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THE COST CHROMATIC NUMBER AND HYPERGRAPH PARAMETERS*

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Abstract

In a graph, by definition, the weight of a (proper) coloring with positive integers is the sum of the colors. The chromatic sum is the minimum weight, taken over all the proper colorings. The minimum number of colors in a coloring of minimum weight is the cost chromatic number or *strength* of the graph. We derive general upper bounds for the strength, in terms of a new parameter of representations by edge intersections of hypergraphs.

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1. Introduction

Though even the traditional notion of proper coloring and minimum proper coloring yields many complex problems, in the last decades a lot of modified versions and generalizations of them were defined. One is the so-called *strength* of graphs. (See the definitions in the Preliminary Section.) This concept was invented by Kubicka [4], in connection with VLSI problems. The paper [9] deals also with the latter subject. Some basic properties of the chromatic sum have been described by Thomassen *et al.* [1] and by

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Kubicka and Schwenk [5]. Its behavior in trees has been investigated by the second author [10] and by Gionfriddo *et al.* [3]. In particular, the former paper describes a condition in terms of (induced) minors. Vizing's theorem and Vizing's conjecture are also related to the subject. The monograph [8] contains important results in this direction. For various open problems in the area, see e.g. [11].

In this work we give some upper and lower bounds for the strength, using a hypergraph parameter that we introduce here. As an intermediate step, we obtain new estimates on the Grundy number of graphs.

2. Preliminaries

A *coloring* of a graph $G = (V, E)$ with $|V| = n$ is a mapping $f : V \to \{1, 2, \ldots, n\}$ such that for all $uv \in E$, $f(u) \neq f(v)$. The color classes are the vertices having the same f value. The *weight* $w(f)$ of a coloring f is $\sum_{v\in V} f(v)$. Thus, the minimum possible weight of a coloring with color classes $C_1, C_2, \ldots, C_k$, in non-decreasing order of their cardinalities, is $k|C_1| + (k-1)|C_2| + \ldots + |C_k|$. The *chromatic sum* $\Sigma(G)$ of a graph G is the minimum weight of a coloring in G. If $w(f) = \Sigma(G)$, then we call f *minimal*.

For a graph G, the minimum number of colors in a coloring of minimum weight is called the *strength* of G and is denoted by $s(G)$.

A *Grundy coloring* is an ordered coloring (where the color classes have indices) in which, for every color class, each vertex of that class has some neighbor in every color class of smaller index. Let us consider every coloring as an ordered one like above. Then the following assertion is obvious.

Lemma 1. *Every coloring of minimum weight is a Grundy coloring.* ■

Let $\Gamma(G)$ denote the largest number of colors in a Grundy coloring of G. The lemma above immediately implies

$$(*)\qquad s(G) \leq \Gamma(G)$$

for every graph G.

Next, let us introduce some new concepts on hypergraphs.

Definitions. Given a hypergraph $\mathcal{F}$ and a hyperedge F, a subset $B \subseteq F$ is a *guard set* of F if every hyperedge, intersecting F but not contained in F intersects B, too.

Remark. A hyperedge may have several guard sets and multiple hyperedges may occur.

Notation. We call the intersection graph of the hypergraph $\mathcal{F}$ briefly its *line graph* and denote it by $L\mathcal{F}$.

Definition. A hypergraph is *c-small* if all of its hyperedges have some guard set of size at most c. A graph G is *c-representable* if there exists a c-small hypergraph $\mathcal{F}$ such that $L\mathcal{F} = G$. The minimum of c such that G is c-representable is called the *representation number* of G and is denoted by $c(G)$.

Some notation. $D(G)$ denotes the maximum degree in the graph G. The clique number and the (ordinary) chromatic number of the graph G will be denoted by $\omega(G)$ and $\chi(G)$, respectively. A graph is called Δ*-free* if it does not contain 3 pairwise adjacent vertices. For a vertex v, $N(v)$ is the *open* neighborhood of v, i.e., the set of all vertices adjacent to v.

Definition. A graph is an *interval graph* if it is the intersection graph of a system of intervals on a line. A graph is *chordal* if all of its cycles of length more than 3 has at least one chord.

3. A General Upper Bound

Remark. $c(G) = 0$ can occur only when every component of G is a clique. This means $\omega(G) = \chi(G) = s(G) = \Gamma(G)$.

