Discussiones Mathematicae Graph Theory 26 (2006) 377–387

ON PARTITIONS OF HEREDITARY PROPERTIES OF GRAPHS

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Abstract

In this paper a concept Q-Ramsey Class of graphs is introduced, where Q is a class of bipartite graphs. It is a generalization of wellknown concept of Ramsey Class of graphs. Some Q-Ramsey Classes of graphs are presented (Theorem 1 and 2). We proved that \mathcal{T}_2 , the class of all outerplanar graphs, is not \mathcal{D}_1 -Ramsey Class (Theorem 3). This results leads us to the concept of acyclic reducible bounds for a hereditary property \mathcal{P} . For \mathcal{T}_2 we found two bounds (Theorem 4). An improvement, in some sense, of that in Theorem 5 is given.

Keywords: hereditary property, acyclic colouring, Ramsey class. **2000 Mathematics Subject Classification:** 05C15, 05C75.

1. INTRODUCTION

We consider only finite undirected graphs without loops or multiple edges. For a graph G = (V, E) and $U \subseteq V$, G[U] denotes the subgraph of G induced by vertices of U.

A k-colouring of a graph G is a mapping f from the set of vertices of G to the set of k colours such that adjacent vertices receive distinct colours. An acyclic k-colouring of a graph G is a k-colouring of G satisfying the subgraph

induced by every pair of colour classes has no cycle. The minimum k such that G has an acyclic k-colouring is called the *acyclic chromatic number* of G, denoted by $\chi_a(G)$.

Similarly, for a class \mathcal{P} of graphs, the *acyclic chromatic number of* \mathcal{P} , denoted by $\chi_a(\mathcal{P})$, is defined as the maximum $\chi_a(G)$ over all graphs $G \in \mathcal{P}$.

This number has been studied extensively over past thirty years. Several authors have been able to determine $\chi_a(\mathcal{P})$ for some classes \mathcal{P} of graphs such as graphs of maximum degree 3, considered by Grűnbaum in [10] and of maximum degree 4, studied by Burstein in [7]. The acyclic chromatic number of planar graphs was found by Borodin in 1979, see [3] for details. Planar graphs with "large" girth, outerplanar and 1-planar graphs also were considered, see for instance [4, 5], etc.

In nineties Sopena *at al.*, have begun their studies on acyclic colourings of graphs with respect to hereditary properties of graphs. Namely, they have considered outerplanar, planar graphs and graphs with bounded degree, see [1, 2]. To precise this notion, we need some definitions. We follow [6].

Let \mathcal{I} denote the class of all finite simple graphs. A property of graphs is any nonempty class of graphs from \mathcal{I} , which is closed under isomorphisms. A property \mathcal{P} of graphs is called *hereditary* if it is closed under subgraphs, i.e., if $H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property \mathcal{P} is called *additive* if for each graph G all of whose components have the property \mathcal{P} it follows that $G \in \mathcal{P}$, too. By \mathbb{L}^a we denote the set of all additive hereditary properties of graphs. We list some additive hereditary properties:

- $\mathcal{O} = \{ G \in \mathcal{I} : E(G) = \emptyset \},\$
- $\mathcal{O}^k = \{ G \in \mathcal{I} : \chi(G) \le k \},\$
- \mathcal{T}_2 = the class of all outerplanar graphs,
- \mathcal{D}_1 = the class of all acyclic graphs.

A hereditary property \mathcal{P} can be uniquely determined by the set of *minimal* forbidden subgraphs which can be defined as follows:

 $\boldsymbol{F}(\mathcal{P}) = \{ G \in \mathcal{I} : \ G \notin \mathcal{P}, \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P} \}.$

Let \mathcal{F} be a family of graphs, $Forb(\mathcal{F})$ is defined to be the property of all graphs having no subgraph isomorphic to any graph of \mathcal{F} . Thus, $\mathcal{P} = Forb(\mathbf{F}(\mathcal{P}))$.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ be hereditary properties of graphs. A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$ -colouring of a graph G is a mapping f from the set of vertices of G to a set of k colours such that for every colour i, the subgraph induced by the *i*-coloured vertices has property \mathcal{P}_i .

Suppose \mathcal{F} is a nonempty family of connected bipartite graphs, each with at least 2 vertices.

