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ARBITRARILY VERTEX DECOMPOSABLE CATERPILLARS WITH FOUR OR FIVE LEAVES

Sylwia Cichacz, Agnieszka Görlich, Antoni Marczyk
Jakub Przybyło

Faculty of Applied Mathematics
AGH University of Science and Technology
Al. Mickiewicza 30, 30–059 Kraków, Poland

e-mail: marczyk@uci.agh.edu.pl

and

Mariusz Woźniak

Institute of Mathematics of Polish Academy of Sciences
(*on leave from AGH*)

Abstract

A graph G of order n is called arbitrarily vertex decomposable if for each sequence $(a_1, \ldots, a_k)$ of positive integers such that $a_1+\ldots+a_k = n$ there exists a partition $(V_1, \ldots, V_k)$ of the vertex set of G such that for each $i \in \{1, \ldots, k\}$, V_i induces a connected subgraph of G on a_i vertices.

D. Barth and H. Fournier showed that if a tree T is arbitrarily vertex decomposable, then T has maximum degree at most 4. In this paper we give a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. We also describe two families of arbitrarily vertex decomposable trees with maximum degree three or four.

Keywords: arbitrarily vertex decomposable graphs, trees, caterpillars, star-like trees.

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1. Introduction

Let $G = (V, E)$ be a graph of order n. A sequence $\tau = (a_1, \ldots, a_k)$ of positive integers is called *admissible for* G if it adds up to n. If $\tau = (a_1, \ldots, a_k)$ is an admissible sequence for G and there exists a partition $(V_1, \ldots, V_k)$ of the vertex set V such that for each $i \in \{1, \ldots, k\}$, $|V_i| = a_i$ and a subgraph induced by V_i is connected then τ is called *realizable in* G and the sequence $(V_1, \ldots, V_k)$ is said to be *a* G*-realization of* τ or *a realization of* τ *in* G. A graph G is *arbitrarily vertex decomposable* (avd for short) if for each admissible sequence τ for G there exists a G-realization of τ.

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers (see for example [1]–[4]). Generally, this problem is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees.

However, it is obvious that each path and each traceable graph is avd. The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd. In [4] M. Horňák and M. Woźniak conjectured that if T is a tree with maximum degree $\Delta(T)$ at least five, then T is not avd. This conjecture was proved by D. Barth and H. Fournier [2].

Theorem 1. *If a tree* T *is arbitrarily vertex decomposable, then* $\Delta(T) \leq 4$. *Moreover, every vertex of degree four of* T *is adjacent to a leaf.*

In [1] D. Barth, O. Baudon and J. Puech studied a family of trees each of them being homeomorphic to $K_{1,3}$ (they call them tripodes) and showed that determining if such a tree is avd can be done using a polynomial algorithm.

There is an interesting motivation for investigation of avd graphs. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are k different users, where i-th one needs n_i resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and a resource should be attributed to at most one user. So we have the problem of seeking a realization of the sequence $(n_1, \ldots, n_k)$ in this graph. Note also that one can find in [4] some references concerning arbitrarily edge decomposable graphs. The aim of this article is a characterization of avd trees with maximum degree at most four that have a very simple structure. Namely, we consider caterpillars or trees which are homeomorphic to a star $K_{1,q}$, where q is three or four.

2. Terminology and Results

In this paper, we deal with finite, simple and undirected graphs.

Let $T = (V, E)$ be a tree. A vertex $x \in V$ is called *primary* if $d(x) \geq 3$. A *leaf* is a vertex of degree one. A path P of T is an *arm* if one of its endvertices is a leaf in T, the other one is primary and all internal vertices of P have degree two in T. A tree T is called *primary* if it contains a primary vertex.

A graph T is a *star-like tree* if it is a tree homeomorphic to a star $K_{1,q}$ for some $q \geq 3$. Such a tree has one primary vertex (let us denote it by c) and q arms (let us denote them by A_i, $i \in \{1, \ldots, q\}$). For each A_i let α_i be the order of A_i. The structure of a star-like tree is (up to a isomorphism) determined by this sequence $(\alpha_1, \ldots, \alpha_q)$ of orders of its arms. Since the ordering of this sequence is not important, we will always assume that $2 \leq \alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_q$ and will denote the above defined star-like tree by $S(\alpha_1, \ldots, \alpha_q)$. Notice that an order of this star-like tree is equal to $1 + \sum_{i=1}^{q}(\alpha_i - 1)$.

A tree T is *a caterpillar* if the set of vertices of degree at least two induces a path. Let T be a caterpillar such that $\Delta(T) \leq 4$. Let us note that if there are two or more vertices of degree four in T, then the sequences $(2, 2, \ldots, 2)$ if n is even or $(1, 2, 2, \ldots, 2)$ if n is odd are not realizable in T, hence T is not avd. Clearly, these particular sequences are realizable in T if there is a perfect matching or a quasi-perfect matching in T. According to the above remark we will consider only caterpillars of maximum degree at most four having at most one vertex of degree four.

Let T be a caterpillar with $\Delta(T) = 3$ and let $\{y_1, \ldots, y_s\}$ be the set of primary vertices of T. We call T a *caterpillar with s single legs attached at $y_1, \ldots, y_s$*.

