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CHVÁTAL-ERDŐS CONDITION AND PANCYCLISM

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Abstract

The well-known Chvátal-Erdős theorem states that if the stability number α of a graph G is not greater than its connectivity then G is hamiltonian. In 1974 Erdős showed that if, additionally, the order of the graph is sufficiently large with respect to α, then G is pancyclic. His proof is based on the properties of cycle-complete graph Ramsey numbers. In this paper we show that a similar result can be easily proved by applying only classical Ramsey numbers.

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1. Introduction

We use Bondy and Murty's book [5] for terminology and notation not defined here and consider finite, undirected and simple graphs only. For a graph G we denote by $V = V(G)$ its vertex-set and by $E = E(G)$ its set of edges. The symbols $\alpha = \alpha(G)$ and $\kappa = \kappa(G)$ stand for the stability number and the connectivity of G. By C_p we denote a p-cycle of G, i.e., a cycle of length p. The order of G will be denoted by n. A graph of order n is said to be pancyclic if it contains cycles of every length p with $3 \leq p \leq n$.

In 1971 Bondy [2] suggested the famous "metaconjecture":
almost all nontrivial sufficient conditions for a graph to be hamiltonian also imply that it is pancyclic except for maybe a simple family of exceptional graphs.

There are various conditions for hamiltonicity that were examined in light of this conjecture, see [16]. Recall now the well-known Chvátal-Erdős [8] theorem.

Theorem 1. *Every k-connected graph on $n \geq 3$ vertices with stability number $\alpha \leq k$ is hamiltonian.*

There is a large family of triangle-free graphs (see for example the survey [7]) that satisfy the Chvátal-Erdős condition ($\alpha(G) \leq \kappa(G)$), thus they are not pancyclic. This family contains the complete bipartite graphs as well as the Andrásfai graphs $G_i = \overline{C^i_{3i+2}}$, $i \geq 1$, i.e., each G_i is the complement of the i-th power of the cycle C_{3i+2}. For example $G_1 = C_5$ and G_8 is a cycle on 8 vertices with the longest chords. The lexicographic product $G_i[\bar{K}_s]$ ($s \geq 1$) is a triangle-free $r = s(i+1)$-regular graph with stability number $\alpha = r$, connectivity r and order $3\alpha - s \leq 3\alpha - 1$, so it also satisfies the Chvátal-Erdös condition and is not pancyclic.

There are several articles that investigate the set of cycle lengths in graphs satisfying this condition. We cite below some results of importance for us. Amar, Fournier and Germa [1]) proved the following.

Theorem 2 (Amar, Fournier and Germa [1]). *Let G be a k-connected graph of stability $\alpha \leq k$ and of order n. If $G \neq K_{k,k}$ and $G \neq C_5$, then G has a C_{n-1}.*

The next result due to Lou [13] was conjectured by Amar, Germa and Fournier [1].

Theorem 3. *If a triangle-free graph G satisfies $\alpha(G) \leq \kappa(G)$, then G has cycles of all length between four and the order of G, unless $G = K_{r,r}$ or $G = C_5$.*

But if $\alpha(G) < \kappa(G)$ then G has to contain a C_3. Taking into account this observation Jackson and Ordaz [12] formulated the following conjecture.

Conjecture 1. Let G be a k-connected graph with stability number α. If $\alpha < k$, then G is pancyclic.

There are few results about this conjecture. By results due to Amar, Fournier and Germa [1] and Chakroun, Sotteau [9] the conjecture is valid for every graph G with $\alpha(G) \leq 3$ while Marczyk and Saclé [14] proved it for any graph G satisfying $\alpha(G) \leq 4$.

The most beautiful result related to both Bondy's "metaconjecture" and the Jackson-Ordaz conjecture is due to Erdős [11]. Applying the properties of cycle-complete graph Ramsey numbers [4] he proved the following result which had been conjectured by Zarins.

Theorem 4. *Every hamiltonian graph with the stability number less than p and the order greater than $4p^4$ is pancyclic.*

Note that the Erdős' proof is not complete and has a small gap which is filled in Section 2.

From the last result and the Chvátal-Erdős theorem we get at once the following corollary.

Corollary 1. *If the stability number α of a graph G does not exceed its connectivity and the order of G is greater than $4(\alpha+1)^4$, then G is pancyclic.*

The purpose of this paper is to present a simple proof of a similar result which use only the classical Ramsey numbers $R(l, m)$, i.e., to show the pancyclicity of every graph satisfying the Chvátal-Erdős condition and having sufficiently large order in relation to α. Our proof is quite different and simpler than that of Erdős though our bound is not as good as that of Corollary 1. Let us recall the simplest version of the Ramsey theorem [15].