A $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$ -colouring of a graph G is said to be \mathcal{F} -free if for every two distinct colours i and j, the subgraph induced by all the edges linking an *i*-coloured vertex and a *j*-coloured vertex does not contain a subgraph isomorphic to any graph F in \mathcal{F} . These \mathcal{F} -free $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k)$ -colourings are a natural generalization of acyclic colourings if $\mathcal{F} = \{C_{2p} : p \geq 2\}$, star-forest colourings if $\mathcal{F} = \{P_4\}$, and so on.

We assume that \mathcal{F} is a minimal set of forbidden subgraphs for a property \mathcal{Q} , i.e., $\mathcal{F} = \mathbf{F}(\mathcal{Q})$.

A property $\mathcal{R} = \mathcal{P}_1 {}_{\mathcal{Q}} \mathcal{P}_2 {}_{\mathcal{Q}} \cdots {}_{\mathcal{Q}} \mathcal{P}_n$ is defined as the set of all graphs having an $F(\mathcal{Q})$ -free $(\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_n)$ -colouring.

If $\mathcal{Q} = \mathcal{D}_1$ then we use the notation $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2 \odot \cdots \odot \mathcal{P}_n$.

A partition of V(G) generated by an \mathcal{F} -free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ -colouring of G is called an \mathcal{F} -free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ -partition. An \mathcal{F} -free $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$ -colouring, where $\mathcal{P}_i = \mathcal{I}$ for i = 1, ..., k, will be called briefly an \mathcal{F} -free colouring. If \mathcal{F} consists of a single graph F, then it will be called an F-free colouring (partition) for short.

For definitions and notations not presented here, we refer to [6, 9].

2. RAMSEY CLASSES OF GRAPHS

A hereditary property \mathcal{P} is called a \mathcal{Q} -Ramsey Class if for every $G \in \mathcal{P}$ there is an $H \in \mathcal{P}$ such that $H_{\overrightarrow{\mathcal{Q}}}G$, i.e., for every $F(\mathcal{Q})$ -free bicolouring of H there is a monochromatic subgraph isomorphic to G.

It is easy to see that if \mathcal{P} is a \mathcal{Q} -Ramsey Class and $\mathcal{Q}' \subseteq \mathcal{Q}$, then \mathcal{P} is a \mathcal{Q}' -Ramsey Class.

Proposition 1. Let $k \geq 2$. Then \mathcal{O}^k is a \mathcal{D}_1 -Ramsey Class.

Proof. Let $G \in \mathcal{O}^k$ and $\alpha(G) = \alpha$. It is easy to see that the graph $H = K_{k \times (\alpha+2)}$, the complete k-partite balanced (i.e., each colour class has the same cardinality $\alpha + 2$) graph satisfies the requirements of Theorem.

Theorem 1. Let F be a connected bipartite graph and $\mathcal{Q} = Forb(F)$. Then \mathcal{O}^k is a \mathcal{Q} -Ramsey Class for $k \geq 2$.

Proof. Let F be a given connected bipartite graph and let F be a subgraph of $K_{r,s}$ with $r \leq s$, and let $G \in \mathcal{O}^k$. Consider a graph $H = K_{k \times n}$, where

 $n \geq s + \alpha(G) + 1$. Suppose that $\{X, Y\}$ is an *F*-free bipartition of *H*. It implies that one of sets *X*, *Y*, say *X*, has at most *s* elements in each colour class V_i of *H*. Similarly, each colour class V_i of *H* has at least $\alpha(G)$ elements in *Y*, thus *G* is a subgraph of H[Y].

Let k be a positive integer. A k-clique is a complete graph of order k. A k-tree is a graph defined inductively as follows: A k-clique is a k-tree. If G is a k-tree, and K is a subgraph of G isomorphic to a k-clique, then a graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a partial k-tree. The tree-width of a graph G is zero if G is edgeless; otherwise it is a smallest integer k such that G is a partial k-tree, and will be denoted by $t_w(G)$. Nontrivial forests have tree-width 1, while every graph has some tree-width.

Let us denote by

$$\mathcal{TW}_k = \{ G \in \mathcal{I} : t_w(G) \le k \}.$$

According to G. Ding, B. Oporowski, D.P. Sanders and D. Vertigan, see [8], we recall the notion of "large" k-trees.