Similarly, if T is a caterpillar and $\{x, y_1, \ldots, y_s\}$ the set of primary vertices of T such that $d(x) = 4$ and $d(y_i) = 3$ for all $i \in \{1, \ldots, s\}$, then

T is called a *caterpillar with one double leg attached at* x *and* s *single legs attached at* $y_1, \ldots, y_s$. For simplicity of notation we say sometimes that we have a caterpillar with s single legs or a caterpillar with one double leg and s single legs. We present two examples of such caterpillars in Figure 1 and Figure 4.

Here and subsequently, we assume that every admissible sequence for a graph G is non-decreasing and we write d^λ for the sequence $(\underbrace{d, \ldots, d}_{\lambda})$ and $d^\lambda \cdot g^\mu$ for the sequence $(\underbrace{d, d, \ldots, d}_{\lambda}, \underbrace{g, g, \ldots, g}_{\mu})$, the concatenation of λ times d and μ times g. We will note $d \cdot g^\mu$ and $d^\lambda \cdot g$ instead of $d^1 \cdot g^\mu$ and $d^\lambda \cdot g^1$.

We denote by (a, b) the greatest common divisor of two positive integers a and b and we write $t(i, j)$ for the transposition of the elements i and j of the set $\{1, 2, \ldots, k\}$. Note that if $i = j$, then by transposition $t(i, j)$ we mean the identity.

Let T be a tree, and let (V_1, V_2) and (V_1', V_2') be two partitions of $V(T)$ such that each V_i and each V_i' induces a tree in T. We say that we can *transpose* V_1 and V_2 (into V_1' and V_2') if $|V_i'| = |V_i|$ $(i = 1, 2)$.

Let $P = y_1, \ldots, y_q$ be a subpath of a tree T and U, W two disjoint subsets of $V(T)$. We shall say that U and W are *neighbouring* in P if for some $j \in \{1, \ldots, q-1\}$, $y_j \in U$ and $y_{j+1} \in W$ or $y_j \in W$ and $y_{j+1} \in U$.

The first result characterizing avd star-like trees (i.e., caterpillars with one single leg) was found by D. Barth, O. Baudon and J. Puech [1] and, independently, by M. Horňák and M. Woźniak [3].

Proposition 2. *The star-like tree* $S(2, a, b)$*, with* $2 \leq a \leq b$ *is avd if and only if* $(a, b) = 1$*. Moreover, each admissible and non-realizable sequence in* $S(2, a, b)$ *is of the form* d^λ*, where* $a \equiv b \equiv 0 \pmod d$ *and* $d > 1$.

In [1] D. Barth, O. Baudon and J. Puech proved the following proposition. In the statement of this result the sequence $(3, a, b)$ is not assumed to be non-decreasing.

Proposition 3. *Each star-like tree* $S(2, 2, a, b)$*, with* $2 \leq a \leq b$ *is avd if and only if*

1^0 *the star-like tree* $S(3, a, b)$ *is avd;*

2^0 *a, b are odd;*

3^0 $a \neq 2 \pmod 3$ *or* $b \neq 2 \pmod 3$.

The next result due to D. Barth and H. Fournier [2] shows that the structure of avd caterpillars is not obvious.

Theorem 4. *For every $s \geq 1$ there exists an avd caterpillar with s single legs.*

The main results of this paper are Theorems 5 and 6 of Sections 3 and 4 which give a complete characterization of avd caterpillars with two single legs and avd star-like trees $S(3, a, b)$. In Section 4 we also give a necessary and sufficient condition for a star-like tree $S(2, 2, a, b)$ to be avd. Thus, we describe the family of avd caterpillars with four leaves. In Section 5 we describe an infinite family of avd caterpillars with one double and one single leg (Proposition 9).

3. Arbitrarily Vertex Decomposable Caterpillars with Two Single Legs

Every caterpillar T of order n with two single legs attached at x and y can be obtained by taking a path $P = x_1, \ldots, x_{n-2}$, where $x = x_i$ and $y = x_j$ $(i < j)$ are two internal vertices of P, adding two vertices u and v, and joining u to x and v to y (see Figure 1). For such a graph let us define $l_x(T) := i$, $r_x(T) := n - i$ and, analogously, $l_y(T) := j + 1$ and $r_y(T) := n - j - 1$.

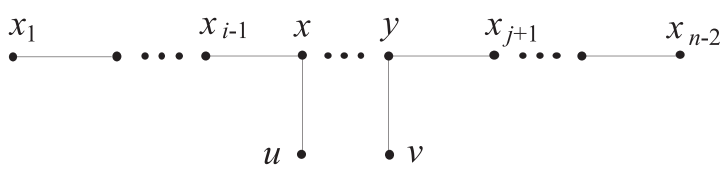

Figure 1. A caterpillar with two single legs.

Theorem 5. *Let $T = (V, E)$ be a caterpillar of order n with two single legs attached at x and y. Then T is avd if and only if the following conditions hold:*

1^0 $(l_x(T), r_x(T)) = 1$;

2^0 $(l_y(T), r_y(T)) = 1$;

3^0 $(l_x(T), r_y(T)) = 1$;

4^0 $(l_y(T), r_x(T)) < l_y(T) - l_x(T)$ *or* $n \equiv 1 \ (\mathrm{mod}(l_y(T), r_x(T)))$;
5^0 $n \neq \alpha l_x(T) + \beta l_y(T)$ *for any* $\alpha, \beta \in \mathbf{N}$;
6^0 $n \neq \alpha r_x(T) + \beta r_y(T)$ *for any* $\alpha, \beta \in \mathbf{N}$.

