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# ARBITRARILY VERTEX DECOMPOSABLE CATERPILLARS WITH FOUR OR FIVE LEAVES 

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#### Abstract

A graph $G$ of order $n$ is called arbitrarily vertex decomposable if for each sequence $\left(a_{1}, \ldots, a_{k}\right)$ of positive integers such that $a_{1}+\ldots+a_{k}=n$ there exists a partition $\left(V_{1}, \ldots, V_{k}\right)$ of the vertex set of $G$ such that for each $i \in\{1, \ldots, k\}, V_{i}$ induces a connected subgraph of $G$ on $a_{i}$ vertices. D. Barth and H. Fournier showed that if a tree $T$ is arbitrarily vertex decomposable, then $T$ has maximum degree at most 4 . In this paper we give a complete characterization of arbitrarily vertex decomposable caterpillars with four leaves. We also describe two families of


arbitrarily vertex decomposable trees with maximum degree three or four.
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## 1. Introduction

Let $G=(V, E)$ be a graph of order $n$. A sequence $\tau=\left(a_{1}, \ldots, a_{k}\right)$ of positive integers is called admissible for $G$ if it adds up to $n$. If $\tau=\left(a_{1}, \ldots, a_{k}\right)$ is an admissible sequence for $G$ and there exists a partition $\left(V_{1}, \ldots, V_{k}\right)$ of the vertex set $V$ such that for each $i \in\{1, \ldots, k\},\left|V_{i}\right|=a_{i}$ and a subgraph induced by $V_{i}$ is connected then $\tau$ is called realizable in $G$ and the sequence $\left(V_{1}, \ldots, V_{k}\right)$ is said to be a G-realization of $\tau$ or a realization of $\tau$ in $G$. A graph $G$ is arbitrarily vertex decomposable (avd for short) if for each admissible sequence $\tau$ for $G$ there exists a $G$-realization of $\tau$.

The problem of deciding whether a given graph is arbitrarily vertex decomposable has been considered in several papers (see for example [1]-[4]). Generally, this problem is NP-complete [1] but we do not know if this problem is NP-complete when restricted to trees.

However, it is obvious that each path and each traceable graph is avd. The investigation of avd trees is motivated by the fact that a connected graph is avd if its spanning tree is avd. In [4] M. Horňák and M. Woźniak conjectured that if $T$ is a tree with maximum degree $\Delta(T)$ at least five, then $T$ is not avd. This conjecture was proved by D. Barth and H. Fournier [2].

Theorem 1. If a tree $T$ is arbitrarily vertex decomposable, then $\Delta(T) \leq 4$. Moreover, every vertex of degree four of $T$ is adjacent to a leaf.

In [1] D. Barth, O. Baudon and J. Puech studied a family of trees each of them being homeomorphic to $K_{1,3}$ (they call them tripodes) and showed that determining if such a tree is avd can be done using a polynomial algorithm.

There is an interesting motivation for investigation of avd graphs. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are $k$ different users, where $i$-th one needs $n_{i}$ resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and a resource should
be attributed to at most one user. So we have the problem of seeking a realization of the sequence $\left(n_{1}, \ldots, n_{k}\right)$ in this graph. Note also that one can find in [4] some references concerning arbitrarily edge decomposable graphs. The aim of this article is a characterization of avd trees with maximum degree at most four that have a very simple structure. Namely, we consider caterpillars or trees which are homeomorphic to a star $K_{1, q}$, where $q$ is three or four.

## 2. Terminology and Results

In this paper, we deal with finite, simple and undirected graphs.
Let $T=(V, E)$ be a tree. A vertex $x \in V$ is called primary if $d(x) \geq 3$. A leaf is a vertex of degree one. A path $P$ of $T$ is an arm if one of its endvertices is a leaf in $T$, the other one is primary and all internal vertices of $P$ have degree two in $T$. A tree $T$ is called primary if it contains a primary vertex.

A graph $T$ is a star-like tree if it is a tree homeomorphic to a star $K_{1, q}$ for some $q \geq 3$. Such a tree has one primary vertex (let us denote it by $c$ ) and $q$ arms (let us denote them by $A_{i}, i \in\{1, \ldots, q\}$ ). For each $A_{i}$ let $\alpha_{i}$ be the order of $A_{i}$. The structure of a star-like tree is (up to a isomorphism) determined by this sequence ( $\alpha_{1}, \ldots, \alpha_{q}$ ) of orders of its arms. Since the ordering of this sequence is not important, we will always assume that $2 \leq \alpha_{1} \leq \alpha_{2} \leq \ldots \leq \alpha_{q}$ and will denote the above defined star-like tree by $S\left(\alpha_{1}, \ldots, \alpha_{q}\right)$. Notice that an order of this star-like tree is equal to $1+\sum_{i=1}^{q}\left(\alpha_{i}-1\right)$.

