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EXTREMUM DEGREE SETS OF IRREGULAR ORIENTED GRAPHS AND PSEUDODIGRAPHS

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Abstract

A digraph in which any two vertices have distinct degree pairs is called irregular. Sets of degree pairs for all irregular oriented graphs (also loopless digraphs and pseudodigraphs) with minimum and maximum size are determined. Moreover, a method of constructing corresponding irregular realizations of those sets is given.

Keywords: irregular digraphs, degree sequences, degree sets.

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1. Introduction

Finite digraphs are considered. The word digraph stands for a general digraph with restriction that multiple arcs are forbidden. Independent loops are allowed. Thus the class of digraphs coincides with that of pseudodigraphs. In what follows, however, any specified digraph G is called a *pseudodigraph* precisely if G has a (directed) loop. A digraph G is called *complete* if G is loopless and with all possible arcs. Adding a single loop to each vertex of a complete digraph gives a *complete pseudodigraph*. An *oriented graph* is a loopless digraph in which any two vertices are joined by at most one arc. Let $G = (V, E)$ be a digraph with vertex set $V = V(G)$ and arc set $E = E(G)$. The cardinalities of V and E are called the *order* and *size* of G, respectively. For undefined terminology and notation we refer to [3].

Let u be a vertex of G. The symbols $\mathrm{od}_G(u)$ and $\mathrm{id}_G(u)$ denote the *outdegree* and *indegree* of u in G, respectively. Then the number $\deg_G(u) = \mathrm{od}_G(u) + \mathrm{id}_G(u)$ is the *degree* of the vertex u in G. The ordered pair $(\mathrm{od}_G(u), \mathrm{id}_G(u))$ is called the *degree pair* of the vertex u in G. A digraph G is said to be *irregular* if its distinct vertices have distinct degree pairs, i.e., the following implication holds

$$u, v \in V(G) \text{ and } u \neq v \Rightarrow (\mathrm{od}_G(u), \mathrm{id}_G(u)) \neq (\mathrm{od}_G(v), \mathrm{id}_G(v)).$$

This kind of irregularity, defined and studied by Gargano *et al.* in [5], is a global irregularity. Graphs and digraphs with different variants of local irregularity are investigated in many papers, for example [1, 2, 9, 10, 11, 13]. Irregular digraphs, called fully irregular in our former publications [12, 14], are studied in [4, 6, 7], too.

An irregular digraph G is called *minimum* (*maximum*) if the size of G is so among all irregular digraphs of the fixed order $|V(G)|$. A digraph with a fixed property is said to be *smallest* if the order and next the size of the digraph are the smallest possible. A smallest irregular digraph (resp. oriented graph) containing a given loopless digraph as an induced subdigraph is constructed in [6, 7]. The (asymptotics of the) maximum independence number and the cardinality of the set of irregular n-vertex digraphs are investigated in these papers.

For each positive integer n the *minimum size*, ϵ_n, of irregular oriented graphs of order n is found in [12]. The number ϵ_n is also the minimum size for n-vertex irregular digraphs. However, the corresponding sets of degree pairs are not characterized there yet.

In this paper we determine all sets D of degree pairs of minimum as well as maximum irregular digraphs in general, and also of oriented graphs in particular. It appears that all sets of degree pairs of minimum irregular loopless digraphs are realized by irregular oriented graphs, too. Our proof of realizability is by construction because we do not know of any better characterization of sequences (sets) of degree pairs in (irregular) oriented graphs, cp. [8].

Each maximum n-vertex irregular oriented graph is clearly an orientation of the complete graph, has the unique set of degree pairs, and therefore is seen to be the transitive tournament T_n. All sets of degree pairs of the maximum irregular digraphs (either loopless or not) are determined by the observation that the corresponding complement of a minimum irregular digraph is a maximum irregular digraph.

Also minimum digraphic sets D which are uniquely irregularly realizable and those which have some special irregular realizations are characterized.

2. PRELIMINARIES

Recall some notations and definitions from paper [12]. Given a digraph G, the symbol D_G denotes the *set of degree pairs of* G, i.e.,

$$D_G = \{(\mathrm{od}_G(u), \mathrm{id}_G(u)) \colon u \in V(G)\}.$$

If $B = \{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\}$ is the set of ordered pairs of integers then the number $s(B) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i$ is called the *sum* of B. The set B is called *balanced* if $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$.