Proof. For abbreviation we write $l_x = l_x(T)$, $r_x = r_x(T)$, $l_y = l_y(T)$ and $r_y = r_y(T)$. Observe first that $n = l_x + r_x = l_y + r_y$ and there is no loss of generality in assuming $l_x \leq r_y$.

Necessity. Suppose that $(l_x, r_x) = d > 1$ ($(l_y, r_y) = d' > 1$, resp.). Then $n = \lambda \cdot d$ ($n = \lambda' \cdot d'$, resp.) for some $\lambda \in \mathbf{N}$ ($\lambda' \in \mathbf{N}$, resp.). It can be easily seen that the sequence d^λ ($d'^{\lambda'}$, resp.) is not realizable in T, so the conditions 1^0 and 2^0 are necessary for T to be avd.

Suppose now $l_x = \alpha \cdot d$, $r_y = \beta \cdot d$ for some integers $\alpha, \beta \geq 1$ and $d > 1$. Hence $n = (\alpha + \beta) \cdot d + r$ and, by 1^0, d does not divide r. Let us consider the sequence $r \cdot d^\lambda$ if $r \leq d$ or $d^\lambda \cdot r$ otherwise. Let S be a subtree of T of order r. It can be easily seen that the graph $T - S$ has a connected component C being a star-like tree $S(2, a, b)$ with $(a, b) = \mu d$ for some integer $\mu \geq 1$ or a path of length which is not divisible by d or else a caterpillar T' with two single legs attached at x and y such that d divides $(l_y(T'), r_y(T'))$ or $(l_x(T'), r_x(T'))$. Thus, using the previous argument or Proposition 2 we may deduce that such a sequence is not realizable in C and this implies the necessity of the condition 3^0.

Assume then $(l_y, r_x) = d \geq l_y - l_x \geq 2$ and n is not congruent to 1 modulo d. If $d = l_y - l_x$, then $l_x \equiv 0 \ (\mathrm{mod}\, d)$ and we can show as above that T is not avd. Assume $d > l_y - l_x$ and let λ and $r \in \{1, \ldots, d-1\}$ be two integers such that $l_x = \lambda d + r$. Thus, $r_x = \alpha d$, $l_y = \beta d$ for some integers α, β and $n = \lambda d + \alpha d + r$. Hence $r \geq 2$ and, because $l_y - l_x < d$, $\beta = \lambda + 1$. Consider now the sequence $\tau = r \cdot d^{\alpha + \lambda}$. Taking the graph $T - S$, where S is a subtree of T on r vertices and using a similar argument as in the previous situation we deduce that τ is not realizable in T, so the condition 4^0 is necessary for T to be avd.

Finally, if $n = \alpha l_x + \beta l_y$ for some $\alpha, \beta \in \mathbf{N}$ (or $n = \alpha r_x + \beta r_y$), then the sequence $l_x^\alpha \cdot l_y^\beta$ (or $r_y^\beta \cdot r_x^\alpha$, resp.) is not realizable in T and this implies the necessity of the conditions 5^0 and 6^0.

Sufficiency. Suppose the conditions 1^0–6^0 hold and let $\tau = (a_1, \ldots, a_k)$ be an admissible sequence for T. We first show that if $a_1 = 1$, then there exists a T-realization of τ. Indeed, consider a caterpillar $T' = T - u$ i.e., a caterpillar with one leg attached at y and an admissible sequence $\tau' =$

$(a_2, a_3, \ldots, a_k)$ for T'. Obviously, if τ' is a realizable sequence for T', then τ is realizable for T. Suppose then, that τ' is not realizable for T'. It follows from Proposition 2 that $(l_y - 1, r_y) = d$ for some integer $d > 1$ and $\tau' = (d, \ldots, d)$. Thus d divides r_y and, by 3^0, l_x is not divisible by d, so τ' is realizable in the tree $T'' = T - v$. It follows that $\tau = (1, d, \ldots, d)$ is realizable in T as claimed.

From now on we will assume that $a_1 \geq 2$, i.e., for every $i = 1, \ldots, k$, $a_i \geq 2$.

Observe that T is avd if and only if for any admissible sequence $\tau = (a_1, \ldots, a_k)$ for T there exists a permutation $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that for all $s \in \{1, \ldots, k\}$

$$(*) \qquad \sum_{i=1}^{s} a_{\sigma(i)} \notin \{l_x, l_y\}.$$

Let m be the minimum number $j \in \{1, .., k\}$ such that $a_1 + \ldots + a_j \geq l_x$. Thus, for $m > 1$ we get $a_1 + \ldots + a_{m-1} < l_x$.

Case 1. $a_1 + \ldots + a_m = l_x$. If $a_j = a_1$ for all $j \in \{1, \ldots, k\}$, then we have a contradiction with condition 1^0. Therefore, there exists $j_0 \geq m + 1$ such that $a_{j_0} > a_1$. We may assume that j_0 is minimal with this property. Let σ be the product of three transpositions: $t(1, m)$, $t(m+1, j_0)$ and $t(m, m+1)$ taken in this order. It can be easily seen that $a_{\sigma(1)} + \ldots + a_{\sigma(m)} > l_x$ and $a_{\sigma(1)} + \ldots + a_{\sigma(m-1)} = a_2 + \ldots + a_m < l_x$ for $m > 1$.

