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GRAPHS WITH CONVEX DOMINATION NUMBER CLOSE TO THEIR ORDER

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Abstract

For a connected graph $G = (V, E)$, a set $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) - D$ has at least one neighbour in D. The *distance* $d_G(u, v)$ between two vertices u and v is the length of a shortest $(u - v)$ path in G. An $(u - v)$ path of length $d_G(u, v)$ is called an $(u - v)$-*geodesic*. A set $X \subseteq V(G)$ is *convex* in G if vertices from all $(a - b)$-geodesics belong to X for any two vertices $a, b \in X$. A set X is a *convex dominating set* if it is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G is the minimum cardinality of a convex dominating set in G. Graphs with the convex domination number close to their order are studied. The convex domination number of a Cartesian product of graphs is also considered.

Keywords: convex domination, Cartesian product.

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1. Terminology

Let $G = (V, E)$ be a simple connected undirected graph with $|V(G)| = n(G)$. The *open neighbourhood* of a vertex $v \in V(G)$ in G is the set $N_G(v)$ of all vertices adjacent to v in G and the *closed neighbourhood* is $N_G[v] = N_G(v) \cup \{v\}$.

The *degree* $\deg_G(v)$ of a vertex v in G is the number of edges incident to v, that is $\deg_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\delta(G) = \Delta(G) = 2$, then G is a cycle and the cycle on n vertices is denoted by C_n.

A set $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G) - D$ has at least one neighbour in D. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set in G.

The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $(u - v)$ path in G. An $(u - v)$ path of length $d_G(u, v)$ is called an $(u - v)$-*geodesic*. A set $X \subseteq V(G)$ is *convex* in G if vertices from all $(a - b)$-geodesics belong to X for every two vertices $a, b \in X$. A set X is a *convex dominating set* if X is convex and dominating. The *convex domination number* $\gamma_{con}(G)$ of a graph G equals the minimum cardinality of a convex dominating set in G. The convex domination number was first introduced in 2002 by Jerzy Topp (Gdańsk University of Technology).

The *Cartesian product* of two graphs G_1, G_2 is the graph $G = G_1 \times G_2$ with the vertex set $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2), (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if we have one of two possibilities: $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

The length of a shortest cycle in G is the *girth* of G and is denoted $g(G)$ and the *circumference* $c(G)$ is the length of a longest cycle in G. The *eccentricity* $e(v)$ of a vertex v of a connected graph G is the number $\max_{u \in V(G)} d_G(u, v)$. Define an *end-vertex* to be a vertex of degree one. The neighbour of an end-vertex is called *a support*.

For a connected graph G, a vertex $x \in V(G)$ is called *a cut-vertex* if $G - x$ is no longer connected and an edge $e \in E(G)$ is called *a bridge* if $G - e$ is not connected. A connected subgraph B of G is called *a block* if B has no cut-vertex and every subgraph $B' \subseteq G$ with $B \subseteq B'$ and $B \neq B'$ has at least one cut-vertex. A graph G is called *a block graph* if every block in G is a clique. A vertex $v \in V(G)$ is called *simplicial* if the subgraph $\langle N_G[v] \rangle$ induced by $N_G[v]$ is a clique.

2. Results

We consider only connected graphs. Let us begin with an example and some obvious observations.

Example 1. For a cycle C_n on $n \geq 6$ vertices, $\gamma_{con}(C_n) = n$.

Observation 1. If $G \neq K_n$ and D is a minimum convex dominating set of G, then

1. every cut-vertex belongs to D,
2. no simplicial vertex belongs to D,
3. for any $x, y \in D$ such that $d_G(x, y) \geq 2$, we have $N_G(x) \cap N_G(y) \subseteq D$.

Observation 2. If G is a block graph and $G \neq K_n$, then $\gamma_{con}(G) = n(G) - s$, where s is the cardinality of the set of all simplicial vertices of G.

The following theorems describe graphs with the convex domination number equal to their order.