Let N_0 be the set of nonnegative integers, $p \in N_0$ and let

$$B_p = \{(a, b) : \ a + b = p, \ \ a, b \in N_0\}.$$

Obviously, $|B_p| = p+1$ and each set B_p is balanced. Note that

Proposition 1. *The transitive tournament T_n is an irregular oriented graph and its set of degree pairs is $D_{T_n} = B_{n-1}$.*

In [12], for a positive integer n, the nonnegative integers τ_n and m_n are defined (see [7] for more information on τ_n) such that $\tau_1 = 0$ and $m_1 = 1$ and

$$n = 1 + 2 + \ldots + \tau_n + m_n, \qquad 1 \le m_n \le \tau_n + 1 \tag{1}$$

whence

$$\tau_n = \left\lfloor \sqrt{2n} - \frac{1}{2} \right\rfloor, \qquad m_n = n - \frac{1}{2}\tau_n(\tau_n + 1). \tag{2}$$

3. Minimum Sets

For a given positive integer n by $\mathcal{D}_n^{\min}$ we denote the class of all sets D of ordered pairs of nonnegative integers such that the following three conditions hold:

(α) D is an n-element set,

(β) D is balanced,

(γ) D has the minimum sum $s(D)$ among all D's satisfying (α) and (β).

By $B_s[k]$, where $0 \le k \le s+1$, we denote a k-element subset of the set B_s. The following lemma gives a more detailed description of all sets in the class $\mathcal{D}_n^{\min}$.

Lemma 2. *Let n be a positive integer and let τ_n and m_n be given by* (2). *Then*

(o) *$D = \{(0,0)\}$ is the only member of $\mathcal{D}_n^{\min}$ for $n = 1$, $D_1^{\min} = \{B_0\}$.*

For $n \ge 2$, $D \in \mathcal{D}_n^{\min}$ if and only if D has one of the following forms:

(i) *$D = \bigcup_{i=0}^{\tau_n - 1} B_i \cup B_{\tau_n}[m_n]$, where the set $B_{\tau_n}[m_n]$ is balanced and m_n is even or τ_n is even,*

(ii) *$D = \bigcup_{i=0}^{\tau_n - 1} B_i \setminus \{(a,b)\} \cup B_{\tau_n}[m_n + 1]$, where $(a,b) \in B_{\tau_n - 1}$, the set $B_{\tau_n}[m_n + 1] \cup \{(b,a)\}$ is balanced and both numbers τ_n and m_n are odd,*

(iii) $D = \bigcup_{i=0}^{\tau_n - 1} B_i \cup B_{\tau_n}[m_n - 1] \cup \{(a,b)\}$ *where* $(a,b) \in B_{\tau_n+1}$*, the set* $B_{\tau_n}[m_n - 1] \cup \{(a,b)\}$ *is balanced and both numbers* τ_n *and* m_n *are odd.*

Proof. Due to (1) and (2) it is easy to see that an n-element set of ordered pairs of nonnegative integers for $n > 1$ has the minimum sum if and only if it has the form:

$$\bigcup_{i=0}^{\tau_n - 1} B_i \cup B_{\tau_n}[m_n]. \tag{3}$$

Moreover, a set of the form (3) is balanced if and only if $B_{\tau_n}[m_n]$ is balanced. On the other hand, one can easily see that a balanced set $B_{\tau_n}[m_n]$ exists if and only if τ_n is even or m_n is even. Thus Lemma 2 is true if τ_n is even or m_n is even. Consider the remaining case that both numbers m_n and τ_n are odd. Let s be the sum of a set of the form (3). Hence $s(D) > s$ for $D \in \mathcal{D}_n^{\min}$ because no set of the form (3) is balanced. In fact, $s(D) = s + 1$ because $s(D)$ is so for all sets D of either form (ii) and (iii), and the existence of such sets D of either form in case $a = b$ is clear. On the other hand, any required set D is obtainable from a set of the form (3) by replacing one pair with the sum of components $\tau_n - 1$ or τ_n by a new pair with the sum of components one greater. In case under consideration the replacement can be carried out in two ways leading precisely to (ii) and (iii). ∎

Proposition 3. *Let* n *be a positive integer such that* τ_n *and* m_n *are both odd. Then a set* D *described in case* (ii) *of Lemma* 2 *exists for any pair* $(a,b) \in B_{\tau_n - 1}$ *if* $m_n < \tau_n$*, otherwise* $m_n = \tau_n$ *and then such* D *exists for* $a = b = \frac{1}{2}(\tau_n - 1)$ *only. Moreover, a set* D *described in case* (iii) *exists for any* $(a,b) \in B_{\tau_n+1}$ *if* $m_n \neq 1$*, otherwise* D *exists for* $a = b = \frac{1}{2}(\tau_n + 1)$ *only.*

