectively y), then $V(F) = \{x, a, b, c, e, h\}$ (respectively $V(F) = \{y, a, b, c, d, e\}$). Colouring rules and (1) implies that the obtained colouring is acyclic. From (2) and (3) we see that blue vertices induce in A_{n+1} a graph with the property $\mathcal{R}$. Red vertices are independent, which is clear from the colouring rules.

If we apply the colouring rules to each such a pair of vertices of level $n+1$, then we obtain an acyclic colouring of A_{n+1}. ■

A maximal outerplanar graph G with at least 3 vertices is called a *2-path of order* $n = 2p$, if G consist of two paths $P_1 = (x_1, x_2, \ldots, x_p)$, $P_2 = (y_1, y_2, \ldots, y_p)$ and additional edges: $x_i y_i$, $i = 1, \ldots, p$ and $x_j y_{j+1}$ for $j = 1, \ldots, p-1$. For an odd $n = 2p-1$ a 2-path H is defined as $H = G - x_p$, where G is 2-path of even order.

A maximal outerplanar graph G with at least 3 vertices is called a *fan of order* n, if G is obtained from a star $K_{1,n-1}$ by joining all vertices of degree one by a path.

Additionally we assume that the graph K_1 and K_2 is a *trivial 2-path* and a *trivial fan*. For each $n \leq 5$ there is exactly one (up to isomorphism) maximal outerplanar graph of order n which is a 2-path and a fan.

Lemma 3. *Let G be a maximal outerplanar graph of order $n \geq 3$. Then*

(a) *G is a fan if and only if neither $G_1 \subseteq G$ nor $G_2 \subseteq G$.*

(b) *G is a 2-path if and only if neither $G_1 \subseteq G$ nor $G_3 \subseteq G$.*

Proof. (a) The fact that any fan contains neither G_1 nor G_2 follows immediately by the definition. For the converse, we employ induction on n, the order of G. Clearly, for $n \leq 6$ it is true. Assume every graph with fewer than $n \geq 7$ vertices is a fan, and suppose G has order n and does not contain a subgraph isomorphic to G_i, $i = 1, 2$. Let x be the vertex of degree 2 in G. By the inductive hypothesis, $G' = G - x$ is a fan of order $n - 1 \geq 6$. Let y

be the unique vertex of maximum degree in G'. If x is not adjacent to y in G, then G contains G_1 or G_2. If x is adjacent to y in G, then G is a fan.

(b) Again, it is easy to see that any 2-path contains neither G_1 nor G_3. It follows immediately by the definition. For the converse, we use again induction on n. Clearly, for $n \leq 6$ it is true. Let us assume that every graph with fewer than $n \geq 7$ vertices is a 2-path, and suppose G has order n and does not contain a subgraph isomorphic to G_i, $i = 1, 3$. It is easy to see, that if G has a vertex of degree greater than 4, then by the maximality of G we get that G contains a subgraph isomorphic to G_3. Hence we can assume that all vertices of G are of degree at most 4. Let x be the vertex of degree 2 in G. The graph $G' = G - x$ is a maximal outerplanar graph with less than n vertices, then by the inductive hypothesis, G' is a 2-path. It is clear that G' has only four vertices of degree less than 4 and x has to be adjacent in G to exactly two of them. From maximality of G we get that x and its neighbours induce a triangle in G i.e., G is a 2-path. ■

Let us recall that a *block* of a given graph G is defined to be a maximal connected subgraph of G without a cutvertex.

A *fan* (*2-path*) *tree* is a connected graph G every block of each is a fan (2-path).

Let us define the property $\mathcal{FT}$ ($\mathcal{PT}$) as the family of all fan (2-path) trees and their subgraphs. Each property is additive hereditary and a proper subfamily of all outerplanar graphs.

From the definition of $\mathcal{FT}$ it follows that G_1 and G_2 do not belong to $\mathcal{FT}$. Similarly, G_1 and G_3 do not belong to $\mathcal{PT}$. It implies the following corollary.

Corollary 1.

$$\mathcal{FT} \subseteq Forb(G_1, G_2),$$

$$\mathcal{PT} \subseteq Forb(G_1, G_3).$$

Because of above Corollary, the next theorem gives a little better than in Theorem 4 two acyclic reducible bounds for outerplanar graphs.

Theorem 5.

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{FT},$$

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot \mathcal{PT}.$$

Proof. We will prove only the first bound, the second one can be proved similarly. By Lemma 1, it is enough to show that each graph of the family $\mathcal{G}$ has the property $\mathcal{O} \odot \mathcal{FT}$. On the contrary, suppose that there is a graph $G \in \mathcal{G}$ such that in every acyclic bipartition $\{U_1, U_2\}$ of $V(G)$, with U_1 being independent, $G[U_2]$ has a subgraph isomorphic to a graph from $\mathcal{T}_2 - \mathcal{FT}$. Let F be its block which is not a fan. Since any maximal outerplanar graph of order ≤ 5 is a fan, thus F has order at least 6. Lemma 3 implies that F contains a subgraph isomorphic to G_1 or to G_2. This fact contradicts Theorem 4. ∎

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