Theorem 3. *If G is a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 6$, then $\gamma_{con}(G) = n(G)$.*

Proof. Let G be a connected graph with $\delta(G) \geq 2$ and $g(G) \geq 6$. Suppose that $\gamma_{con}(G) < n(G)$. Let D be a minimum convex dominating set of G. Since $\gamma_{con}(G) < n(G)$, there exists a vertex $x \in V(G)$ such that $x \notin D$. Denote $N_G(x) = \{x_1, \ldots, x_p\}$, where $p \geq 2$. Since $g(G) \geq 6$, for every $i, j \in \{1, \ldots, p\}$ we have $x_i x_j \notin E(G)$ and $|N_G(x_i) \cap N_G(x_j)| = 1$ (x is the only common neighbour of vertices x_i and x_j).

Notice that for every $i, j \in \{1, \ldots, p\}, i \neq j$, we have $d_G(x_i, x_j) = 2$ and every shortest path between x_i and x_j contains x.

If there were vertices $x_i, x_j \in N_G(x)$ such that $x_i, x_j \in D$, then, by Observation 1, $x \in D$, a contradiction. Thus $|N_G(x) \cap D| \leq 1$ and since x is dominated, we have $|N_G(x) \cap D| = 1$. Without loss of generality assume that $x_1 \in N_G(x) \cap D$. Hence $x_2 \notin D$. Since x_2 is dominated, there exists a vertex $y \in N_G(x_2)$ such that $y \neq x$ and $y \in D$. Since $g(G) \geq 6$, we have $N_G(y) \cap N_G(x) = \{x_2\}$ and $N_G(y) \cap N_G(x_i) = \emptyset$, where $1 \leq i \leq p$. Thus $d_G(y, x_1) = 3$ and the path (y, x_2, x, x_1) is a $(y - x_1)$-geodesic such that two vertices from this path do not belong to D, which contradicts the convexity of D. Thus $\gamma_{con}(G) = n(G)$. ∎

Theorem 4. *$\gamma_{con}(G) < n(G)$ for a connected graph G with $c(G) \leq 5$.*

Proof. Let G be a graph with $c(G) \leq 5$. If there is an end-vertex in G, then by Observation 1, $\gamma_{con}(G) < n(G)$, so from now on we assume that $\delta(G) \geq 2$. Let $C^r = (x_1, \ldots, x_r, x_1)$, $r \leq 5$, be a longest cycle in G. We consider two cases.

Case 1. For every vertex $v \in V(G)$, $|N_G(v) \cap V(C^r)| \geq 2$.
If $r = 3$, then $G = C_3$, for otherwise C_4 is obtained. In this case $\gamma_{con}(G) = 1 < n(G)$.

If $r = 4$, then $D = \{x_1, x_2\}$ is a convex set in G, because every vertex $x \in V(G) - V(C^r)$ has exactly two non-adjacent neighbours belonging to $V(C^r)$ and thus each vertex of $v \in V(G) - V(C^r)$ has exactly one neighbour among $\{x_1, x_2\}$. Hence D is dominating and $\gamma_{con}(G) \leq 2 < n(G)$.

Assume that $r = 5$. If $x_1x_4 \in E(G)$ and $x_3x_5 \in E(G)$, then $D = \{x_1, x_4, x_5\}$ is a convex dominating set of G. Otherwise $D = \{x_1, x_3\} \cup S$, where $S = N_G(x_1) \cap N_G(x_3)$, is a dominating set of G. Suppose D is not convex. Then for some $x, y \in S$ there exists a vertex $z \in N_G(x) \cap N_G(y)$ such that $z \notin D$. Then $(x_1, x, z, y, x_3, x_4, x_5, x_1)$ is a cycle of length seven, a contradiction. Thus D is convex and we obtained a convex dominating set of G of cardinality smaller than $n(G)$.

Case 2. There exists a vertex $v \in V(G)$ such that $|N_G(v) \cap V(C^r)| \leq 1$. Let $A = \{u \in V(G) : d_G(u, v) = e(v)\}$ and let $\mathcal{C}$ be the family of cycles such that $|A \cap V(C)| \geq 1$ for every $C \in \mathcal{C}$. Let C_s be a longest cycle belonging to $\mathcal{C}$ such that $d_G(v, V(C^s))$ is minimum (notice that C^s does not have to be a longest cycle in G). Observe that v could not have more than one neighbour belonging to C^s, as otherwise we would obtain a cycle $C \in \mathcal{C}$ not shorter than C^s such that $d_G(v, V(C)) < d_G(v, V(C^s))$. We show that there exists exactly one vertex $a_1 \in V(C^s)$ such that $d_G(V(C^s), v) = d_G(a_1, v)$ and a_1 is a cut-vertex.