A tree $T$ is a caterpillar if the set of vertices of degree at least two induces a path. Let $T$ be a caterpillar such that $\Delta(T) \leq 4$. Let us note that if there are two or more vertices of degree four in $T$, then the sequences $(2,2, \ldots, 2)$ if $n$ is even or $(1,2,2, \ldots, 2)$ if $n$ is odd are not realizable in $T$, hence $T$ is not avd. Clearly, these particular sequences are realizable in $T$ if there is a perfect matching or a quasi-perfect matching in $T$. According to the above remark we will consider only caterpillars of maximum degree at most four having at most one vertex of degree four.

Let $T$ be a caterpillar with $\Delta(T)=3$ and let $\left\{y_{1}, \ldots, y_{s}\right\}$ be the set of primary vertices of $T$. We call $T$ a caterpillar with $s$ single legs attached at $y_{1}, \ldots, y_{s}$.

Similarly, if $T$ is a caterpillar and $\left\{x, y_{1}, \ldots, y_{s}\right\}$ the set of primary vertices of $T$ such that $d(x)=4$ and $d\left(y_{i}\right)=3$ for all $i \in\{1, \ldots, s\}$, then
$T$ is called a caterpillar with one double leg attached at $x$ and $s$ single legs attached at $y_{1}, \ldots, y_{s}$. For simplicity of notation we say sometimes that we have a caterpillar with $s$ single legs or a caterpillar with one double leg and $s$ single legs. We present two examples of such caterpillars in Figure 1 and Figure 4.

Here and subsequently, we assume that every admissible sequence for a graph $G$ is non-decreasing and we write $d^{\lambda}$ for the sequence $(\underbrace{d, \ldots, d}_{\lambda})$ and $d^{\lambda} \cdot g^{\mu}$ for the sequence $(\underbrace{d, d, \ldots, d}_{\lambda}, \underbrace{g, g, \ldots, g}_{\mu})$, the concatenation of $\lambda$ times $d$ and $\mu$ times $g$. We will note $d \cdot g^{\mu}$ and $d^{\lambda} \cdot g$ instead of $d^{1} \cdot g^{\mu}$ and $d^{\lambda} \cdot g^{1}$.

We denote by $(a, b)$ the greatest common divisor of two positive integers $a$ and $b$ and we write $t(i, j)$ for the transposition of the elements $i$ and $j$ of the set $\{1,2, \ldots, k\}$. Note that if $i=j$, then by transposition $t(i, j)$ we mean the identity.

Let $T$ be a tree, and let $\left(V_{1}, V_{2}\right)$ and $\left(V_{1}^{\prime}, V_{2}^{\prime}\right)$ be two partitions of $V(T)$ such that each $V_{i}$ and each $V_{i}^{\prime}$ induces a tree in $T$. We say that we can transpose $V_{1}$ and $V_{2}$ (into $V_{1}^{\prime}$ and $V_{2}^{\prime}$ ) if $\left|V_{i}^{\prime}\right|=\left|V_{i}\right|(i=1,2)$.

Let $P=y_{1}, \ldots, y_{q}$ be a subpath of a tree $T$ and $U, W$ two disjoint subsets of $V(T)$. We shall say that $U$ and $W$ are neighbouring in $P$ if for some $j \in\{1, \ldots, q-1\}, y_{j} \in U$ and $y_{j+1} \in W$ or $y_{j} \in W$ and $y_{j+1} \in U$.

The first result characterizing avd star-like trees (i.e., caterpillars with one single leg) was found by D. Barth, O. Baudon and J. Puech [1] and, independently, by M. Horňák and M. Woźniak [3].

Proposition 2. The star-like tree $S(2, a, b)$, with $2 \leq a \leq b$ is avd if and only if $(a, b)=1$. Moreover, each admissible and non-realizable sequence in $S(2, a, b)$ is of the form $d^{\lambda}$, where $a \equiv b \equiv 0(\bmod d)$ and $d>1$.

In [1] D. Barth, O. Baudon and J. Puech proved the following proposition. In the statement of this result the sequence $(3, a, b)$ is not assumed to be non-decreasing.

Proposition 3. Each star-like tree $S(2,2, a, b)$, with $2 \leq a \leq b$ is avd if and only if
$1^{0}$ the star-like tree $S(3, a, b)$ is avd;
$2^{0}$ a,b are odd;
$3^{0} a \neq 2(\bmod 3)$ or $b \neq 2(\bmod 3)$.