Let k be a positive integer. We will define some classes of k-trees, each with a *level* function λ defined on its vertices. Let the level of a subgraph of a graph with a level function be the maximum level of its vertices. Let T(k, 0, 0) be the k-clique, and each of its vertices have level zero. Let l, rbe non-negative integers. We will proceed by induction on l. The k-tree T(k, l, r) and its level function are obtained from k-tree T = T(k, l - 1, r)(or T = T(k, 0, 0) if l = 1) and its level function by the following: For each k-clique K of T that has level l - 1, add r new vertices, join each of them to all vertices of K, and declare the new vertices to be at level l. For a new vertex v added, let K(v) denote this k-clique K of level l - 1.

Proposition 2 [8]. The graph T(k, l, r) is a k-tree and every k-tree is a subgraph of T(k, l, r), for some l, r.

Theorem 2. Let $k \geq 2$. Then \mathcal{TW}_k is a \mathcal{D}_1 -Ramsey Class.

Proof. Let $G \in \mathcal{TW}_k$. By Proposition 2 it follows that $G \subseteq T(k, l, r)$ for some integers l, r. Let H = T(k, p, s), s, p > 1. We claim that if p and s are large enough, then in every acyclic bicolouring f with a bipartition $\{U_1, U_2\}$ of $V(H), H[U_1] \supseteq T(k, l, r)$ or $H[U_2] \supseteq T(k, l, r)$.

Firstly, let us observe that if J is any k-clique in H then J has at least k-1 monochromatic vertices, say $|V(J) \cap U_1| \ge k-1$, in any acyclic bicolouring f of H.

Secondly, if a k-clique J of level j < p has exactly one vertex in U_2 then there is a monochromatic k-clique J' of the level j + 1 with $V(J') \subseteq U_1$.

Since we choose p much larger than l therefore, without loss of generality, assume that k-clique K of the level zero is monochromatic, say $V(K) \subseteq U_1$.

Now let x be the vertex of level one in H and $K(x) = K \subseteq U_1$. If $x \in U_2$ then $y \in U_1$ for all vertices $y \neq x$ of level one. Therefore all, except at most one, k-cliques of level one have vertices in U_1 . Now, if we consider a vertex x' of level two, with $K(x') \subseteq U_1$, we get that all, except at most one, k-cliques of level two, having common vertices with K(x') have vertices in U_1 , and so on. If s is large enough then $T(k, l, r) \subseteq H[U_1]$.

Now we consider the property \mathcal{T}_2 which is a proper subclass of \mathcal{TW}_2 . For \mathcal{T}_2 we have the following results.

Theorem 3. T_2 is not a D_1 -Ramsey Class.

To prove Theorem 3 we need some notations and lemmas.

Let $\mathcal{G} = \{A_l : A_l = T(2, l, 1), l > 0, A_0 = T(2, 0, 0)\}$. It is easy to see that \mathcal{G} is a family of maximal outerplanar graphs with a level function and $A_l \subset A_{l+1}$ for all l.

Lemma 1. If $G \in \mathcal{T}_2$ then there is an integer $k \ge 0$ such that $G \subseteq A_k$.

Proof. Clearly, it is enough to consider only maximal outerplanar graphs G with at least 3 vertices. The proof is by induction on the number of vertices of G. If |V(G)| = 3 then G is isomorphic to A_1 . Assume that for all maximal outerplanar graphs with less than n vertices, $n \ge 3$, the lemma is true. Consider a maximal outerplanar graph G with n vertices. Let $x \in V(G)$ such that $deg_G(x) = 2$ and G' = G - x. Applying the induction hypothesis to G' we get $A_k \in \mathcal{G}$ such that $G' \subseteq A_k$. If $G \not\subseteq A_k$ then we construct a graph A_{k+1} from A_k . Since $deg_G x = 2$ it is clear that $G \subseteq A_{k+1}$.