Theorem 5. *For every pair $l, m \geq 2$ of integers there exists an integer $r(l, m)$ such that each graph of order $n \geq r(l, m)$ contains a clique on l vertices or a stable set of cardinality m.*

The Ramsey number $R(l, m)$ is defined to be the smallest number $r(l, m)$ with this property. Our main result reads as follows:

Theorem A. *Let G be k-connected graph with stability number α. If $\alpha \leq k$ and the order of G is at least $2R(4\alpha, \alpha+1)$, then G is pancyclic.*

In the last theorem the order of the graph G satisfying the hypothesis is very large and our Theorem is weaker than Corollary 1, however we feel that the bound $2R(4\alpha, \alpha+1)$ can be considerably lowered (see Section 2). The proof of Theorem A is given in Section 4.

2. Some Remarks on the Theorem by Erdős

It is surprising, but true, that the beautiful theorem by Erdős (Theorem 4) was forgotten for a long period. For example, it was not mentioned in the survey [12]. Consequently, we obtained our main result of the present paper without any knowledge of this theorem.

In his proof Erdős used the Ramsey number $R(C_m, K_p)$ i.e., the smallest number such that each graph of order $n \geq R(C_m, K_p)$ contains a cycle of length m or a stable set of cardinality p. A theorem of Bondy and Erdős [4] states that $R(C_m, K_p) = (m-1)(p-1)+1$ for $m \geq p^2 - 2$. Thus, if $p^2 - 2 \leq m \leq \frac{n}{p}$ then $n \geq (m-1)(p-1)+1$ and any graph of order n contains a cycle C_m for $p^2 - 2 \leq m \leq \frac{n}{p}$ (provided $p^2 - 2 \leq \frac{n}{p}$). For $\frac{n}{p} < m < n$ Erdős gave an original proof. However, he forgot to write down the case $3 \leq m \leq p^2 - 3$. The existence of C_m belonging to this interval follows easily from another result of Bondy and Erdős published in the same paper: $R(C_m, K_p) \leq mp^2$ for all m and p. Indeed, if $m \leq p^2 - 3$, then $mp^2 \leq p^4 - 3p^2 < 4p^4 < n$, so if the stability number is at most $p-1$, a C_m exists in G.

In his paper Erdős conjectured that the same conclusion holds if we replace the bound $4p^4$ by Cp^2, where C is a constant (sufficiently large). He also wrote that a simple example shows that it certainly fails for $n < \frac{p^2}{4}$, but did not present it in the article. Consider now another example. Take $p-1$ disjoint copies $A_1, \ldots, A_{p-1}$ of the complete graph K_{2p-4}, where $p \geq 3$. Choose two vertices x_i, y_i in each copy A_i and add $p-1$ independent edges $x_i y_{i+1}$ (indices are taken modulo $p-1$). It can be easily seen that the stability number of this hamiltonian graph is $p-1$ and there exist cycles

C_m for every m except $m = 2p-3$, therefore, we cannot lower the bound of the theorem of Erdős below the number $(p-1)(2p-4) = 2p^2 - 6p + 4$.

However, for graphs satisfying the Chvátal-Erdős condition perhaps the following is true: there exist two constants c and C, $c < 2$, such that every graph G with $\alpha(G) = \alpha = \kappa(G)$ and $|V(G)| > C\alpha^c$ is pancyclic ([6, 10]). The graphs $G_i[\bar{K}_s]$ show that such the constant c must be at least one.

3. Notation

Let C be a cycle of G and a a vertex of C. We shall denote by $\vec{C}$ the cycle C with a given orientation, by $a^+ = a^{+1}$ the successor of a on $\vec{C}$ and by $a^- = a^{-1}$ its predecessor. We write a^{++} for $(a^+)^+$, a^{+k} for $(a^{+(k-1)})^+$ and a^{-k} for $(a^{-(k-1)})^-$.

Let a and b be two vertices of C. By $a \vec{C} b$ we denote the set of consecutive vertices of C from a to b (a and b included) in the direction specified by the orientation of C. It will be called the *segment of* $\vec{C}$ *from* a *to* b. The orientation of C defines the natural relation of order in $a \vec{C} b$ (denoted by $\prec$). When $a = b$ the symbol $a \vec{C} b$ means the one-vertex subset $\{a\}$ of $V(C)$.