The next result due to D. Barth and H. Fournier [2] shows that the structure of avd caterpillars is not obvious.

Theorem 4. For every $s \geq 1$ there exists an avd caterpillar with $s$ single legs.

The main results of this paper are Theorems 5 and 6 of Sections 3 and 4 which give a complete characterization of avd caterpillars with two single legs and avd star-like trees $S(3, a, b)$. In Section 4 we also give a necessary and sufficient condition for a star-like tree $S(2,2, a, b)$ to be avd. Thus, we describe the family of avd caterpillars with four leaves. In Section 5 we describe an infinite family of avd caterpillars with one double and one single leg (Proposition 9).

## 3. Arbitrarily Vertex Decomposable Caterpillars with Two Single Legs

Every caterpillar $T$ of order $n$ with two single legs attached at $x$ and $y$ can be obtained by taking a path $P=x_{1}, \ldots, x_{n-2}$, where $x=x_{i}$ and $y=x_{j}(i<j)$ are two internal vertices of $P$, adding two vertices $u$ and $v$, and joining $u$ to $x$ and $v$ to $y$ (see Figure 1). For such a graph let us define $l_{x}(T):=i$, $r_{x}(T):=n-i$ and, analogously, $l_{y}(T):=j+1$ and $r_{y}(T):=n-j-1$.


Figure 1. A caterpillar with two single legs.
Theorem 5. Let $T=(V, E)$ be a caterpillar of order $n$ with two single legs attached at $x$ and $y$. Then $T$ is avd if and only if the following conditions hold:
$1^{0}\left(l_{x}(T), r_{x}(T)\right)=1 ;$
$2^{0}\left(l_{y}(T), r_{y}(T)\right)=1 ;$
$3^{0}\left(l_{x}(T), r_{y}(T)\right)=1 ;$

$$
\begin{aligned}
& 4^{0} \quad\left(l_{y}(T), r_{x}(T)\right)<l_{y}(T)-l_{x}(T) \text { or } n \equiv 1\left(\bmod \left(l_{y}(T), r_{x}(T)\right)\right) \\
& 5^{0} \quad n \neq \alpha l_{x}(T)+\beta l_{y}(T) \text { for any } \alpha, \beta \in \mathbf{N} \\
& 6^{0} \quad n \neq \alpha r_{x}(T)+\beta r_{y}(T) \text { for any } \alpha, \beta \in \mathbf{N}
\end{aligned}
$$

Proof. For abbreviation we write $l_{x}=l_{x}(T), r_{x}=r_{x}(T), l_{y}=l_{y}(T)$ and $r_{y}=r_{y}(T)$. Observe first that $n=l_{x}+r_{x}=l_{y}+r_{y}$ and there is no loss of generality in assuming $l_{x} \leq r_{y}$.

Necessity. Suppose that $\left(l_{x}, r_{x}\right)=d>1\left(\left(l_{y}, r_{y}\right)=d^{\prime}>1\right.$, resp. $)$. Then $n=\lambda \cdot d$ ( $n=\lambda^{\prime} \cdot d^{\prime}$, resp.) for some $\lambda \in \mathbf{N}\left(\lambda^{\prime} \in \mathbf{N}\right.$, resp.). It can be easily seen that the sequence $d^{\lambda}$ ( $d^{\lambda^{\prime}}$, resp.) is not realizable in $T$, so the conditions $1^{0}$ and $2^{0}$ are necessary for $T$ to be avd.

Suppose now $l_{x}=\alpha \cdot d, r_{y}=\beta \cdot d$ for some integers $\alpha, \beta \geq 1$ and $d>1$. Hence $n=(\alpha+\beta) \cdot d+r$ and, by $1^{0}, d$ does not divide $r$. Let us consider the sequence $r \cdot d^{\lambda}$ if $r \leq d$ or $d^{\lambda} \cdot r$ otherwise. Let $S$ be a subtree of $T$ of order $r$. It can be easily seen that the graph $T-S$ has a connected component $C$ being a star-like tree $S(2, a, b)$ with $(a, b)=\mu d$ for some integer $\mu \geq 1$ or a path of length which is not divisible by $d$ or else a caterpillar $T^{\prime}$ with two single legs attached at $x$ and $y$ such that $d$ divides $\left(l_{y}\left(T^{\prime}\right), r_{y}\left(T^{\prime}\right)\right)$ or $\left(l_{x}\left(T^{\prime}\right), r_{x}\left(T^{\prime}\right)\right)$. Thus, using the previous argument or Proposition 2 we may deduce that such a sequence is not realizable in $C$ and this implies the necessity of the condition $3^{0}$.