Theorem 1. *Let $\omega(G) = \omega$ and let G be c-representable. Then for $c \geq 2$*

$$s(G) \leq \Gamma(G) \leq (c/2)\omega^2. \tag{1}$$

For $c = 1$, we have $s(G) \leq \Gamma(G) \leq (\omega^2 + \omega)/2$.

Proof. We shall prove both upper bounds for $\Gamma(G)$. This will be sufficient, by $(*)$.

Let $\mathcal{F}$ be a c-small hypergraph, representing G.

Let the color classes $C_1, C_2, \ldots, C_k$ be labeled so that the sequence in the *reverse* order is a Grundy coloring. Let us pick a vertex v_1 of C_1 and a neighbor of v_1 from every other color class (by definition, they exist). Let W_1 be the hyperedge representing v_1 in $\mathcal{F}$, and let B_1 be a minimum guard set of W_1. The neighbors chosen above are represented by hyperedges intersecting W_1. Let us denote by F_i the hyperedge representing a vertex in C_i, adjacent to v_1.

Our next goal is to find an i for which F_i is contained in W_1. By the definition of a guard set, each F_i not contained in W_1 intersects B_1 in some point b. But the set of hyperedges containing b represents a clique in G, thus their number is at most ω. Since $|B_1| \leq c$, the number of hyperedges, not contained in W_1 but intersecting it is at most $c\,(\omega - 1)$. Consequently, if the number of classes is large, then we find an F_i which is contained in W_1; in the remaining part of the proof, we call it W_2. Furthermore, i can be chosen to be small, namely $i \leq 2 + c\,(\omega - 1)$ is possible. Similarly, there exists a hyperedge W_3, contained in W_2, representing a vertex in a color class C_j, with $j \leq i+1+c\,(\omega-2)$. The reason of the latter bound is that the number of hyperedges F_j, containing a vertex in a guard set of W_2, with $j > i$, is at most $\omega - 2$, since together with W_1 and W_2, they form a clique in G.

We continue the process, and we get a series of hyperedges $W_1 \supseteq W_2 \supseteq W_3 \supseteq \ldots \supseteq W_\omega$ such that the index of the corresponding color class increases by at most $1 + c\,(\omega - \ell)$ where the hyperedge just investigated is W_ℓ. If the graph had one color class more, we would get a contradiction, because the last represented vertex v_ω has some neighbor in the next color class, a hyperedge would intersect W_ω and thus all the W_i, consequently it would form a clique of size $\omega + 1$ in G.

Thus, the number of color classes is at most
$1 + (\omega - 1) + c\,[(\omega - 1) + (\omega - 2) + \ldots + 2 + 1] \leq (c/2)\omega^2 - ((c-2)/2)\omega$.
For $c \geq 2$, we get $\Gamma(G) \leq (c/2)\omega^2$. For c=1, we get $\Gamma(G) \leq (\omega^2 + \omega)/2$.

We have proved Theorem 1. ■

4. Applications of the Upper Bound

Corollary 1. *If G is an interval graph with $\omega(G) = \omega$, then $s(G) \leq \Gamma(G) \leq \omega^2$.*

Proof. In an interval representation by closed intervals on the set of integers, the set $\{a, b\}$ is a guard set of the interval $[a, b]$. Thus, every interval graph is 2-representable. ■

It is known that every chordal graph is isomorphic to the vertex-intersection graph of subtrees of a tree.

Corollary 2. *Chordal graphs G with $\omega(G) = \omega$ representable by subtrees of a tree with at most c endpoints satisfy $s(G) \le \Gamma(G) \le (c/2)\omega^2$.*

Proof. A c-small guard set of a subtree T is formed by the endpoints of T, together with those internal points which have at least one neighbor outside T. ■

Finally, a trivial consequence of Theorem 1:

Corollary 3. *If G is the line graph of a hypergraph with rank at most c (i.e., each hyperedge has at most c vertices), and $\omega(G) = \omega$, then $s(G) \le \Gamma(G) \le (c/2)\omega^2$.*

Proof. Each hyperedge is a guard set of itself. ■

5. Evaluation of the Upper Bound

5.1. Triangle-free graphs

One may ask how sharp the upper bound in Theorem 1 is. At least for $\omega = 2$, we can give some answer to this question.