Suppose, on the contrary, that for $b, c \in V(C^s)$,

$$d_G(V(C^s), v) = d_G(b, v) = d_G(c, v) = t.$$

Let P be a longest $(b-c)$-path containing only vertices of C^s. We denote $(b - v) = (b_0, b_1, \ldots, b_t)$, where $b_0 = b, b_t = v$ and $(c - v) = (c_0, c_1, \ldots, c_t)$, where $c_0 = c, c_t = v$, $l = \min\{k : b_k = c_k\}$. If $l \geq 2$, we obtain a cycle of length longer than 5, a contradiction. If $a_1 = v$, then $d_G(v, V(C^r)) \geq$

$1 > d_G(v, V(C^s)) = 0$ and for this reason $V(C^s) \cap A = \emptyset$, a contradiction. Thus $l = 1$. Now, if $bc \in E(C^s)$, then $V(C^s) \cup \{b_1\}$ forms a cycle longer than C^s, which has the same properties as C^s, a contradiction. Otherwise $(C^s - \{z\}) \cup \{b_1\}$, where $z \in V(C^s) - P$, forms a cycle $C \in \mathcal{C}$ such that $d_G(v, V(C)) < d_G(v, V(C^s))$, a contradiction. Hence a_1 is unique.

Now we show that a_1 is a cut-vertex. Let $w \in V(C^s) \cap N_G(a_1)$. It is easy to observe that if there is $(w - v)$-path not containing a_1, then we obtain a longer cycle, which has the same properties as C^s. Hence each $(w - v)$-path contains a_1 and thus a_1 is a cut-vertex.

Denote $C^s = (a_1, \ldots, a_s, a_1)$ and let S be the component of $G - \{a_1\}$ containing the vertices of $V(C^s) - \{a_1\}$. We consider three subcases.

Subcase 2.1. There exists a vertex $w \in V(S)$ such that $N_G(w) \cap V(C^s) = \emptyset$. Then $d_G(w, a_1) = 2$, as otherwise $A \cap V(C^s) = \emptyset$. Moreover, there are $k \in N_G(a_1) \cap N_G(w)$ and $x \in N_G(w)$, $x \neq k$. Since S is connected, there is a path between x and $V(C^s) - \{a_1\}$. If there is a path not containing k, then for a_i such that $d_G(x, V(C^s)) = d_G(x, a_i)$, $i \neq 1$, either $(a_1, k, w, x, a_i, \ldots, a_s, a_1)$ or $(a_1, a_2, \ldots, a_s, x, w, k, a_1)$ is a cycle longer than 5, a contradiction. Hence every path from x to $V(C^s) - \{a_1\}$ contains k. Without loss of generality we can assume that $ka_3 \in E(G)$. Observe that $N_G[w] \subset N_G[k]$ and hence $(V(G) - N_G(k)) \cup \{a_1, a_3\}$ is a convex dominating set of G.

Subcase 2.2. There exists a vertex $w \in V(S)$ such that $|N_G(w) \cap V(C^s)| = 1$.

Let $w \in N_G(a_i)$ for some i and $t \in N_G(w)$, $k \neq a_i$. If $k \in N_G(a_j)$, $j \neq i$, or if there exists a $(k - a_j)$-path not containing a_i, then we obtain a cycle longer than C^s, which has the same properties as C^s, a contradiction. Hence, since $d_G(w, a_1) \leq 2$, we have $N_G[w] \subset N_G[a_i]$ and $(V(G) - N_G[w]) \cup \{a_i\}$ is a convex dominating set of G.

Subcase 2.3. All vertices from S have at least two neighbours in $V(C^s)$. If $s = 3$, then $V(G) - V(S)$ is a convex dominating set of G and thus $\gamma_{con}(G) < n$.

If $s = 4$, then $(V(G) - V(S)) \cup \{a_2\}$ is a convex dominating set of G and again $\gamma_{con}(G) < n$.

Consider the case $s = 5$. Assume that $a_2a_4 \notin E(G)$ or $a_3a_5 \notin E(G)$. Then $D = (V(G) - V(S)) \cup \{a_2, a_5\} \cup (N_G(a_2) \cap N_G(a_5))$ is a convex dominating set of G with $|D| < n(G)$.