Assume then $\left(l_{y}, r_{x}\right)=d \geq l_{y}-l_{x} \geq 2$ and $n$ is not congruent to 1 modulo $d$. If $d=l_{y}-l_{x}$, then $l_{x} \equiv 0(\bmod d)$ and we can show as above that $T$ is not avd. Assume $d>l_{y}-l_{x}$ and let $\lambda$ and $r \in\{1, \ldots, d-1\}$ be two integers such that $l_{x}=\lambda d+r$. Thus, $r_{x}=\alpha d, l_{y}=\beta d$ for some integers $\alpha$, $\beta$ and $n=\lambda d+\alpha d+r$. Hence $r \geq 2$ and, because $l_{y}-l_{x}<d, \beta=\lambda+1$. Consider now the sequence $\tau=r \cdot d^{\alpha+\lambda}$. Taking the graph $T-S$, where $S$ is a subtree of $T$ on $r$ vertices and using a similar argument as in the previous situation we deduce that $\tau$ is not realizable in $T$, so the condition $4^{0}$ is necessary for $T$ to be avd.

Finally, if $n=\alpha l_{x}+\beta l_{y}$ for some $\alpha, \beta \in \mathbf{N}$ (or $n=\alpha r_{x}+\beta r_{y}$ ), then the sequence $l_{x}^{\alpha} \cdot l_{y}^{\beta}$ (or $r_{y}^{\beta} \cdot r_{x}^{\alpha}$, resp.) is not realizable in $T$ and this implies the necessity of the conditions $5^{0}$ and $6^{0}$.

Sufficiency. Suppose the conditions $1^{0}-6^{0}$ hold and let $\tau=\left(a_{1}, \ldots, a_{k}\right)$ be an admissible sequence for $T$. We first show that if $a_{1}=1$, then there exists a $T$-realization of $\tau$. Indeed, consider a caterpillar $T^{\prime}=T-u$ i.e., a caterpillar with one leg attached at $y$ and an admissible sequence $\tau^{\prime}=$
$\left(a_{2}, a_{3}, \ldots, a_{k}\right)$ for $T^{\prime}$. Obviously, if $\tau^{\prime}$ is a realizable sequence for $T^{\prime}$, then $\tau$ is realizable for $T$. Suppose then, that $\tau^{\prime}$ is not realizable for $T^{\prime}$. It follows from Proposition 2 that $\left(l_{y}-1, r_{y}\right)=d$ for some integer $d>1$ and $\tau^{\prime}=(d, \ldots, d)$. Thus $d$ divides $r_{y}$ and, by $3^{0}, l_{x}$ is not divisible by $d$, so $\tau^{\prime}$ is realizable in the tree $T^{\prime \prime}=T-v$. It follows that $\tau=(1, d, \ldots, d)$ is realizable in $T$ as claimed.

From now on we will assume that $a_{1} \geq 2$, i.e., for every $i=1, \ldots, k$, $a_{i} \geq 2$.

Observe that $T$ is avd if and only if for any admissible sequence $\tau=$ $\left(a_{1}, \ldots, a_{k}\right)$ for $T$ there exists a permutation $\sigma:\{1, \ldots, k\} \rightarrow\{1, \ldots, k\}$ such that for all $s \in\{1, \ldots, k\}$

$$
\begin{equation*}
\sum_{i=1}^{s} a_{\sigma(i)} \notin\left\{l_{x}, l_{y}\right\} . \tag{*}
\end{equation*}
$$

Let $m$ be the minimum number $j \in\{1, . ., k\}$ such that $a_{1}+\ldots+a_{j} \geq l_{x}$. Thus, for $m>1$ we get $a_{1}+\ldots+a_{m-1}<l_{x}$.

Case 1. $a_{1}+\ldots+a_{m}=l_{x}$. If $a_{j}=a_{1}$ for all $j \in\{1, \ldots, k\}$, then we have a contradiction with condition $1^{0}$. Therefore, there exists $j_{0} \geq m+1$ such that $a_{j_{0}}>a_{1}$. We may assume that $j_{0}$ is minimal with this property. Let $\sigma$ be the product of three transpositions: $t(1, m), t\left(m+1, j_{0}\right)$ and $t(m, m+1)$ taken in this order. It can be easily seen that $a_{\sigma(1)}+\ldots+a_{\sigma(m)}>l_{x}$ and $a_{\sigma(1)}+\ldots+a_{\sigma(m-1)}=a_{2}+\ldots+a_{m}<l_{x}$ for $m>1$.