Assume that there exists $m' \geq m$ such that $a_{\sigma(1)} + \ldots + a_{\sigma(m')} = l_y$. Now, if $a_{\sigma(j)} = a_1$ for each $j \in \{m'+1, \ldots, k\}$ then $r_y \geq 2a_1$ $(k-1 > m')$, because $l_x \leq r_y$ and $(l_x, r_y) = 1$. So $j_0 = k$ and $a_i = a_1$ for each $i < k$. It follows that $l_x = ma_1$ and $l_y = (m'-1)a_1 + a_k$; consequently $r_y = n - l_y = \alpha a_1$ for some α which contradicts 3^0. Hence, we can also assume there exists $s \in \{m'+1, \ldots, k\}$ such that $a_{\sigma(s)} > a_1$.

Case 1.1. $m = m'$. Hence $a_{\sigma(m)} \geq l_y - l_x + 1$. If $a_{\sigma(j)} > a_{\sigma(m)}$ for some $j > m$ then we can take the permutation $t(m, m+1) \circ t(m+1, j) \circ \sigma$ satisfying $(*)$. Thus we may assume that if $j > m$ then $a_{\sigma(j)}$ can take only two values: a_1 and $a_{\sigma(m)}$. Moreover, by 5^0, we have $m \geq 2$. Set

$d = a_{\sigma(m)}$,

$r = \sum_{i=2}^{m-1} a_i$ for $m > 2$ and

$r = 0$ for $m = 2$.

Hence $l_x = a_1 + r + a_m$ and $l_y = r + a_m + d$.

Case 1.1.1. $d > a_m$. Suppose first $a_m > a_1$ and take the permutation $\sigma' = t(1, m+1) \circ \sigma$ (recall that $a_{\sigma(1)} = a_m$ and $a_{\sigma(m+1)} = a_1$). We have now $a_{\sigma'(1)} + \ldots + a_{\sigma'(m-1)} = a_1 + r < a_1 + r + a_m = l_x$, $l_y = r + a_m + d > a_{\sigma'(1)} + \ldots + a_{\sigma'(m)} = a_1 + r + d > l_x$ (because $a_m > a_1$ and $d > a_m$), $a_{\sigma'(1)} + \ldots + a_{\sigma'(m+1)} = a_1 + r + d + a_m = l_y + a_1 > l_y$, therefore σ' verifies $(*)$. Suppose then $a_1 = a_m$, i.e., $a_j = a_1$ for all $j \in \{1, \ldots, m\}$ and $l_x = \lambda a_1$ for some integer $\lambda \geq 2$. Therefore, by 3^0, there exists $i_0 \geq m+1$, $i_0 \neq j_0$, such that $a_{i_0} = d$. Consider now the permutation $\sigma' = t(m-1, i_0) \circ \sigma$. We have $a_{\sigma'(1)} + \ldots + a_{\sigma'(m)} = (\lambda - 2)a_1 + 2d > l_y = (\lambda - 1)a_1 + d$. Thus, if $(\lambda - 2)a_1 + d \neq l_x = \lambda a_1$, i.e., $d \neq 2a_1$, then σ' satisfies $(*)$. But if $d = 2a_1$, then r_y is divisible by a_1 and we get a contradiction with 3^0.

Case 1.1.2. $d = a_m$. By construction of our permutation σ, we get $a_j = d$, for all $j \geq m$, so $r_x = (k-m)d$ and $a_1 < d$. Instead of our permutation σ take another permutation ρ given by the following formula: $\rho(i) = k - i + 1$, $i = 1, 2, \ldots, k$. Clearly, $a_{\rho(i)} = a_m = d$ for $i = 1, \ldots, k-m$ and, since $l_y < r_x$, we obtain $\sum_{i=1}^{k-m} a_{\rho(i)} > l_y$. From 1^0, l_x is not divisible by d, therefore the condition $(*)$ does not hold for ρ if $l_y = \gamma d$ for some integer γ. But in this case there are three positive integers w, α', β' such that $(l_y, r_x) = wd \geq d > d - a_1 = l_y - l_x$ and $n = r_x + l_x = r_x + l_y - d + a_1 = \alpha' wd + \beta' wd - d + a_1 = (\alpha' + \beta' - 1)wd + (w-1)d + a_1 \neq 1 (\mathrm{mod}\, wd)$ (because $d > a_1 \geq 2$) and we obtain a contradiction with 4^0.

Case 1.2. $m < m'$. Suppose that there exists $s_0 \in \{m'+1, \ldots, k\}$ such that $a_{\sigma(s_0)} \neq a_{\sigma(m')}$. Without loss of generality we can assume that $s_0 = m'+1$ (if necessary, we can perform an appropriate transposition). Now taking the transposition $t(m', m'+1)$ we get a permutation that satisfies $(*)$. Assume then $a_{\sigma(s)} = a_{\sigma(m')}$ for all $s \in \{m'+1, \ldots, k\}$.

Now, if $m+1 < m'$ and for some $i \in \{m+1, m'-1\}$ we have $a_{\sigma(i)} \neq a_{\sigma(m')}$, then we can take the permutation $t(m', m'+1) \circ t(i, m') \circ \sigma$ that verifies $(*)$. Therefore, we can assume that $a_{\sigma(s)} = a_{\sigma(m')}$ for all $s \in \{m+1, \ldots, m'\}$, so $a_{\sigma(s)} = a_1$ for $s \in \{m+1, \ldots, k\}$ and $l_x = ma_1$, which is impossible by 3^0.