Proof. The set D is balanced. Therefore the pair (a,b) is balanced if the intersection $I_n = D \cap B_{\tau_n}$ is so. Hence the pair (a,b) is as is stated in case (ii) if $m_n = \tau_n$ (i.e., if $I_n = B_{\tau_n}$ and in case (iii) if $m_n = 1$ (i.e., if $I_n = \emptyset$). Consider any remaining possibility for the value of m_n. It is enough to show the existence of the intersection I_n. Now the cardinality $|I_n| = m_n \pm 1$ is even and $|I_n| \leq \tau_n - 1$. Let (a,b) be any of possible pairs in question. Then $|a - b| \leq \tau_n + 1$ and $|a - b|$ is even. On the other hand, if the pair (c_i, d_i) ranges over the set B_{τ_n} then the difference $c_i - d_i$ is odd and ranges bijectively over the set $\{-\tau_n, 2 - \tau_n, \ldots, -1, +1, \ldots, \tau_n\}$. Therefore two pairs (c_i, d_i), $i = 1, 2$, can be found so that they together with the pair

(b,a) in case (ii) or with the pair (a,b) in case (iii) make up a balanced triple. The set I_n which comprises the two pairs (c_i,d_i) as well as $\frac{1}{2}|I_n|-1$ balanced twos of pairs selected from remaining pairs in B_{τ_n} will do. ■

A few examples of sets from $\mathcal{D}_n^{\min}$ follow.

For $n=24$ we have $\tau_n=6$ and $m_n=3$, so sets from $\mathcal{D}_{24}^{min}$ have the form (i), for example

$$\bigcup_{i=0}^{5} B_i \cup \{(6,0),(2,4),(1,5)\}\,, \quad \bigcup_{i=0}^{5} B_i \cup \{(5,1),(3,3),(1,5)\}\,.$$

For $n=18$ we have $\tau_n=5$ and $m_n=3$, so sets from $\mathcal{D}_{18}^{min}$ have the form (ii) or (iii), for example

$$\bigcup_{i=0}^{4} B_i \setminus \{(1,3)\} \cup \{(4,1),(3,2),(2,3),(0,5)\}\,,$$

$$\bigcup_{i=0}^{4} B_i \setminus \{(2,2)\} \cup \{(4,1),(3,2),(2,3),(1,4)\}\,,$$

$$\bigcup_{i=0}^{4} B_i \cup \{(2,3),(1,4)\} \cup \{(5,1)\}, \bigcup_{i=0}^{4} B_i \cup \{(0,5),(4,1)\} \cup \{(4,2)\}.$$

4. The Operation $\diamond_M$ on Digraphs

Let G and H be vertex disjoint digraphs and let M be a set of arcs in G, $M \subseteq E(G)$. For each arc $(x_i,y_i) \in M$, we choose one or two vertices in H, say $v_i', v_i'' \in V(H)$. In the union $G \cup H$ we replace each (x_i,y_i) by two arcs (x_i,v_i') and (v_i'',y_i). If the resulting structure, say F, does not have any multiple arc, i.e., F is a digraph, we write $F \in G \diamond_M H$ and we say that F is obtained from G and H by using the operation $\diamond_M$. Note that the resulting F is an oriented graph if both G and H are so and the arc set M is a matching in G.

In Figure 1 we present an example of the graph $F \in G \diamond_M H$ in case $M=\{(x_1,y_1),(x_2,y_2)\}$.

Proposition 4. *Let G and H be vertex disjoint oriented graphs and let M be a matching in G. If $F \in G \diamond_M H$ then F is an oriented graph, degree pairs of vertices of G remain unchanged in F, and the total increase of outdegrees and that of indegrees of vertices of H on passing on to F are both equal to $|M|$.*

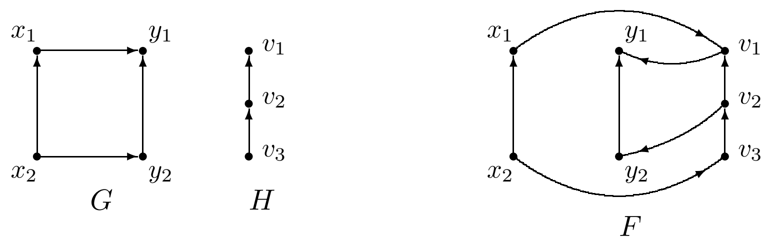

Figure 1

Proposition 5. *Let G and H be vertex disjoint digraphs. Let M be a matching of cardinality r $(r > 0)$ in G and let $f^+, f^-\colon V(H) \to N_0$ be functions such that*

$$\sum_{v \in V(H)} f^+(v) = \sum_{v \in V(H)} f^-(v) = r.$$

Then there exists a digraph $F \in G \diamond_M H$ such that for each $v \in V(H)$ the following equalities hold:

$$\text{(4)} \qquad \mathrm{od}_F(v) = \mathrm{od}_H(v) + f^+(v) \qquad \textit{and} \qquad \mathrm{id}_F(v) = \mathrm{id}_H(v) + f^-(v).$$