If $a_4a_2 \in E(G)$ and $a_3a_5 \in E(G)$, then $D = (V(G) - V(S)) \cup \{a_2\}$ is a convex dominating set of G of cardinality smaller than $n(G)$, for otherwise there would exist a vertex $u \in S$ such that $N_G(u) \cap \{a_1, a_2\} = \emptyset$. But then $u \in N(a_3) \cap N(a_5)$ and $(a_1, a_2, a_4, a_3, u, a_5, a_1)$ is a cycle longer than 5, a contradiction. ■

Theorem 5. *Let G be a connected graph with $n(G) \geq 5$. If $\gamma_{con}(G) = n(G)$, then $\Delta(G) \leq n(G) - 4$.*

Proof. We shall prove that if $\Delta(G) > n(G) - 4$, then $\gamma_{con}(G) < n(G)$.

If $\Delta(G) = n(G) - 1$, then obviously $\gamma_{con}(G) = 1 < n(G)$.

Let x be a vertex with $d_G(x) = \Delta(G) = n(G) - 2$ and let $u \notin N_G[x]$. Since G is connected, there exists a vertex $y \in N_G(u) \cap N_G(x)$ and $\{x, y\}$ is a minimum convex dominating set in G. Thus $\gamma_{con}(G) = 2 < n(G)$.

Assume now that $d_G(x) = \Delta(G) = n(G) - 3$ and let $u, v \notin N_G[x]$. If $d_G(u) = 1$ or $d_G(v) = 1$, say $d_G(u) = 1$, then $V(G) - \{u\}$ is a convex dominating set of G, which implies $\gamma_{con}(G) < n(G)$. Hence assume $d_G(u) > 1$ and $d_G(v) > 1$. If there exists a vertex $w \in N_G(x) \cap N_G(u) \cap N_G(v)$, then $\{x, w\}$ is a minimum convex dominating set in G and $\gamma_{con}(G) = 2 < n(G)$. Otherwise, if u and v have no common neighbour in $N_G(x)$, then there are $y \in N_G(x) \cap N_G(u)$ and $z \in N_G(x) \cap N_G(v)$. Then $\{x, y, z\} \cup (N_G(y) \cap N_G(z))$ is a convex dominating set in G and once again $\gamma_{con}(G) < n(G)$. ■

For connected graphs G with $\gamma_{con}(G) = n(G)$, the inequality $\Delta(G) \leq n(G) - 4$ is best possible and equality $\Delta(G) = n(G) - 4$ holds for example for $G = C_6$.

The next corollary follows directly from Theorem 3.

Corollary 6. *If $\gamma_{con}(G) = n(G)$ and $G \neq K_1$, then $2 \leq \delta(G) \leq \Delta(G) \leq n(G) - 4$.*

Hence we immediately have:

Corollary 7. *If $\gamma_{con}(G) = n(G)$ and $G \neq K_1$, then $n(G) \geq 6$.*

Now we follow with two observations. The straightforward proofs are omitted.

Observation 8. If $\gamma_{con}(G) = n(G)$ and G_1 is the graph obtained from G by adding a vertex v and edges $e_1, \ldots, e_k$ incident to v and to vertices of a k-clique in G, then $\gamma_{con}(G_1) = n(G_1) - 1 = n(G)$.

Observation 9. If $\gamma_{con}(G) = n(G)$ and G_2 is the graph obtained from G by adding vertices of a k-clique and edges $e_1, \ldots, e_k$ joining all vertices of the clique with any vertex of G, then $\gamma_{con}(G_2) = n(G_2) - k = n(G)$.

Now let G be a graph in which u, v, w induce a path P in G and let X be a non-empty set of vertices with $V(G) \cap X = \emptyset$. Denote by $F(G, P, X)$ an operation of adding to G the vertices of X and edges xu and xw for each $x \in X$.