Case 2. $a_1 + \ldots + a_m > l_x$. We may assume that there exists $m' \geq m$ such that $a_1 + \ldots + a_{m'} = l_y$, because otherwise the identity permutation satisfies $(*)$. Now, since $a_i \geq a_{m'}$ for $i > m'$, it is enough to consider only the case where $a_i = a_{m'}$ for $i > m'$, i.e., $r_y = \alpha a_{m'}$ for some integer α. Using the same method as in Case 1.2 we see that if there is no permutation verifying $(*)$, then $a_i = a_{m'}$ for all $i > m$. Notice that if $a_{m+1} > a_m$ then the transposition $\sigma = t(m, m'+1)$ satisfies $(*)$. So assume $a_i = a_{m'}$ for all $i \geq m$. Hence $l_y < r_x < (k-m+1)a_{m'}$. Now take the permutation ρ defined as follows: $\rho(i) = k-i+1$, $i = 1, 2, \ldots, k$. Since $r_y = \alpha a_{m'}$, for some integer α, it follows by 3^0 and 2^0 that the condition $(*)$ holds for ρ and we are done. This finishes the proof of the theorem. ■

4. Arbitrarily Vertex Decomposable $S(3, a, b)$ and $S(2, 2, a, b)$

Theorem 6. *Let a, b, $3 \leq a \leq b$, be two integers and $T = S(3, a, b)$ a star-like tree with three arms. Then T is avd if and only if the following conditions hold:*

1^0 $(a, b) \leq 2$;
2^0 $(a+1, b) \leq 2$;
3^0 $(a, b+1) \leq 2$;
4^0 $(a+1, b+1) \leq 3$;
5^0 $n \neq \alpha \cdot a + \beta \cdot (a+1)$ *for* $\alpha, \beta \in \mathbf{N}$.

Proof. Let c be the primary vertex of degree three of T and A_1, A_2, A_3 its arms. The vertices of three arms will be denoted as follows:

$$V(A_1) = \{c, x, y\},$$
$$V(A_2) = \{x_1, \ldots, x_a = c\},$$
$$V(A_3) = \{x_a = c, x_{a+1}, \ldots, x_{a+b-1}\},$$

(see Figure 2).

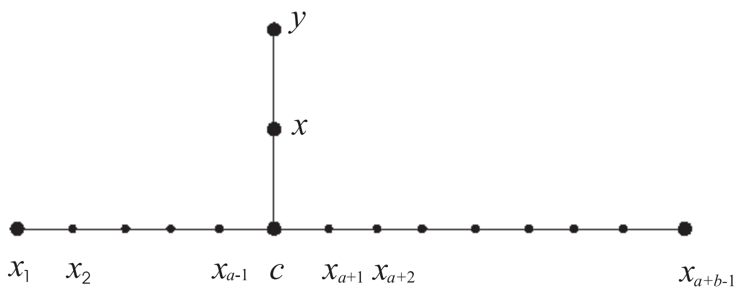

Figure 2. $S(3, a, b)$

Necessity. Suppose that $(a, b) = d > 2$. Then $n = \lambda \cdot d + 1$ for some integer $\lambda \geq 2$, and it can be easily seen that the sequence $d^{\lambda-1} \cdot (d + 1)$ is not realizable in T.

Let $(a + 1, b) = d \geq 3$ ($(a, b + 1) = d' \geq 3$). We have $n = \lambda \cdot d$, $\lambda \in \mathbf{N}, \lambda \geq 2$ ($n = \lambda' \cdot d'$, $\lambda' \in \mathbf{N}, \lambda' \geq 2$, resp.) and it is easy to check that the sequence d^{λ} ($(d')^{\lambda'}$, resp.) is not realizable in T.

Similarly, if $(a + 1, b + 1) = d > 3$, then $n = \lambda \cdot d - 1$, $\lambda \in \mathbf{N}$, so the sequence $(d - 1) \cdot d^{\lambda-1}$ is not realizable in T.

We now turn to the case $n = \alpha \cdot a + \beta \cdot (a + 1)$, $\alpha, \beta \in \mathbf{N}$. This implies that the sequence $a^{\alpha} \cdot (a + 1)^{\beta}$ is not realizable in T.

Sufficiency. Suppose that conditions 1^0–5^0 hold and let $\tau = (m_1, \ldots, m_k)$ be an admissible sequence for the tree T. Such a sequence is realizable in T if $m_k = 1$ (because it is ordered in a non-decreasing way), so we will assume $m_k > 1$. Let $\hat{\tau} = (n_1, \ldots, n_k)$ be a non-decreasing ordering of the sequence $(m_1, \ldots, m_{k-1}, m_k - 1)$, with $n_s = m_k - 1$. Consider the tree $\hat{T} = T - y$ which is isomorphic to the star-like tree $S(2, a, b)$. Clearly, the sequence $\hat{\tau}$ is admissible for the tree $\hat{T}$. Suppose $\hat{\tau}$ is not realizable in $\hat{T}$. Then, by *1*^{0} and Proposition 2, $(a, b) = 2$ and $\hat{\tau} = 2^k$. Hence $\tau = 2^{k-1} \cdot 3$ is obviously realizable in T. From now on we will assume that $\hat{\tau}$ is realizable in $\hat{T}$.

Furthermore, since τ is realizable in T if $m_i \in \{1, 2\}$ for some i, we can assume $n_j \geq 3$ for all $j \neq s$ and $n_s \geq 2$. Let $\hat{M} = (V_1, \ldots, V_s, \ldots, V_k)$ be a $\hat{T}$-realization of $\hat{\tau}$ such that $|V_i| = n_i$ for $i = 1, \ldots, k$, and V_p induces the primary tree of $\hat{T}$. Observe that if

$$(*) \qquad |V_p| = m_k - 1,$$

then the sequence M, obtained from $\hat{M}$ by replacing V_p by $V_p \cup \{y\}$, is a T-realization of τ. Therefore, we will assume that the condition $(*)$ does not hold (so $V_p \neq V_s$).