Lemma 2.

$$\mathcal{T}_2 \subseteq \mathcal{O} \odot Forb(G_1).$$



Figure 1. Graph G_1

Proof. From Lemma 1 it follows that it is enough to consider only graphs $A_0, A_1, \ldots, A_n, \ldots$ from the family \mathcal{G} . The proof is by induction on n. Obviously, the lemma is true if n = 0, 1, 2. We consider the graph A_n and A_{n+1} , for $n \geq 3$, assuming that f is an acyclic colouring of A_n , with a bipartition $\{U_1, U_2\}$ of $V(A_n)$, such that U_1 (the set of red vertices) is independent and $A_n[U_2]$ (the subgraph induced by blue vertices) has the property $\mathcal{R} = Forb(G_1)$. We use f to construct an acyclic colouring f' of A_{n+1} , with a bipartition $\{U'_1, U'_2\}$ of $V(A_{n+1})$, such that U'_1 (the set of red vertices) is independent and $A_{n+1}[U'_2]$ (the subgraph induced by blue vertices) has the property \mathcal{R} . First, let f'(v) = f(v) for all vertices in A_{n+1} of level less than n+1.

Let x, y be a pair of uncoloured vertices of the level n + 1 in A_{n+1} . Let us assume, without loss of generality, that x is adjacent to a and b, and yis adjacent to b and c such that they form a triangle in A_n . It is clear that a, b, c have level less than n+1 and form a triangle in A_n , and f' has already coloured them.

To colour the vertices of the level n + 1 we apply the following rules.

Rule 1. If one of the vertices a, b, c is red, then both x and y should be blue.

Rule 2. If a, b, c are blue, then x should be red and y should be blue.

From the construction of the graph A_{n+1} it follows that:

(1) If C is a cycle in A_{n+1} containing x (respectively y), then C contains also either the path (x, b, y) or (x, b, c) (either the path (y, b, x) or (y, b, a)respectively);

(2) If G is a subgraph of A_{n+1} containing x (respectively y), then G contains also a, b, c and y (respectively x).

Colouring rules and (1) implies that the obtained colouring is acyclic.

Similarly, by the rules and (2) we see that blue vertices induce in A_{n+1} a graph with the property \mathcal{R} and red vertices are independent.

It is clear that if we apply these colouring rules to each such a pair of vertices of the level n + 1, we will obtain a required colouring of A_{k+1} .

Now we are ready to prove Theorem 3.

Proof of Theorem 3. We only need to find an outerplanar graph F such that for an arbitrary outerplanar graph H there is an acyclic bipartition $\{U_1, U_2\}$ of V(H), with U_1 being independent and $H[U_2] \not\supseteq F$. Clearly, Lemma 2 yields the graph G_1 (Figure 1), which satisfies the requirements of Theorem.

3. Acyclic Reducible Bounds

In this section we give some acyclic reducible bounds for the class of outerplanar graphs. We start with a few definitions.

An additive hereditary property \mathcal{R} is said to be *acyclic reducible* in \mathbb{L}^a if there are nontrivial additive hereditary properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \odot \mathcal{P}_2$ and *acyclic irreducible* in \mathbb{L}^a , otherwise.

Obviously, the smallest acyclic reducible property in \mathbb{L}^a is the property $\mathcal{O}^{(2)} = \mathcal{D}_1$.

Theorem 4.

$$\mathcal{I}_2 \subseteq \mathcal{O} \odot Forb(G_1, G_2),$$

$$\mathcal{I}_2 \subseteq \mathcal{O} \odot Forb(G_1, G_3).$$

Proof. We will proof only the first inclusion, the second one can be proved similarly. As in the proof of Lemma 2, we use Lemma 1 to restrict our attention only to graphs $A_0, A_1, \ldots, A_n, \ldots$ from \mathcal{G} . The proof is by induction on n. Obviously, it holds for n = 0, 1, 2. We consider the graph A_n and A_{n+1} , for $n \geq 3$, assuming that f is an acyclic colouring of A_n , with a bipartition $\{U_1, U_2\}$ of $V(A_n)$, such that U_1 (the set of red vertices) is independent and $A_n[U_2]$ (the subgraph induced by blue vertices) has the property $\mathcal{R} = Forb(G_1, G_2)$. We use f to construct an acyclic colouring f'of A_{n+1} , with a bipartition $\{U'_1, U'_2\}$ of $V(A_{n+1})$, such that U'_1 (the set of red vertices) is independent and $A_{n+1}[U'_2]$ (the subgraph induced by blue vertices) has the property \mathcal{R} . First, let f'(v) = f(v) for all vertices in A_{n+1} of level less than n + 1.

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Figure 2. Graphs G_1 , G_2 and G_3 .

Let x, y be a pair of uncoloured vertices of the level n + 1 in A_{n+1} . Let us assume that x is adjacent to a and b, and y is adjacent to b and c such that the vertices a, b, c form a triangle in A_n and are coloured.