Proof. Assume that $M = \{(x_i, y_i) : \ i = 1, 2, \ldots, r\}$ is a matching in G. By X and Y we denote the sets of initial and terminal vertices of arcs from M, respectively. Let $V(H) = \{v_1, v_2, \ldots, v_k\}$. It is easy to see that both sets X and Y can be represented as unions of k pairwise disjoint sets $X_1, X_2, \ldots, X_k$ and $Y_1, Y_2, \ldots, Y_k$ respectively such that $|X_j| = f^-(v_j)$ and $|Y_j| = f^+(v_j)$ for $j = 1, 2, \ldots, k$. Hence, for each $i = 1, 2, \ldots, r$, there exists exactly one pair (p, q), $\ p, q \in \{1, 2, \ldots, k\}$ such that $x_i \in X_p$ and $y_i \in Y_q$. Let F be the digraph obtained from $G \cup H$ by replacing the arc (x_i, y_i) by the two arcs (x_i, v_p) and (v_q, y_i) for $i \in \{1, 2, \ldots, r\}$. It is not difficult to check that $F \in G \diamond_M H$ and equalities (4) hold. ■

Recall that the *matching number* of a digraph G, denoted by $\mu(G)$, is the maximum cardinality among all matchings in G. Usually the operation $\diamond_M$ will be used for $G = \bigcup_{i=1}^{k} T_i$, where T_1,T_2, …,T_k are vertex disjoint transitive tournaments. It is easy to note that

$$\text{(5)} \qquad \mu\left(\bigcup_{i=1}^{k} T_i\right) = \left\lfloor \frac{k^2}{4} \right\rfloor.$$

5. Irregular Realizability of Minimum Sets

Note that the irregular realization of the set $\{(0,0),(1,1)\}$ is unique and is pseudodigraphic. No other member of $\mathcal{D}_n^{\min}$ has this property.

Theorem 6. *An n-set D of ordered pairs of integers is realized by a minimum irregular oriented graph G if and only if $D \in \mathcal{D}_n^{\min}$ and $D \neq \{(0,0),(1,1)\}$.*

Proof. *Necessity* is clear, $D = D_G \in \mathcal{D}_n^{\min}$ which follows from the construction of $\mathcal{D}_n^{\min}$.

Sufficiency. Refer to (2) for τ_n and m_n. Cases (i)–(iii) we are going to refer to are those in Lemma 2.

If $n \leq 3$ then D is one of the following three sets: $\{(0,0)\}$, $\{(1,0),(0,1)\}$ and $\{(0,0),(1,0),(0,1)\}$. Then the oriented graphs T_1, T_2 and $T_1 \cup T_2$ are respective realizations of D.

Let $n \geq 4$. Then $\tau_n \geq 2$ by (2). By Lemma 2, we can represent the set D as the disjoint union $B \cup C$ where:

$$B = \bigcup_{i=0}^{\tau_n - 1} B_i \ \text{ and } \ C = B_{\tau_n}[m_n] \text{ in case (i)},$$

$$B = \bigcup_{i=0}^{\tau_n - 2} B_i \ \text{ and } \ C = B_{\tau_n - 1} \setminus \{(a,b)\} \cup B_{\tau_n}[m_n + 1] \text{ in case (ii)},$$

$$B = \bigcup_{i=0}^{\tau_n - 1} B_i \ \text{ and } \ C = B_{\tau_n}[m_n - 1] \cup \{(a,b)\} \text{ in case (iii)}.$$

Note that the set C is balanced because both D and B are balanced.

Consider the vertex disjoint tournaments $T_1, T_2, \ldots, T_{\tau_n+1}, T_{\tau_n+2}$. Put

$$(6) \qquad G_0 = \bigcup_{i=1}^{\tau_n} T_i \quad \text{in cases (i) and (iii)}, \qquad G_0 = \bigcup_{i=1}^{\tau_n - 1} T_i \quad \text{in case (ii)},$$

$T = T_{\tau_n} \cup T_{\tau_n+1} \cup T_{\tau_n+2}$,

$H = \langle V' \rangle_T$, where $V' = \{v \in V(T) \colon \ (\mathrm{od}_T(v), \mathrm{id}_T(v)) \in C\}$

and the symbol $\langle V' \rangle_T$ denotes the subdigraph of T induced by V'.