Case 1. $V_s \subset V(A_2)$. Suppose $x_{a-1} \in V_p$. Because $A_2 - V_p$ is a path in $\hat{T}$, we can arrange the sets V_i's covering this path in such a way that V_p and V_s are neighboring in A_2. Therefore, the subtree of T induced by $V_p \cup V_s \cup \{y\}$ can be covered by $(V_s \cup \{z\}, V_p \setminus \{z\} \cup \{y\})$, where z is the first vertex of V_p on A_2. Adding the remaining sets V_i we get a T-realization of τ. Thus, let us assume that V_p induces a path in $\hat{T}$ such that $V_p \setminus \{x\} \subset V(A_3)$ (see Figure 3).

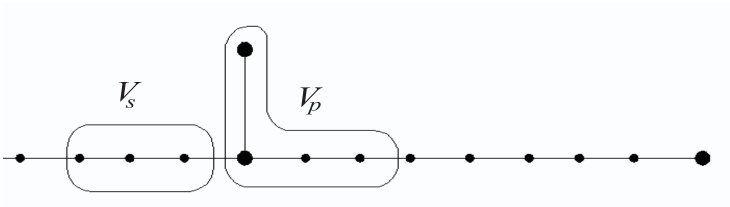

Figure 3. V_p and V_s are neighboring in A_2.

Suppose now $n_s > n_p$. Since $A_2 - V_p$ is a path in $\hat{T}$, we can assume without loss of generality that V_p and V_s are neighboring in A_2 (see Figure 3). Now the subtree of $\hat{T}$ induced by the set $V_s \cup V_p$ can be covered by (V'_s, V'_p), where V'_s induces a subpath of A_2 on n_p vertices, and V'_p a star-like tree on $n_s = m_k - 1$ vertices that contains c. Put $V'_i = V_i$ for $i \neq p, s$. It is easy to see that $(V'_1, \ldots, V'_k)$ is a $\hat{T}$-realization of $\hat{\tau}$ satisfying $(*)$ and we can easily obtain a T-realization of τ. Hence, by the choice of n_s, we can assume that $n_s = n_p - 1 = m_k - 1$. Then $n_i \leq n_p$ for all i's. If for some $i \neq s, p$, $V_i \subset A_2$ and $|V_i| \leq n_p - 2$, then, assuming that V_i and V_p are neighboring in A_2, we can cover $V_i \cup V_p$ by the pair (V'_i, V'_p), where V'_i induces a subpath of $A_3 - c$ on n_i vertices and V'_p induces a tree containing c. Applying the same argument as above we get a T-realization of τ. Hence, $n_p - 1 \leq |V_i| \leq n_p$ for all i's such that $V_i \subset V(A_2)$. Suppose that for some j, $V_j \subset V(A_3)$ and $|V_j| < n_p$. Now, because V_p induces a path in $\hat{T}$, we can place this V_j at the beginning of the path $xcx_{a+1} \ldots x_{a+b-1}$ and find a T-realization of τ as in the previous cases. Thus, $|V_i| = n_p$ for all i's such that $V_i \subset V(A_3)$.

Let $q := n_p$. We have now $a = \lambda q + \mu(q-1)$ and $b + 1 = \nu q$, for some integers $\lambda > 0$, $\mu \geq 0$ and $\nu > 0$. Moreover, the sequence τ is of the form

$$(q-1)^{\mu} \cdot q^{\lambda+\nu}.$$

If $\mu = 0$, then, by 3^0, $q \leq 2$, a contradiction with our assumption on n_p. Suppose $\mu = 1$. Then $a + 1 = (\lambda+1)q$, hence, by 4^0, $q = 3$, so $\tau = 2 \cdot 3^k$ and this sequence is clearly T-realizable. So consider the case $\mu \geq 2$. Because $a < b$, it follows that $\nu \geq 2$, so the sequence $(q-1)^2 \cdot q^{\nu-2}$ is realizable in $A_3 - c$ and the sequence $(q-1)^{\mu-2} \cdot q^{\lambda+2}$ is realizable in the tree induced by $A_2 \cup \{x, y\}$, hence τ is realizable in T.

Case 2. $V_s \subset V(A_3)$. As in Case 1 we assume that $x_{a+1} \notin V_p$, $q - 1 \leq |V_i| \leq q$ for $V_i \subset V(A_3)$ and $|V_j| = q$ for $V_j \subset V(A_2)$, where $q = n_p$. Now we can write $b = \lambda q + \mu(q-1)$ and $a+1 = \nu q$, for some integers $\lambda > 0$, $\mu \geq 0$ and $\nu > 0$. If $\mu = 0$, then, by 2^0, $q \leq 2$, and we get a contradiction with our assumption on n_p. For $\mu = 1$ we proceed as in Case 1 and show that τ is realizable in T. Suppose then $\mu \geq 2$. If $\nu \geq 2$ we proceed as in Case 1 and we show that τ is realizable in T. If $\nu = 1$ (the essential difference between Case 1 and Case 2), then $q = a+1$ and $n = (\lambda+1)(a+1) + \mu a$, a contradiction. This finishes the proof of the theorem. ■

Corollary 7. *Let a, b, $3 \leq a \leq b$ be two integers and $T = S(2,2,a,b)$ a star-like tree on n vertices. Then T is avd if and only if the following conditions hold:*

- $1'$ $(a,b) = 1$;
- $2'$ $(a+1,b) = 1$;
- $3'$ $(a,b+1) = 1$;
- $4'$ $(a+1,b+1) = 2$;
- $5'$ $n \neq \alpha \cdot a + \beta \cdot (a+1)$ *for* $\alpha, \beta \in \mathbf{N}$.