Lemma 10. *If $\gamma_{con}(G) = n(G)$ and the vertices u, v, w induce a path P in G, then for the graph $H = F(G, P, \{x\})$,*

$$\gamma_{con}(H) = n(H) = n(G) + 1.$$

Proof. Suppose that $\gamma_{con}(G) = n(G)$ and $\gamma_{con}(H) < n(H)$. Let D_H be a minimum convex dominating set in H. Observe that if $u, w \notin D_H$, then x belongs to D_H, which implies that x is isolated in subgraph $\langle D_H \rangle$ induced by D_H and thus D_H is not convex. Hence $u \in D_H$ or $w \in D_H$. Without loss of generality we can assume that $u \in D_H$.

If $w \notin D_H$ and $x \in D_H$, then $v \notin D_H$, because otherwise w belongs to a $(v - x)-$geodesic and $w \in D_H$. But then $D_G = D_H - \{x\} \cup \{v\}$ is a convex dominating set of G, a contradiction.

If $w \notin D_H$ and $x \notin D_H$, then $D_G = D_H$ is a convex dominating set of G of cardinality smaller than $n(G)$, a contradiction.

If $u, w \in D_H$, then $v, x \in D_H$, because D_H is convex. Hence there exists $y \notin N_H[x]$ such that $y \notin D_H$. But then $D_G = D_H - \{x\}$ is a convex dominating set of G of cardinality smaller than $n(G)$, a contradiction. ∎

Corollary 11. *Let X be a non-empty set of vertices. If $\gamma_{con}(G) = n(G)$ and $u, v, w \in V(G)$ induce a path P in G, then $\gamma_{con}(H) = n(H) = n(G) + |X|$ for a graph $H = F(G, P, X)$.*

For two disjoint connected graphs G and G_1 denote by $J(G, P, G_1) = F(G, P, V(G_1)) \cup E(G_1)$, where u, v, x induce a path P in G.

Theorem 12. *If G and G_1 are disjoint connected graphs such that $\gamma_{con}(G) = n(G)$ and $u, v, w \in V(G)$ induce a path P in G, then for the graph $H = J(G, P, G_1)$ is $\gamma_{con}(H) = n(H) = n(G) + n(G_1)$.*

Proof. Let $H_0 = F(G, P, V(G_1))$. Corollary 11 implies that $\gamma_{con}(H_0) = n(H_0) = n(G) + n(G_1)$. Moreover, each $x \in V(G_1)$ belongs to $(u - w)$-geodesic. It follows that for $H = H_0 \cup E(G_1)$, $\gamma_{con}(H) = n(H) = n(G) + n(G_1)$. ∎

Corollary 13. *For every integer $k \geq 3$ there exists a graph H such that $\Delta(H) = k$ and $\gamma_{con}(H) = n(H)$.*

Proof. Let G be a cycle on six vertices and let $G_1 = K_{k-2}$. Then $\gamma_{con}(G) = n(G)$. If u, v, w are any consecutive vertices of the cycle, then u, v, w induce a path P in G and Theorem 12 implies that for $H = J(G, P, G_1)$ is $\gamma_{con}(H) = n(H)$. Obviously, $\Delta(H) = k$. ∎

From Theorem 12 we obtain that a forbidden subgraph characterization for graphs with $\gamma_{con}(G) = n(G)$ cannot be obtained since for any graph G_1 there exists a graph H, namely $H = J(C_6, P, G_1)$, such that $\gamma_{con}(H) = n(H)$ and G_1 is an induced subgraph of H.

Lemma 14. *If G_1 and G_2 are connected graphs such that $\gamma_{con}(G_1) = n(G_1) > 1$ and $\gamma_{con}(G_2) = n(G_2) > 1$, then for a graph H obtained from G_1 and G_2 by adding an edge e joining any vertex of G_1 to any vertex of G_2,*

$$\gamma_{con}(H) = n(H) = n(G_1) + n(G_2).$$

Proof. In such a graph H, e is a bridge. As $\gamma_{con}(G_1) = n(G_1) > 1$ and $\gamma_{con}(G_2) = n(G_2) > 1$, the statement follows. ∎

Instead of an edge e, we can also add a path $P_k = (v_1 \ldots v_k)$ such that $G_1 \cap P_k = \{v_1\}$ and $G_2 \cap P_k = \{v_k\}$.

For a graph G and a cycle C_p, let $H = G \circ C_p$ be the graph obtained from G and $n(G)$ copies $C_p^1, C_p^2, \ldots, C_p^{n(G)}$ of C_p by joining each $v_i \in V(G)$ with exactly one vertex of C_p^i for $i = 1, 2, \ldots, n(G)$.