Assume that there exists $m^{\prime} \geq m$ such that $a_{\sigma(1)}+\ldots+a_{\sigma\left(m^{\prime}\right)}=l_{y}$. Now, if $a_{\sigma(j)}=a_{1}$ for each $j \in\left\{m^{\prime}+1, \ldots, k\right\}$ then $r_{y} \geq 2 a_{1}\left(k-1>m^{\prime}\right)$, because $l_{x} \leq r_{y}$ and $\left(l_{x}, r_{y}\right)=1$. So $j_{0}=k$ and $a_{i}=a_{1}$ for each $i<k$. It follows that $l_{x}=m a_{1}$ and $l_{y}=\left(m^{\prime}-1\right) a_{1}+a_{k}$; consequently $r_{y}=n-l_{y}=\alpha a_{1}$ for some $\alpha$ which contradicts $3^{0}$. Hence, we can also assume there exists $s \in\left\{m^{\prime}+1, \ldots, k\right\}$ such that $a_{\sigma(s)}>a_{1}$.

Case 1.1. $m=m^{\prime}$. Hence $a_{\sigma(m)} \geq l_{y}-l_{x}+1$. If $a_{\sigma(j)}>a_{\sigma(m)}$ for some $j>m$ then we can take the permutation $t(m, m+1) \circ t(m+1, j) \circ \sigma$ satisfying $(*)$. Thus we may assume that if $j>m$ then $a_{\sigma(j)}$ can take only two values: $a_{1}$ and $a_{\sigma(m)}$. Moreover, by $5^{0}$, we have $m \geq 2$. Set

$$
\begin{aligned}
& d=a_{\sigma(m)}, \\
& r=\sum_{i=2}^{m-1} a_{i} \text { for } m>2 \text { and } \\
& r=0 \text { for } m=2 .
\end{aligned}
$$

Hence $l_{x}=a_{1}+r+a_{m}$ and $l_{y}=r+a_{m}+d$.

Case 1.1.1. $d>a_{m}$. Suppose first $a_{m}>a_{1}$ and take the permutation $\sigma^{\prime}=t(1, m+1) \circ \sigma$ (recall that $a_{\sigma(1)}=a_{m}$ and $\left.a_{\sigma(m+1)}=a_{1}\right)$. We have now $a_{\sigma^{\prime}(1)}+\ldots+a_{\sigma^{\prime}(m-1)}=a_{1}+r<a_{1}+r+a_{m}=l_{x}, l_{y}=r+a_{m}+d>$ $a_{\sigma^{\prime}(1)}+\ldots+a_{\sigma^{\prime}(m)}=a_{1}+r+d>l_{x}$ (because $a_{m}>a_{1}$ and $d>a_{m}$ ), $a_{\sigma^{\prime}(1)}+\ldots+a_{\sigma^{\prime}(m+1)}=a_{1}+r+d+a_{m}=l_{y}+a_{1}>l_{y}$, therefore $\sigma^{\prime}$ verifies (*). Suppose then $a_{1}=a_{m}$, i.e., $a_{j}=a_{1}$ for all $j \in\{1, \ldots, m\}$ and $l_{x}=\lambda a_{1}$ for some integer $\lambda \geq 2$. Therefore, by $3^{0}$, there exists $i_{0} \geq m+1, i_{0} \neq j_{0}$, such that $a_{i_{0}}=d$. Consider now the permutation $\sigma^{\prime}=t\left(m-1, i_{0}\right) \circ \sigma$. We have $a_{\sigma^{\prime}(1)}+\ldots+a_{\sigma^{\prime}(m)}=(\lambda-2) a_{1}+2 d>l_{y}=(\lambda-1) a_{1}+d$. Thus, if $(\lambda-2) a_{1}+d \neq l_{x}=\lambda a_{1}$, i.e., $d \neq 2 a_{1}$, then $\sigma^{\prime}$ satisfies (*). But if $d=2 a_{1}$, then $r_{y}$ is divisible by $a_{1}$ and we get a contradiction with $3^{0}$.