Now we denote some vertices of A_n . Let $e \neq b$ be a unique vertex adjacent to both a and c, let d be be the vertex different from c adjacent to both a and e (it is a unique vertex in A_n with these properties), and let h be a vertex different from a adjacent to both c and e (it is only one such a vertex in A_n).

It is clear that b has level n. The level of one from $\{a, c\}$ is equal to n-1, say c; then the level of a is less than or equal to n-2.

To colour the vertices of the level n + 1 we apply the following rules.

Rule 1. If one of the vertices a, b, c is red, then both x and y should be blue.

Rule 2. If a, b, c are blue, then

(a) if f'(e) = red, then f'(x) = f'(y) = blue.

(b) if f'(e) = blue, then

(b₁) if f'(d) = red, then f'(x) = red and f'(y) = blue;

(b₂) if f'(d) = blue, then f'(x) = blue and f'(y) = red.

(Notice that the case f'(d) = f'(h) = blue is impossible, in such case all

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vertices $\{a, b, c, e, d, h\}$ would be coloured blue and a graph induced by this set in A_n would be isomorphic to $G_{1.}$)

From the construction of the graph A_{n+1} it follows that:

(1) If C is a cycle in A_{n+1} containing x (respectively y), then C contains also either the path (x, b, y) or (x, b, c) (either the path (y, b, x) or (y, b, a)respectively);

(2) If F is a subgraph of A_{n+1} isomorphic to G_1 , containing x or y, then $V(F) = \{x, y, a, b, c, e\};$

(3) If F is a subgraph of A_{n+1} isomorphic to G_2 , containing x (respectively y), then $V(F) = \{x, a, b, c, e, h\}$ (respectively $V(F) = \{y, a, b, c, d, e\}$). Colouring rules and (1) implies that the obtained colouring is acyclic. From (2) and (3) we see that blue vertices induce in A_{n+1} a graph with the property \mathcal{R} . Red vertices are independent, which is clear from the colouring rules.

If we apply the colouring rules to each such a pair of vertices of level n+1, then we obtain an acyclic colouring of A_{n+1} .

A maximal outerplanar graph G with at least 3 vertices is called a 2-path of order n = 2p, if G consist of two paths $P_1 = (x_1, x_2, \ldots, x_p)$, $P_2 = (y_1, y_2, \ldots, y_p)$ and additional edges: $x_i y_i$, $i = 1, \ldots, p$ and $x_j y_{j+1}$ for $j = 1, \ldots, p-1$. For an odd n = 2p - 1 a 2-path H is defined as $H = G - x_p$, where G is 2-path of even order.

A maximal outerplanar graph G with at least 3 vertices is called a *fan of* order n, if G is obtained from a star $K_{1,n-1}$ by joining all vertices of degree one by a path.

Additionally we assume that the graph K_1 and K_2 is a trivial 2-path and a trivial fan. For each $n \leq 5$ there is exactly one (up to isomorphism) maximal outerplanar graph of order n which is a 2-path and a fan.

Lemma 3. Let G be a maximal outerplanar graph of order $n \ge 3$. Then

- (a) G is a fan if and only if neither $G_1 \subseteq G$ nor $G_2 \subseteq G$.
- (b) G is a 2-path if and only if neither $G_1 \subseteq G$ nor $G_3 \subseteq G$.

Proof. (a) The fact that any fan contains neither G_1 nor G_2 follows immediately by the definition. For the converse, we employ induction on n, the order of G. Clearly, for $n \leq 6$ it is true. Assume every graph with fewer than $n \geq 7$ vertices is a fan, and suppose G has order n and does not contain a subgraph isomorphic to G_i , i = 1, 2. Let x be the vertex of degree 2 in G. By the inductive hypothesis, G' = G - x is a fan of order $n - 1 \geq 6$. Let y

be the unique vertex of maximum degree in G'. If x is not adjacent to y in G, then G contains G_1 or G_2 . If x is adjacent to y in G, then G is a fan.