Since T_{i+1} realizes the set B_i, the digraph G_0 realizes the set B. A realization of D will be obtained by passing on from $G_0 \cup H$ to $G_0 \diamond_M H$

(where M is a matching in G_0) so that the operation $\diamond_M$ restores the degree pairs in the part H to their values in T if possible. This works in general and, for a few sets D only, the realization of D is of the form $G_0 \diamond_M \tilde{H}$ where $\tilde{H}$ is H together with one or two arcs joining nonadjacent vertices in H. Instead of functions f^+, f^- (Proposition 5 above) we use the following parameters for $v \in V(H)$.

$$r^+_{(T,H)}(v) = \mathrm{od}_T(v) - \mathrm{od}_H(v), \quad r^-_{(T,H)}(v) = \mathrm{id}_T(v) - \mathrm{id}_H(v).$$

It is clear that $r^+_{(T,H)}(v) \geq 0$ and $r^-_{(T,H)}(v) \geq 0$ for every $v \in V'$ and, since the sets C and D are balanced,

$$\sum_{v \in V'} r^+_{(T,H)}(v) = \sum_{v \in V'} r^-_{(T,H)}(v). \tag{7}$$

Put

$$r = \sum_{v \in V'} r^+_{(T,H)}(v). \tag{8}$$

By definitions of H, G_0, r and by (5), we have

$$r = \begin{cases} m_n(\tau_n - m_n + 1)/2 & \text{in case (i)}, \\ \left((\tau_n - 1) + (m_n + 1)(\tau_n - m_n)\right)/2 & \text{in case (ii)}, \\ \left((m_n - 1)(\tau_n - m_n + 2) + (\tau_n + 1)\right)/2 & \text{in case (iii)} \end{cases} \tag{9}$$

and

$$\mu(G_0) - r = \begin{cases} \frac{1}{4}\left((\tau_n - m_n)^2 + (m_n - 1)^2 - 1\right) & \text{for even } \tau_n \text{ in case (i)}, \\ \frac{1}{4}\left((\tau_n - m_n)^2 + (m_n - 1)^2 - 2\right) & \text{for odd } \tau_n \text{ in case (i)}, \\ \frac{1}{4}\left((\tau_n - m_n - 3)^2 + (m_n - 2)^2 - 10\right) & \text{in case (ii)}, \\ \frac{1}{4}\left((\tau_n - m_n)^2 + (m_n - 3)^2 - 8\right) & \text{in case (iii)}. \end{cases}$$

Hence

$$\mu(G_0) - r = \begin{cases} -1 & \text{in case (iii) if } (\tau_n, m_n) \in \{(5,5),(5,3)\}, \\ -2 & \text{in case (iii) if } (\tau_n, m_n) = (3,3), \\ -2 & \text{in case (ii) if } (\tau_n, m_n) \in \{(7,3),(5,1),(5,3),(3,1)\}, \end{cases}$$

and $\mu(G_0) - r \geq 0$ in remaining cases.

If $r = 0$ then $H = T_{\tau_n+1}$. So $G_0 \cup H$ is an oriented graph which realizes the set D.

Let $0 < r \le \mu(G_0)$. Choose a matching M of cardinality r in G_0 and put $f^+(v) = r^+_{(T,H)}(v)$ and $f^-(v) = r^-_{(T,H)}(v)$. The construction presented in the proof of Proposition 5 gives an oriented graph $G \in G_0 \diamond_M H$ which realizes the set D.

Let $\mu(G_0) - r$ be equal to -1 or -2. Then, for the particular pairs (τ_n, m_n), adding one or two arcs to the graph H can give a supergraph $\widetilde{H}$ such that

$$r^+_{(T,\widetilde{H})}(v) \ge 0 \quad \text{and} \quad r^-_{(T,\widetilde{H})}(v) \ge 0 \quad \text{for} \quad v \in V(\widetilde{H}) \tag{10}$$

and

$$\sum_{v \in V(\widetilde{H})} r^+_{(T,\widetilde{H})}(v) = \sum_{v \in V(\widetilde{H})} r^-_{(T,\widetilde{H})}(v) = \mu(G_0). \tag{11}$$

Then any oriented graph from the class $G_0 \diamond_M \widetilde{H}$, where M is a maximum matching in G_0, is a realization of D. ■

Obviously, any oriented graph is a loopless digraph.

Corollary 7. $\bigcup_{n=1}^{+\infty} \mathcal{D}_n^{\min} \setminus \{(0,0),(1,1)\}$ *and* $\bigcup_{n=1}^{+\infty} \mathcal{D}_n^{\min}$ *are the classes of all sets of degree pairs for minimum irregular respectively loopless digraphs and pseudodigraphs.*

6. Specialized Irregular Realizations

By a 2-*cycle* we mean the complete 2-vertex digraph. The following questions arise. Which sets from $\mathcal{D}_n^{\min}$ have not only a realization as an irregular oriented graph but also as an irregular digraph with 2-cycle and which sets have a realization as an irregular pseudodigraph with a loop? The answers to these questions are given below.