Case 1.1.2. $d=a_{m}$. By construction of our permutation $\sigma$, we get $a_{j}=d$, for all $j \geq m$, so $r_{x}=(k-m) d$ and $a_{1}<d$. Instead of our permutation $\sigma$ take another permutation $\rho$ given by the following formula: $\rho(i)=k-i+1, i=1,2, \ldots, k$. Clearly, $a_{\rho(i)}=a_{m}=d$ for $i=1, \ldots, k-m$ and, since $l_{y}<r_{x}$, we obtain $\sum_{i=1}^{k-m} a_{\rho(i)}>l_{y}$. From $1^{0}, l_{x}$ is not divisible by $d$, therefore the condition (*) does not hold for $\rho$ if $l_{y}=\gamma d$ for some integer $\gamma$. But in this case there are three positive integers $w, \alpha^{\prime}, \beta^{\prime}$ such that $\left(l_{y}, r_{x}\right)=w d \geq d>d-a_{1}=l_{y}-l_{x}$ and $n=r_{x}+l_{x}=r_{x}+l_{y}-d+a_{1}=$ $\alpha^{\prime} w d+\beta^{\prime} w d-d+a_{1}=\left(\alpha^{\prime}+\beta^{\prime}-1\right) w d+(w-1) d+a_{1} \neq 1(\bmod w d)$ (because $d>a_{1} \geq 2$ ) and we obtain a contradiction with $4^{0}$.

Case 1.2. $m<m^{\prime}$. Suppose that there exists $s_{0} \in\left\{m^{\prime}+1, \ldots, k\right\}$ such that $a_{\sigma\left(s_{0}\right)} \neq a_{\sigma\left(m^{\prime}\right)}$. Without loss of generality we can assume that $s_{0}=m^{\prime}+1$ (if necessary, we can perform an appropriate transposition). Now taking the transposition $t\left(m^{\prime}, m^{\prime}+1\right)$ we get a permutation that satisfies $(*)$. Assume then $a_{\sigma(s)}=a_{\sigma\left(m^{\prime}\right)}$ for all $s \in\left\{m^{\prime}+1, \ldots, k\right\}$.

Now, if $m+1<m^{\prime}$ and for some $i \in\left\{m+1, m^{\prime}-1\right\}$ we have $a_{\sigma(i)} \neq$ $a_{\sigma\left(m^{\prime}\right)}$, then we can take the permutation $t\left(m^{\prime}, m^{\prime}+1\right) \circ t\left(i, m^{\prime}\right) \circ \sigma$ that verifies $(*)$. Therefore, we can assume that $a_{\sigma(s)}=a_{\sigma\left(m^{\prime}\right)}$ for all $s \in\{m+$ $\left.1, \ldots, m^{\prime}\right\}$, so $a_{\sigma(s)}=a_{1}$ for $s \in\{m+1, \ldots, k\}$ and $l_{x}=m a_{1}$, which is impossible by $3^{0}$.

Case 2. $a_{1}+\ldots+a_{m}>l_{x}$. We may assume that there exists $m^{\prime} \geq m$ such that $a_{1}+\ldots+a_{m^{\prime}}=l_{y}$, because otherwise the identity permutation satisfies (*). Now, since $a_{i} \geq a_{m^{\prime}}$ for $i>m^{\prime}$, it is enough to consider only the case where $a_{i}=a_{m^{\prime}}$ for $i>m^{\prime}$, i.e., $r_{y}=\alpha a_{m^{\prime}}$ for some integer $\alpha$. Using the same method as in Case 1.2 we see that if there is no permutation
verifying $(*)$, then $a_{i}=a_{m^{\prime}}$ for all $i>m$. Notice that if $a_{m+1}>a_{m}$ then the transposition $\sigma=t\left(m, m^{\prime}+1\right)$ satisfies ( $*$ ). So assume $a_{i}=a_{m^{\prime}}$ for all $i \geq m$. Hence $l_{y}<r_{x}<(k-m+1) a_{m^{\prime}}$. Now take the permutation $\rho$ defined as follows: $\rho(i)=k-i+1, i=1,2, \ldots, k$. Since $r_{y}=\alpha a_{m^{\prime}}$, for some integer $\alpha$, it follows by $3^{0}$ and $2^{0}$ that the condition (*) holds for $\rho$ and we are done. This finishes the proof of the theorem.

## 4. Arbitrarily Vertex Decomposable $S(3, a, b)$ and $S(2,2, a, b)$

Theorem 6. Let $a, b, 3 \leq a \leq b$, be two integers and $T=S(3, a, b)$ a starlike tree with three arms. Then $T$ is avd if and only if the following conditions hold:

$$
\begin{aligned}
& 1^{0} \quad(a, b) \leq 2 \\
& 2^{0} \quad(a+1, b) \leq 2 \\
& 3^{0} \quad(a, b+1) \leq 2 \\
& 4^{0} \quad(a+1, b+1) \leq 3 ; \\
& 5^{0} \quad n \neq \alpha \cdot a+\beta \cdot(a+1) \text { for } \alpha, \beta \in \mathbf{N} .
\end{aligned}
$$

Proof. Let $c$ be the primary vertex of degree three of $T$ and $A_{1}, A_{2}, A_{3}$ its arms. The vertices of three arms will be denoted as follows:

$$
\begin{aligned}
& V\left(A_{1}\right)=\{c, x, y\} \\
& V\left(A_{2}\right)=\left\{x_{1}, \ldots, x_{a}=c\right\}, \\
& V\left(A_{3}\right)=\left\{x_{a}=c, x_{a+1}, \ldots, x_{a+b-1}\right\}
\end{aligned}
$$

(see Figure 2).