Throughout the paper the indices of a cycle $C = x_1, x_2, \ldots, x_p$ are to be taken modulo p.

Suppose A is a subset of $V(G-C)$. The symbol $N_C(A)$ stands for the set $\{y \in V(C) |$ there is a vertex $x \in A$ such that $xy \in E(G)\}$. We write $N_C(x)$ for $A = \{x\}$ and we denote by $d_C(x)$ the number $| N_C(x) |$.

Let $P = x_1, x_2, \ldots, x_r$ be an oriented path in G. The symbol $\overleftarrow{P}$ stands for the path obtained by reversing the order of P, i.e., $\overleftarrow{P} = x_r, x_{r-1}, \ldots, x_1$. Consider another path $Q = y_1, y_2, \ldots, y_p$ of G such that $y_1 = x_r$. If Q is vertex-disjoint (except for x_r) from P, then by P, Q we mean the path $x_1, x_2, \ldots, x_r, y_2, \ldots, y_p$.

4. Proof of Theorem A

Suppose that G is a k-connected graph of stability number $\alpha \leq k$ such that $n \geq 2R(4\alpha, \alpha+1)$, where n is the order of G. Obviously we may assume $\alpha \geq 2$, $n \geq 2 \cdot 28 = 56$ and, since $n > 2\alpha$, G is not bipartite.

1. First we shall show that G contains a C_p for each $p > \frac{n}{2} - 2$. Observe that, by Chvátal-Erdős theorem and Theorem 2, this statement is evident

for $p = n$ and $p = n - 1$. Suppose G contains a cycle C_p with $p > \frac{n}{2}$. We shall prove that it contains also a C_{p-2}. Indeed, since $p > \frac{n}{2} \geq R(4\alpha, \alpha + 1)$, and the graph $\langle C_p \rangle$ induced by C_p has no stable set of cardinality $\alpha + 1$, it follows from Ramsey theorem that it contains a clique, say K, having 4α vertices. Let $\overrightarrow{C_p}$ denote the cycle C_p with a given orientation and let $x_1, x_2, \ldots, x_{4\alpha}$ be the vertices of K appearing on $\overrightarrow{C_p}$ in order of their indices. Clearly, for every $l = 1, 2, \ldots, 2\alpha$ (indices are taken modulo 4α) the vertices x_{2l} and x_{2l+2} are separated by at least one vertex on C_p. Consider now the set $= x_2^{++}, x_4^{++}, x_6^{++}, \ldots, x_{2l}^{++}, \ldots, x_{4\alpha}^{++}$ of $2\alpha > \alpha + 1$ vertices. Since the stability number of $\langle C_p \rangle$ is at most α, there is in $\langle C_p \rangle$ an edge of the form $x_{2i}^{++} x_{2j}^{++}$ $(i \neq j)$. Therefore, the following cycle $x_{2i}^{++}, x_{2i}^{+3}, \ldots, x_{2j}, x_{2i}, x_{2i}^{-}, \ldots, x_{2j}^{+3}, x_{2j}^{++}, x_{2i}^{++}$ (we allow that the paths $x_{2i}^{++}, x_{2i}^{+3}, \ldots, x_{2j}$ and $x_{2i}, x_{2i}^{-}, \ldots, x_{2j}^{+3}, x_{2j}^{++}$ are trivial) has $p-2$ vertices and our claim is proved.

Now, using the fact that C_n and C_{n-1} exist, it is a simple matter to prove recursively that G contains a C_p for $p > \frac{n}{2} - 2$.

2. Now we shall show that G contains a cycle on p vertices for every p such that $3 \leq p \leq \frac{n}{2} - 2$. It is evident for $3 \leq p \leq 4\alpha$ because $n > R(4\alpha, \alpha + 1)$ and G has no stable set of cardinality $\alpha + 1$, so it follows from Ramsey's theorem that it contains a clique on 4α vertices.