Proposition 8. *If D is realized by an oriented graph G which includes a path P_4 with nonadjacent endvertices then D is realized by a loopless digraph with exactly one 2-cycle.*

Proof. Let G be an oriented graph which realizes D and contains a path (v_0, v_1, v_2, v_3) such that vertices v_0 and v_3 are not adjacent. Then adding the arcs (v_0, v_3) and (v_2, v_1) together with removing the arcs (v_0, v_1) and (v_2, v_3) results in a required digraph. ■

Proposition 9. *If D is realized by an oriented graph G which includes a path P_3 with nonadjacent endvertices then D is realized by a pseudodigraph with exactly one loop.*

Proof. Let G be an oriented graph G which realizes D and contains an induced path (v_0, v_1, v_2). Then replacing the arcs (v_0, v_1), (v_1, v_2) by the arc (v_0, v_2) and a loop at the vertex v_1 results in a required pseudodigraph. ∎

Lemma 10. *If $D \in \mathcal{D}_n^{\min}$ and D includes at least two pairs whose all components are positive then there exists an irregular oriented graph G which realizes the set D and contains a path P_4 with nonadjacent endvertices.*

Proof. Assume $n \geq 7$, because for $n < 7$ no D has two pairs with positive components. Then $\tau_n \geq 3$. Refer to (6) for G_0. Then $G_0 = \bigcup_{i=1}^{k} T_i$ where $k \geq 2$.

Case 1. $k \geq 4$. Let T_3 and T_4 be transitive tournaments where $V(T_3) = \{u_1, u_2, u_3\}$, $V(T_4) = \{v_1, v_2, v_3, v_4\}$, $E(T_3) = \{(u_i, u_j) \colon i < j\}$, and $E(T_4) = \{(v_i, v_j) \colon i < j\}$. Then the arc (v_1, v_4) joins the endvertices of the path $P_4 = (v_1, v_2, v_3, v_4)$. Let G_0' denote the graph obtained from G_0 by removing two arcs (v_1, v_4) and (u_1, u_2) and by adding arcs $(u_1, v_4), (v_1, u_2)$. Note that the path P_4 has nonadjacent endvertices in G_0', $\mu(G_0') = \mu(G_0)$, and in G_0' we can choose a matching of cardinality $\mu(G_0')$ which does not contain any arc of P_4. Then we can obtain a required G by the method used in the proof of Theorem 6 taking G_0' instead of G_0.

Case 2. $k < 4$. From the definitions of G_0, τ_n and the assumption $n \geq 7$ it follows that $\tau_n = 3$. Then only the following sets D_1–D_{13} have at least two pairs with all positive components.

$D_1 = \{(0,0), (1,0), (0,1), (2,0), (0,2), (2,1), (1,2)\}$,
$D_2 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,3), (2,1)\}$,
$D_3 = \{(0,0), (1,0), (0,1), (1,1), (0,2), (3,0), (1,2)\}$,
$D_4 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (2,2)\}$,
$D_5 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (2,1), (1,2)\}$,
$D_6 = \{(0,0), (1,0), (0,1), (2,0), (0,2), (3,0), (2,1), (1,2), (0,3)\}$,
$D_7 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (0,3), (2,2)\}$,
$D_8 = \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (0,4)\}$,

$$D_9 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(0,3),(1,2),(4,0)\},$$
$$D_{10} = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0),(1,2),(1,3)\},$$
$$D_{11} = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(0,3),(2,1),(3,1)\},$$
$$D_{12} = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(2,1),(1,2),(2,2)\},$$
$$D_{13} = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),(3,0),(2,1),(1,2),(0,3)\}.$$

For $1 \le i \le 13$, $i \ne 3,9,11$ the graph G_i presented in Figure 2 realizes the set D_i. A realization of D_i for $i = 3,9,11$ can be obtained by reversing the arcs in digraphs G_2, G_8, G_{10}, respectively. ■

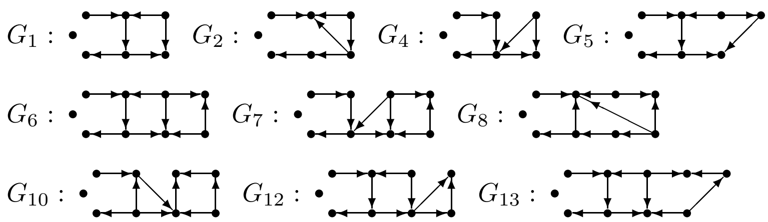

Figure 2. Irregular oriented graphs which include a path P_4 with nonadjacent endvertices.