Figure 2. $S(3, a, b)$

Necessity. Suppose that $(a, b)=d>2$. Then $n=\lambda \cdot d+1$ for some integer $\lambda \geq 2$, and it can be easily seen that the sequence $d^{\lambda-1} \cdot(d+1)$ is not realizable in $T$.

Let $(a+1, b)=d \geq 3\left((a, b+1)=d^{\prime} \geq 3\right)$. We have $n=\lambda \cdot d$, $\lambda \in \mathbf{N}, \lambda \geq 2\left(n=\lambda^{\prime} \cdot d^{\prime}, \lambda^{\prime} \in \mathbf{N}, \lambda^{\prime} \geq 2\right.$, resp. $)$ and it is easy to check that the sequence $d^{\lambda}\left(\left(d^{\prime}\right)^{\lambda^{\prime}}\right.$, resp.) is not realizable in $T$.

Similarly, if $(a+1, b+1)=d>3$, then $n=\lambda \cdot d-1, \lambda \in \mathbf{N}$, so the sequence $(d-1) \cdot d^{\lambda-1}$ is not realizable in $T$.

We now turn to the case $n=\alpha \cdot a+\beta \cdot(a+1), \alpha, \beta \in \mathbf{N}$. This implies that the sequence $a^{\alpha} \cdot(a+1)^{\beta}$ is not realizable in $T$.

Sufficiency. Suppose that conditions $1^{0}-5^{0}$ hold and let $\tau=\left(m_{1}, \ldots, m_{k}\right)$ be an admissible sequence for the tree $T$. Such a sequence is realizable in $T$ if $m_{k}=1$ (because it is ordered in a non-decreasing way), so we will assume $m_{k}>1$. Let $\hat{\tau}=\left(n_{1}, \ldots, n_{k}\right)$ be a non-decreasing ordering of the sequence $\left(m_{1}, \ldots, m_{k-1}, m_{k}-1\right)$, with $n_{s}=m_{k}-1$. Consider the tree $\hat{T}=T-y$ which is isomorphic to the star-like tree $S(2, a, b)$. Clearly, the sequence $\hat{\tau}$ is admissible for the tree $\hat{T}$. Suppose $\hat{\tau}$ is not realizable in $\hat{T}$. Then, by $1^{0}$ and Proposition $2,(a, b)=2$ and $\hat{\tau}=2^{k}$. Hence $\tau=2^{k-1} \cdot 3$ is obviously realizable in $T$. From now on we will assume that $\hat{\tau}$ is realizable in $\hat{T}$.

Furthermore, since $\tau$ is realizable in $T$ if $m_{i} \in\{1,2\}$ for some $i$, we can assume $n_{j} \geq 3$ for all $j \neq s$ and $n_{s} \geq 2$. Let $\hat{M}=\left(V_{1}, \ldots, V_{s}, \ldots, V_{k}\right)$ be a $\hat{T}$-realization of $\hat{\tau}$ such that $\left|V_{i}\right|=n_{i}$ for $i=1, \ldots, k$, and $V_{p}$ induces the primary tree of $\hat{T}$. Observe that if

$$
\begin{equation*}
\left|V_{p}\right|=m_{k}-1 \tag{*}
\end{equation*}
$$

then the sequence $M$, obtained from $\hat{M}$ by replacing $V_{p}$ by $V_{p} \cup\{y\}$, is a $T$-realization of $\tau$. Therefore, we will assume that the condition (*) does not hold (so $V_{p} \neq V_{s}$ ).

Case 1. $V_{s} \subset V\left(A_{2}\right)$. Suppose $x_{a-1} \in V_{p}$. Because $A_{2}-V_{p}$ is a path in $\hat{T}$, we can arrange the sets $V_{i}$ 's covering this path in such a way that $V_{p}$ and $V_{s}$ are neighboring in $A_{2}$. Therefore, the subtree of $T$ induced by $V_{p} \cup V_{s} \cup\{y\}$ can be covered by $\left(V_{s} \cup\{z\}, V_{p} \backslash\{z\} \cup\{y\}\right)$, where $z$ is the first vertex of $V_{p}$ on $A_{2}$. Adding the remaining sets $V_{i}$ we get a $T$-realization of $\tau$. Thus, let us assume that $V_{p}$ induces a path in $\hat{T}$ such that $V_{p} \backslash\{x\} \subset V\left(A_{3}\right)$ (see Figure 3).