Suppose G has a C_p for some p satisfying $p \leq \frac{n}{2} - 4\alpha$. We claim that it contains also a cycle on $p + 4\alpha - 2$ vertices. Indeed, the order of the graph $G - C_p$ is equal to $n - p > n/2$. By Ramsey's theorem it contains a clique, say K, on 4α vertices. Since $\alpha \leq k$, it follows by Menger's theorem that we can choose $r = \min(\alpha, p)$ vertex-disjoint paths, say $P_1, P_2, \ldots, P_r$, that join C_p with K. Denote by $x_i \in V(C_p)$ and $y_i \in V(K)$ the end-vertices of P_i $(i = 1, 2, \ldots, r)$. The vertex x_i will be called starting vertex of P_i $(i = 1, \ldots, r)$. Since the stability number of G is equal to α we may assume that the length of every path P_i is less than or equal to $2\alpha - 1$. Let $\overrightarrow{C_p}$ denote the cycle C_p with a given orientation and suppose there is some i such that x_i and x_{i+1} are consecutive on $\overrightarrow{C_p}$. Then the length of the cycle $x_i^-, x_i, P_i, Q_i, \overleftarrow{P_{i+1}}, x_{i+1}, x_{i+1}^+, \ldots, x_i^-$ is $p + 4\alpha - 2$, where Q_i is a path from y_i to y_{i+1} in K of $4\alpha - |V(P_i)| - |V(P_{i+1})| + 2 \geq 2$ vertices. Suppose then that any two vertices x_i and x_{i+1} are separated by at least one vertex on C_p. Hence $r = \alpha$ and the set $\{x_1^+, x_2^+, \ldots, x_\alpha^+\}$ has α elements. If there are two indices, say i and j, such that $x_i^+ x_j^+$ belongs to $E(G)$, then the cycle $x_i^-, x_i, P_i, Q_{ij}, \overleftarrow{P_j}, x_j, x_j^-, \ldots, x_i^+, x_j^+, x_j^{++}, \ldots x_i^-$, where Q_{ij} is a $y_i - y_j$ path of $4\alpha - |V(P_i)| - |V(P_j)| + 2 \geq 2$ vertices which is contained in K (see Figure 1). Obviously, the length of this cycle is $p + 4\alpha - 2$.

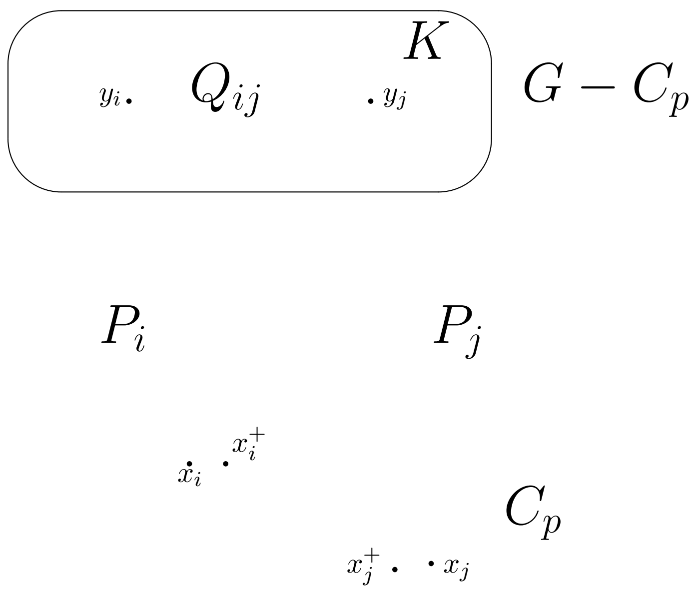

Figure 1

Thus suppose the vertices $x_1^+, x_2^+, \ldots, x_\alpha^+$ are independent and let u be the second vertex on P_1 (starting at x_1). If $ux_i^+ \in E(G)$ for some i, $2 \leq i \leq \alpha$, then the starting vertices of P_i and of the path obtained by replacing in P_1 the vertex x_1 by x_i^+ and the edge ux_1 by ux_i^+ are consecutive on C. So we can construct a cycle of length $p + 4\alpha - 2$ as above. So assume $ux_i^+ \notin E(G)$ for $i = 2, \ldots, \alpha$. Thus, because the stability number of G is α, $ux_1^+ \in E(G)$ and we can replace the path P_1 by another one starting at x_1^+. Repeating this reasoning (if necessary) we obtain $ux_2^- \in E(G)$. Therefore, there are two disjoint paths whose starting vertices are consecutive on the cycle and we can construct a $C_{p+4\alpha-2}$. So our claim is proved.

Because G contains a cycle C_p for every p between 3 and 4α, the existence of a cycle of length p for $3 \leq p \leq n/2 - 2$ follows by induction from our now-proved claim. This completes the proof of the theorem.

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