Lemma 11. *If $D \in \mathcal{D}_n^{\min} \backslash \{(0,0),(1,1)\}$ and D contains a pair whose components are both positive then there exists an irregular oriented graph G which includes a path P_3 with nonadjacent endvertices and realizes D.*

Proof. Refer to (6) and (8) for G_0 and r, respectively. Consider the following two cases.

Case 1. G_0 contains T_3. Let $V(T_3) = \{v_1, v_2, v_3\}$. If $r = 0$ then $G = G_0 \cup T_{\tau_n+1}$ realizes D. This graph includes the only arc of T_2, say $a_1 = (u_1, u_2)$, and G includes three arcs, say (v_1, v_2), $a_2 = (v_1, v_3)$ and (v_2, v_3), of T_3. Therefore replacing in G the two arcs a_1 and a_2 by (u_1, v_3) and (v_1, u_2) results in a required irregular oriented graph G.

If $r > 0$ then we apply the construction described in the proof of Theorem 6. In this construction the operation $\diamond_M$, where M is a matching in G_0, is used. So, if $(v_1, v_3) \in M$ then obtained graph G includes the induced path (v_1, v_2, v_3).

Case 2. G_0 does not contain T_3. Then $\tau_n = 3$ in case (ii), $\tau_n = 2$ in cases (i), and (iii) is not possible. So D is one of the following six sets:

$D_1 = \{(0,0),(1,0),(0,1),(1,1)\}$,
$D_2 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$,
$D_3 = \{(0,0),(1,0),(0,1),(2,0),(0,2),(2,1),(1,2)\}$,
$D_4 = \{(0,0),(1,0),(0,1),(1,1),(0,2),(3,0),(1,2)\}$,
$D_5 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,3),(2,1)\}$,
$D_6 = \{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(2,1),(1,2),(0,3)\}$.

For $1 \le i \le 6$ the graph G_i presented in Figure 3 realizes the set D_i. ■

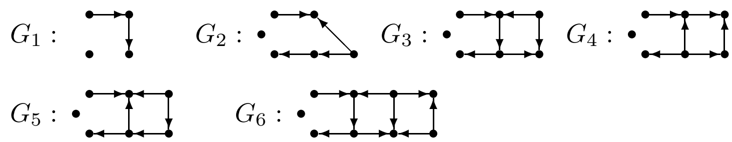

Figure 3. Irregular oriented graphs which include a path P_3 with nonadjacent endvertices.

The above propositions and lemmas imply the following.

Theorem 12. *For $D \in \mathcal{D}_n^{\min}$ there exists an irregular digraph G with a single 2-cycle C_2 which realizes D if and only if D includes at least two pairs whose all components are positive.* ■

Theorem 13. *For $D \in \mathcal{D}_n^{\min}$ there exists an irregular pseudodigraph with a single loop and without 2-cycles which realizes D if and only if D includes a pair whose components are positive.* ■

7. The Unique Irregular Realizations

In this section we describe all sets from $\mathcal{D}_n^{\min}$ which are uniquely realizable in the class of irregular oriented graphs (digraphs).

Examples. Let $D_1 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,3),(2,1)\}$ and $D_2 = \{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(0,3)\}$. Note that $D_1, D_2 \in \mathcal{D}_7^{min}$ and D_1 is not uniquely realizable while D_2 is uniquely realizable (see Figure 4).

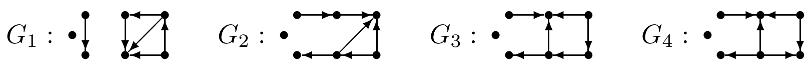

Figure 4. G_1, G_2 and G_3 are nonisomorphic realizations of D_1, G_4 is the unique realization of D_2.

It appears that seven is the greatest integer n such that there exists $D \in \mathcal{D}_n^{\min}$ which has the unique irregular realization.

Theorem 14. *In $\bigcup_{n=1}^{+\infty} \mathcal{D}_n^{\min} \setminus \{(0,0),(1,1)\}$ only the sets $\{(0,0)\}$, $\{(1,0),(0,1)\}$, $\{(0,0),(1,0),(0,1)\}$, $\{(0,0),(1,0),(0,1),(1,1)\}$, $\{(0,0),(1,0),(0,1),(2,0),(0,2)\}$ and $\{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(0,3)\}$ have the unique realization in the class of irregular oriented graphs as well as in that of irregular simple digraphs.*

Proof. Let $n \geq 8$, $D \in \mathcal{D}_n^{\min}$, and let s be the largest integer such that D includes the union $B_0 \cup B_1 \cup \ldots \cup B_s$. Then $s \geq 2$ except of $s = 1$ for $D = D_3$ where

$$D_3 = \{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(2,1),(1,2),(0,3)\}.$$

Nonisomorphic irregular realizations of D_3 are presented in Figure 5.