Let us define the following number.

$$s(c, \omega) := \max \{s(G) : c(G) \le c,\ \omega(G) \le \omega\}.$$

From Theorem 1 we know that for $c \ge 2$, $s(c, \omega) \le (c/2)\omega^2$. If $\omega = 2$, we get

$$s(c, 2) \le 2c. \tag{2}$$

The (ordinary) chromatic number is trivially a lower bound for the strength. On the chromatic number, the following result is valid, see [8, page 124].

For D, g arbitrarily large there exist D-regular graphs with girth at least g and chromatic number at least

$$(1 + o(1))D/2 \ln D. \tag{3}$$

By o(1), we mean here a function tending to zero (possibly from below) when D tends to infinity. From (3), we shall deduce a lower bound on $s(c, 2)$. Namely,

Theorem 2. *For arbitrarily large c,*

$$s(c,2) \geq (1+o(1))c/2\ln c. \tag{4}$$

Here o(1) is meant with respect to c. For the proof, first we verify some simple statements.

Definition. Let x, y be two adjacent vertices. If $N(x)-\{y\} \supseteq N(y)-\{x\}$, we say that x *majorates* y.

Lemma 2. *Suppose that in a graph H, the vertex x has c independent neighbors such that none of them is majorated by x. Then $c(H) \geq c$.*

Proof. Let us represent x by the hyperedge X, with an arbitrary guard set U. Let y be a neighbor among the c independent ones, represented by a hyperedge Y. We state that Y intersects U. The set Y intersects X, by the definition of intersection graph. Furthermore, $Y \subseteq X$ would contradict the fact that x does not majorate y. So, $Y \not\subseteq X$ and by the definition of a guard set, Y intersects U.

Thus, U intersects all the hyperedges, representing the c independent neighbors. They are pairwise disjoint and consequently, $|U| \geq c$. ■

Proposition 1. *In a regular Δ-free connected graph $G \neq K_2$, $c(G) = D(G)$.*

Proof of $c(G) \leq D(G)$ for all graphs. This will be done by a well-known construction. Let the vertex set of the hypergraph $\mathcal{H} = \mathcal{H}_G$ be the edge set $E(G)$ of G. For all $v \in V(G)$, we pick a hyperedge $F = F_v$, consisting of the edges incident with v. Obviously, G is the intersection graph of $\mathcal{H}$, and $\mathcal{H}$ is $D(G)$-small. ■

For proving $c(G) \geq D(G)$, we need Lemma 2 as an auxiliary assertion.

Proof of $c(G) \geq D(G)$ for Δ-free regular graphs. Let us take any vertex x of G. If x does not majorate any of its neighbors then we are done since its neighbors are independent from the Δ-freeness and so the Lemma implies $c(G) \geq D(G)$. Otherwise, let y be a majorated neighbor. Regularity implies that $N(x) - y = N(y) - x$. But, by Δ-freeness, the only case when this can occur is when the whole graph is the edge xy which was excluded. Proposition 1 is proved. ■

Because of $s(G) \geq \chi(G)$, if we apply (3), we have

Corollary 4. *For arbitrarily large c,*

$$(1 + o(1))c/2 \ln c \leq s(c, 2) \leq 2c.$$

Remarks. 1. Corollary 4 means that our upper bound is "not far from the truth", at least in the case of $\omega = 2$. For larger ω, we do not know lower bounds which would be similarly close to the upper bound. In particular, currently, the upper bound on $s(c, \omega)$ is quadratic, while the the lower bound is linear in ω.

2. The lower bound is also true for Γ, by $(*)$.

3. The condition "regular Δ-free" in Proposition 1 can be weakened as follows: There exists a vertex v of maximum degree such that $N(v)$ is independent and every vertex in $N(v)$ has degree at least 2.

4. As proved in [6], the stronger form of Brooks's theorem is also true: $s(G) \leq D(G)$, apart from a few exceptional graphs G. This upper bound is in fact independent from ours, as shown by the graphs which consist of a clique of size ω and some very large stars with their centers in the clique. Such graphs can be shown to have guard number one, thus the maximum degree can be arbitrarily large even if $c\omega^2$ is bounded.

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