Proof. **Necessity.** Assume that T is avd. Hence, from Proposition 3, $S(3,a,b)$ is avd, a, b are odd, and $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$.

Therefore, the odd numbers a and b satisfy the conditions 1^0–5^0 of Theorem 6, hence also the conditions $1'$ and $5'$ of our theorem. Since a and b are odd, it follows by 1^0, 2^0 and 3^0 that $(a,b) = 1$, $(a+1,b) = 1$ and $(a,b+1) = 1$. So a and b satisfy $1'$, $2'$ and $3'$. By 4^0, $(a+1,b+1) \in \{2,3\}$ and since $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$, we have $(a+1,b+1) \neq 3$ and the condition $4'$ holds.

Sufficiency. If a and b verify the conditions $1'$–$5'$ then the conditions 1^0–5^0 of Theorem 6 are satisfied. Thus $S(3,a,b)$ is avd and, by $1'$–$3'$, a and b are odd.

Suppose that $a \equiv 2 \pmod 3$ and $b \equiv 2 \pmod 3$. Then $a+1 \equiv 0 \pmod 3$ and $b+1 \equiv 0 \pmod 3$, so $(a+1,b+1) \geq 3$, a contradiction. This implies that $a \neq 2 \pmod 3$ or $b \neq 2 \pmod 3$, and, by Proposition 3, T is avd. This finishes the proof. ■

Corollary 8. *There are infinitely many arbitrarily vertex decomposable star-like trees $S(3,a,b)$ and $S(2,2,a,b)$.*

Proof. Let $a \geq 5$ be a prime and $b = a + 2$. It can be easily seen that a and b satisfy the conditions $1'$–$5'$ (and also 1^0–5^0) for $n = 2a + 3$. ∎

5. Caterpillars with One Double and One Single Leg

Every caterpillar with one double and one single leg attached at x and y can be constructed in the following way. Take a path $P = x_1, \ldots, x_{n-3}$ where $x = x_a$ and $y = x_j$ $(a < j)$ are two internal vertices of P, add three vertices u,v and z and join u and v to x and v to y (see Figure 4).

Let $L_x = \{x_1, x_2, \ldots, x\}$, $R_x = \{x, x_{a+1}, \ldots, x_{n-3}\} \cup \{z\}$, $L_y = \{x_1, x_2, \ldots, y\} \cup \{u, v\}$, $R_y = \{y, x_{j+1}, \ldots, x_{n-3}\}$ and let $l_x = |L_x|$, $r_x = |R_x|$, $l_y = |L_y|$ and $r_y = |R_y|$.

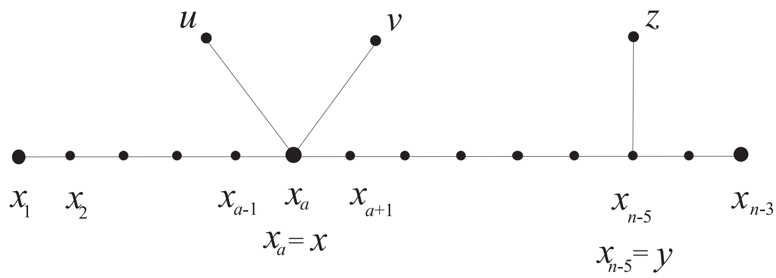

Figure 4. A caterpillar with one double and one single leg.

Proposition 9. *Let T be a caterpillar of order n with one double and one single leg attached at x and y resp. Let $a = l_x$ and $b = r_x$. If $a \equiv 1 \pmod 6$, $b \equiv 0 \pmod 3$, $7 \leq a < b$, $(a - 3, b) = 1$, $n - 1 \neq \alpha a$ $(\alpha \in \mathbf{N})$, $r_y = 3$ and a and b satisfy the conditions $1'$–$5'$ of Corollary 7, then T is avd.*

Proof. Let u and v denote two vertices of degree one adjacent to x and let z be the vertex of degree one adjacent to y (see Figure 4). It follows from our assumptions that $n = a + b + 1 \equiv 2 \pmod 3$. Let $\tau = (a_1, \ldots, a_k)$ be an admissible sequence for the tree T. We will show that it suffices to consider the case where $a_t \geq 2$ for all t. Indeed, the caterpillar $T' = T - v$ with two single legs satisfies $l'_x = a$, $r'_x = b$, $l'_y = a + b - 3 = n - 4 \equiv 1 \pmod 3$, $r'_y = 3$, so the conditions 1^0–3^0 of Theorem 5 are satisfied. We also have $(l'_y, r'_x) = (a + b - 3, b) = (a - 3, b) = 1 < l'_y - l'_x = b - 3$, so the condition

4^0 holds. Furthermore, if $\alpha l'_x + \beta l'_y = \alpha a + (n-4)\beta = n-1$, for some $\alpha, \beta \in \mathbf{N}$, then, since $a \geq 7$, we have $\beta = 0$, which is a contradiction. Assume $n - 1 = \alpha r'_x + \beta r'_y = \alpha b + 3\beta \equiv 0 \pmod 3$ $(\alpha, \beta \in \mathbf{N})$. But $n - 1 = a + b \equiv 1 \pmod 3$, and we get a contradiction. So also 5^0–6^0 of Theorem 5 hold. Now, if $a_1 = 1$, we can put $V_1 = \{v\}$ and the existence of T-realization of τ is obvious. Therefore, we may assume $a_t \geq 2$ for all t.