(b) Again, it is easy to see that any 2-path contains neither G_1 nor G_3 . It follows immediately by the definition. For the converse, we use again induction on n. Clearly, for $n \leq 6$ it is true. Let us assume that every graph with fewer than $n \geq 7$ vertices is a 2-path, and suppose G has order n and does not contain a subgraph isomorphic to G_i , i = 1, 3. It is easy to see, that if G has a vertex of degree greater than 4, then by the maximality of G we get that G contains a subgraph isomorphic to G_3 . Hence we can assume that all vertices of G are of degree at most 4. Let x be the vertex of degree 2 in G. The graph G' = G - x is a maximal outerplanar graph with less than n vertices, then by the inductive hypothesis, G' is a 2-path. It is clear that G' has only four vertices of degree less than 4 and x has to be adjacent in G to exactly two of them. From maximality of G we get that x and its neighbours induce a triangle in G i.e., G is a 2-path.

Let us recall that a *block* of a given graph G is defined to be a maximal connected subgraph of G without a cutvertex.

A fan (2-path) tree is a connected graph G every block of each is a fan (2-path).

Let us define the property \mathcal{FT} (\mathcal{PT}) as the family of all fan (2-path) trees and their subgraphs. Each property is additive hereditary and a proper subfamily of all outerplanar graphs.

From the definition of \mathcal{FT} it follows that G_1 and G_2 do not belong to \mathcal{FT} . Similarly, G_1 and G_3 do not belong to \mathcal{PT} . It implies the following corollary.

Corollary 1.

$$\mathcal{FT} \subseteq Forb(G_1, G_2),$$
$$\mathcal{PT} \subseteq Forb(G_1, G_3).$$

Because of above Corollary, the next theorem gives a little better than in Theorem 4 two acyclic reducible bounds for outerplanar graphs.

Theorem 5.

$$\mathcal{T}_{2} \subseteq \mathcal{O} \odot \mathcal{F}\mathcal{T},$$
$$\mathcal{T}_{2} \subseteq \mathcal{O} \odot \mathcal{P}\mathcal{T}.$$

Proof. We will prove only the first bound, the second one can be proved similarly. By Lemma 1, it is enough to show that each graph of the family \mathcal{G} has the property $\mathcal{O} \odot \mathcal{FT}$. On the contrary, suppose that there is a graph $G \in \mathcal{G}$ such that in every acyclic bipartition $\{U_1, U_2\}$ of V(G), with U_1 being independent, $G[U_2]$ has a subgraph isomorphic to a graph from $\mathcal{T}_2 - \mathcal{FT}$. Let F be its block which is not a fan. Since any maximal outerplanar graph of order ≤ 5 is a fan, thus F has order at least 6. Lemma 3 implies that F contains a subgraph isomorphic to G_1 or to G_2 . This fact contradicts Theorem 4.

References

- P. Boiron, E. Sopena and L. Vignal, Acyclic improper colorings of graphs, J. Graph Theory 32 (1999) 97–107.
- [2] P. Boiron, E. Sopena and L. Vignal, Acyclic improper colourings of graphs with bounded degree, DIMACS Series in Discrete Mathematics and Theoretical Computer Science 49 (1999) 1–9.
- [3] O.V. Borodin, On acyclic colorings of planar graphs, Discrete Math. 25 (1979) 211–236.
- [4] O.V. Borodin, A.V. Kostochka, A. Raspaud and E. Sopena, Acyclic colourings of 1-planar graphs, Discrete Applied Math. 114 (2001) 29–41.
- [5] O.V. Borodin, A.V. Kostochka and D.R. Woodall, Acyclic colorings of planar graphs with large girth, J. London Math. Soc. 60 (1999) 344–352.
- [6] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, A survey of hereditary properties of graphs, Discussiones Mathematicae Graph Theory 17 (1997) 5–50.
- M.I. Burstein, Every 4-valent graph has an acyclic 5-coloring, Soobšč. Akad. Gruzin. SSR 93 (1979) 21-24 (in Russian).
- [8] G. Ding, B. Oporowski, D.P. Sanders and D. Vertigan, Partitioning graphs of bounded tree-width, Combinatorica 18 (1998) 1–12.
- [9] R. Diestel, Graph Theory (Springer, Berlin, 1997).
- [10] B. Grűnbaum, Acyclic coloring of planar graphs, Israel J. Math. 14 (1973) 390–412.
- [11] P. Mihók and G. Semanišin, *Reducible properties of graphs*, Discuss. Math. Graph Theory 15 (1995) 11–18.

Received 11 January 2006 Revised 21 September 2006