Figure 5. Two nonisomorphic realizations of the set D_3.

If $D \setminus \bigcup_{i=0}^{s} B_i = \emptyset$ then the union of vertex disjoint transitive tournaments $T_1, T_2, \ldots, T_{s+1}$ is one of realizations of D. Replacing therein two arcs, say (u_1,u_2) from T_i and (v_1,v_2) from T_j where $i \neq j$, by arcs (u_1,v_2) and (v_1,u_2) gives another realization of D. Now let $D \setminus \bigcup_{i=0}^{s} B_i \neq \emptyset$. In this case we can obtain nonisomorphic realizations of D using the operation $\diamond_M$ with distinct matchings in G_0, cp. (6) and the construction given in the proof of Theorem 6. Distinct matchings in G_0 exist because G_0 contains T_3. Testing all sets $D \in \mathcal{D}_n^{\min}$ for $n \leq 7$ for uniqueness of their realizations we obtain the stated list. Then uniqueness among simple irregular digraphs follows by Theorem 12. ∎

Proposition 15. *In* $\bigcup_{n=1}^{+\infty} \mathcal{D}_n^{\min}$ *only the sets* $\{(0,0)\}$, $\{(0,0),(1,1)\}$, $\{(1,0),(0,1)\}$, $\{(0,0),(1,0),(0,1)\}$, $\{(0,0),(1,0),(0,1),(2,0),(0,2)\}$ *and* $\{(0,0),(1,0),(0,1),(2,0),(0,2),(3,0),(0,3)\}$ *have the unique irregular realization in the class of pseudodigraphs.*

Proof. It is not difficult to check that each set D from $\mathcal{D}_n^{\min}$, which is not on the list, contains a pair whose both components are positive. Then, by Theorems 6 and 13, D has two realizations: one as an oriented graph, another as a pseudodigraph. On the other hand, $D = \{(0,0),(1,1)\}$ is the only set on the list whose unique irregular realization is pseudodigraphic. ■

8. Maximum Sets

The following is quite obvious (cf. Introduction and Proposition 1, or [5]).

Proposition 16. *The set* B_{n-1} *is the unique set of degree pairs among maximum* n*-vertex irregular oriented graphs. The transitive tournament* T_n *is the only maximum* n*-vertex irregular oriented graph,* $D_{T_n} = B_{n-1}$.

Remark 1. The transitive tournament T_n is the only realization of B_{n-1} in the class of n-vertex pseudodigraphs, too. However, T_n is not a maximum irregular n-vertex simple digraph for $n \geq 3$.

If D is an n-set of ordered pairs of nonnegative integers then we put

$$D^c = \{(n-1-a, n-1-b) : (a,b) \in D\},$$

$$D^{c+} = \{(n-a, n-b) : (a,b) \in D\}.$$

Given a digraph G (resp. pseudodigraph G), we let $\overline{G}$ (resp. $\overline{G}^+$) denote the complement of G in the complete digraph (resp. complete pseudodigraph).

Theorem 17. *An* n*-set* D *of ordered pairs of nonnegative integers is realized by a maximum irregular simple digraph* (*resp. pseudodigraph*) *iff* $D^c \in \mathcal{D}_n^{\min} \setminus \{(0,0),(1,1)\}$ (*resp.* $D^{c+} \in \mathcal{D}_n^{\min}$).

Proof. Theorem follows from Corollary 7 and the following observations.

- G is an irregular simple digraph (general digraph) iff $\overline{G}$ (resp. $\overline{G}^+$) is an irregular simple digraph (general digraph),

- G is a maximum irregular simple digraph (pseudodigraph) iff $\overline{G}$ (resp. $\overline{G}^+$) is a minimum irregular simple digraph (general digraph),
- $\mathrm{od}_{\overline{G}}(v) = |V(G)| - 1 - \mathrm{od}_G(v)$ and $\mathrm{id}_{\overline{G}}(v) = |V(G)| - 1 - \mathrm{id}_G(v)$,
- $\mathrm{od}_{\overline{G}^+}(v) = |V(G)| - \mathrm{od}_G(v)$ and $\mathrm{id}_{\overline{G}^+}(v) = |V(G)| - \mathrm{id}_G(v)$. ■

9. Concluding Remark

It can be seen that, for each structure: oriented graph, simple digraph, and pseudodigraph, the size among n-vertex irregular structures ranges over an integer interval. We conjecture that deleting a single arc at a time from a digraph can transform a certain maximum irregular structure to some minimum one so that all intermediate structures are irregular, too.

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