Notice that, by Corollary 7, the star-like tree $\hat{T} = S(2, 2, a, b)$ obtained by deleting the edge zy and adding zx_{n-3} is avd. Let $\hat{M} = (V_1, \ldots, V_k)$ be a $\hat{T}$-realization of τ such that V_p (V_s, resp.) induces a primary tree (a primary tree or a subpath, resp.) of $\hat{T}$ containing x (y, resp.). Observe that if V_s contains x_{n-4} (the vertex that follows y in the path $x_1, \ldots, x_{n-3}$) then τ is T-realizable. Indeed, if $z \in V_s$, then $\hat{M}$ is also a T-realization of τ and if $z \in V_{s'}$ for some $s' \neq s$, then $V_{s'} = \{x_{n-3}, z\}$ and replacing in $\hat{M}$ V_s and $V_{s'}$ by the sets $(V_s \setminus \{x_{n-4}\}) \cup \{z\}$ and $(V_{s'} \setminus \{z\}) \cup \{x_{n-4}\}$, we get a T-realization of τ.

Therefore, we shall assume that V_s does not contain x_{n-4}. Hence, because $a_r \geq 2$ for all r, there is g with $V_g = \{x_{n-4}, x_{n-3}, z\}$ (see Figure 5).

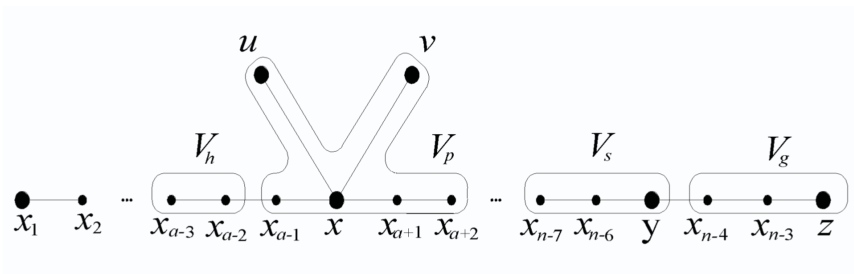

Figure 5. Case 1.1

Notice that for every r such that $V_r \subset R_x$ we have $|V_r| = 3$, for otherwise $g \neq r$ and assuming V_r and V_g are neighboring in R_x we could transpose V_r and V_g into V'_r and V'_g, in such a way that V'_r or V'_g contains the set $\{y, x_{n-4}\}$.

Now, since $|R_x| = b \equiv 0 \pmod 3$, we have $|V_p \cap (R_x \setminus \{x\})| \equiv 2 \pmod 3$, hence $|V_p \cap (R_x \setminus \{x\})| \geq 2$. Furthermore, since $a_r \geq 2$ for all r, we have $u, v \in V_p$ and $|V_p| \geq 5$.

Case 1. There is h such that $V_h \subset L_x$ and $|V_h| \neq 3$. Obviously, we may suppose that V_h and V_p are neighboring in L_x.

Case 1.1. $|V_h| \leq |V_p \cap (R_x \setminus \{x\})|$ (see Figure 5). Now we can transpose V_p and V_h into V'_p and V'_h with $V'_h \subset R_x$. Using the same argument as above, we easily find a T-realization $(V'_1, \ldots, V'_k)$ of τ.

Case 1.2. $|V_h| > |V_p \cap (R_x \setminus \{x\})|$. Let $b = 3q$ and $|V_h| = 3w + r$, where q, w, r are three integers such that $3 \leq q$, $1 \leq w$ and $r \in \{0, 1, 2\}$. We have by assumption $3 < |V_h| = 3w + r < a < b = 3q$, so setting:
$V'_h = \{x_{n-3w-r-1}, x_{n-3w-r}, \ldots, x_{n-3}, z\}$,
$V'_p = \{x_t, x_{t+1}, \ldots, x_a, \ldots, x_{a+2-r}\} \cup \{u, v\}$,
where $|V'_p| = |V_p| = a_p$, we can cover the remaining vertices of R_x by $q - w - 1 \geq 0$ sets of cardinality 3 and the existence of a T-realization $(V'_1, \ldots, V'_k)$ of τ is obvious.

Case 2. $\tau = (3, 3, \ldots, 3, |V_p|)$. Because $a - 1 \equiv 0 \pmod 3$ and $|V_p| > 3$, we can place the set of cardinality $|V_p|$ at the end of the path $x_1, x_2, \ldots, x_{n-3}, z$ and easily construct a realization of τ in T. ■

Theorem 10. *The number of avd caterpillars with one double and one single leg is infinite.*

Proof. Take a such that $b = a+2 = 3p$, where p is a prime number greater than five. Therefore, $a \equiv 1 \pmod 6$, $(b, a-3) = 1$, $n = 2a+3$, $n-1 = 2a+2$ and it is easy to check that the assumptions $1'$–$5'$ of Corollary 7 hold. By Proposition 9 our caterpillar is avd. ■

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