Lemma 15. *If G is a connected graph on $n(G) > 1$ vertices, then for $H = G \circ C_p$, $p \geq 6$,*

$$\gamma_{con}(H) = n(H) = (p + 1)n(G).$$

Proof. Each edge connecting a vertex of $V(G)$ to a vertex of a copy of C_p is a bridge. Moreover, $\gamma_{con}(C_p) = p$. Hence $\gamma_{con}(H) = n(H) = (p+1)n(G)$. ∎

Now let $G \times H$ be the Cartesian product of connected graphs G and H. For a set $D \subseteq V(G \times H)$ we denote:

$$D_G = \{u \in V(G) : (u, v) \in D \text{ for some } v \in V(H)\},$$

$$D_H = \{v \in V(H) : (u, v) \in D \text{ for some } u \in V(G)\}.$$

The *Vizing Conjecture* says that the domination number of the Cartesian product of any two graphs is at least as large as the product of their domination numbers.

The following result can be found in [2]:

Theorem 16. *A set $D \subseteq V(G \times H)$ is convex in $G \times H$ if and only if $D = D_G \times D_H$, where D_G and D_H are convex in G and H, respectively.*

Using this result and the next lemma, we prove that the convex domination number of the Cartesian product of two connected graphs is at least as large as the product of their convex domination numbers.

Lemma 17. *If D is dominating in $G \times H$, then D_G is dominating in G and D_H is dominating in H.*

Proof. Let D be a dominating set in $G \times H$. Then it is easily seen that $D_G = V(G)$ or $D_H = V(H)$. It suffices to show that D_G is dominating in G if $D_H = V(H)$ (similarly we can prove that D_H is dominating in H if $D_G = V(G)$).

Let $x \in V(G) - D_G$. For every $y \in V(H)$ is $(x, y) \in V(G \times H) - D$ and $N_{G \times H}(x, y) \cap D \neq \emptyset$. Since $(\{x\} \times N_H(y)) \cap D = \emptyset$ (because $x \notin D_G$), we have

$$\begin{aligned} \emptyset &\neq N_{G\times H}(x, y) \cap D \\ &= ((N_G(x) \times \{y\}) \cup (\{x\} \times N_H(y))) \cap D \\ &= (N_G(x) \times \{y\}) \cap D \\ &= (N_G(x) \times \{y\}) \cap (D_y \times \{y\}), \end{aligned}$$

where D_y is a projection of $D \cap (V(G) \times \{y\})$ into $V(G)$

$$= (N_G(x) \cap D_y) \times \{y\}.$$

Hence $N_G(x) \cap D_y \neq \emptyset$ and $N_G(x) \cap D_G \neq \emptyset$, because $D_y \subseteq D_G$. It proves that D_G is a dominating set in G. ∎

Theorem 18. *For any connected graphs G and H, we have inequality $\gamma_{con}(G)\gamma_{con}(H) \leq \gamma_{con}(G \times H)$.*

Proof. Let D be a minimum convex dominating set in $G \times H$. Thus, by Theorem 14 and Lemma 15, the sets D_G and D_H are dominating and convex in G and H, respectively. Hence $\gamma_{con}(G) \leq |D_G|$ and $\gamma_{con}(H) \leq |D_H|$. Since $D = D_G \times D_H$ (by Theorem 14), we have equality $|D| = |D_G||D_H|$. Thus we have $\gamma_{con}(G)\gamma_{con}(H) \leq |D_G||D_H| = |D| = \gamma_{con}(G \times H)$. ■

Corollary 19. *For graphs G for which $\gamma(G) = \gamma_{con}(G)$, the Vizing's Conjecture is satisfied.*

Corollary 20. *If G and H are connected graphs and $\gamma_{con}(G) = n(G)$, $\gamma_{con}(H) = n(H)$, then $\gamma_{con}(G \times H) = n(G)n(H) = n(G \times H)$.*

Observation 21. If every vertex of G is a support or an end-vertex, then $\gamma_{con}(G) = \gamma(G)$.

Corollary 22. *If every vertex of G_1 and G_2 is a support or an end-vertex, then $\gamma(G_1)\gamma(G_2) \leq \gamma(G_1 \times G_2)$.*

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