Figure 3. $V_{p}$ and $V_{s}$ are neighboring in $A_{2}$.
Suppose now $n_{s}>n_{p}$. Since $A_{2}-V_{p}$ is a path in $\hat{T}$, we can assume without loss of generality that $V_{p}$ and $V_{s}$ are neighboring in $A_{2}$ (see Figure 3). Now the subtree of $\hat{T}$ induced by the set $V_{s} \cup V_{p}$ can be covered by $\left(V_{s}^{\prime}, V_{p}^{\prime}\right)$, where $V_{s}^{\prime}$ induces a subpath of $A_{2}$ on $n_{p}$ vertices, and $V_{p}^{\prime}$ a star-like tree on $n_{s}=m_{k}-1$ vertices that contains $c$. Put $V_{i}^{\prime}=V_{i}$ for $i \neq p, s$. It is easy to see that $\left(V_{1}^{\prime}, \ldots, V_{k}^{\prime}\right)$ is a $\hat{T}$-realization of $\hat{\tau}$ satisfying $(*)$ and we can easily obtain a $T$-realization of $\tau$. Hence, by the choice of $n_{s}$, we can assume that $n_{s}=n_{p}-1=m_{k}-1$. Then $n_{i} \leq n_{p}$ for all $i$ 's. If for some $i \neq s, p, V_{i} \subset A_{2}$ and $\left|V_{i}\right| \leq n_{p}-2$, then, assuming that $V_{i}$ and $V_{p}$ are neighboring in $A_{2}$, we can cover $V_{i} \cup V_{p}$ by the pair $\left(V_{i}^{\prime}, V_{p}^{\prime}\right)$, where $V_{i}^{\prime}$ induces a subpath of $A_{3}-c$ on $n_{i}$ vertices and $V_{p}^{\prime}$ induces a tree containing $c$. Applying the same argument as above we get a $T$-realization of $\tau$. Hence, $n_{p}-1 \leq\left|V_{i}\right| \leq n_{p}$ for all $i$ 's such that $V_{i} \subset V\left(A_{2}\right)$. Suppose that for some $j, V_{j} \subset V\left(A_{3}\right)$ and $\left|V_{j}\right|<n_{p}$. Now, because $V_{p}$ induces a path in $\hat{T}$, we can place this $V_{j}$ at the beginning of the path $x c x_{a+1} \ldots x_{a+b-1}$ and find a $T$-realization of $\tau$ as in the previous cases. Thus, $\left|V_{i}\right|=n_{p}$ for all $i$ 's such that $V_{i} \subset V\left(A_{3}\right)$.

Let $q:=n_{p}$. We have now $a=\lambda q+\mu(q-1)$ and $b+1=\nu q$, for some integers $\lambda>0, \mu \geq 0$ and $\nu>0$. Moreover, the sequence $\tau$ is of the form

$$
(q-1)^{\mu} \cdot q^{\lambda+\nu}
$$

If $\mu=0$, then, by $3^{0}, q \leq 2$, a contradiction with our assumption on $n_{p}$. Suppose $\mu=1$. Then $a+1=(\lambda+1) q$, hence, by $4^{0}, q=3$, so $\tau=2 \cdot 3^{k}$ and this sequence is clearly $T$-realizable. So consider the case $\mu \geq 2$. Because $a<b$, it follows that $\nu \geq 2$, so the sequence $(q-1)^{2} \cdot q^{\nu-2}$ is realizable in $A_{3}-c$ and the sequence $(q-1)^{\mu-2} \cdot q^{\lambda+2}$ is realizable in the tree induced by $A_{2} \cup\{x, y\}$, hence $\tau$ is realizable in $T$.

Case 2. $V_{s} \subset V\left(A_{3}\right)$. As in Case 1 we assume that $x_{a+1} \notin V_{p}, q-1 \leq$ $\left|V_{i}\right| \leq q$ for $V_{i} \subset V\left(A_{3}\right)$ and $\left|V_{j}\right|=q$ for $V_{j} \subset V\left(A_{2}\right)$, where $q=n_{p}$. Now we
can write $b=\lambda q+\mu(q-1)$ and $a+1=\nu q$, for some integers $\lambda>0, \mu \geq 0$ and $\nu>0$. If $\mu=0$, then, by $2^{0}, q \leq 2$, and we get a contradiction with our assumption on $n_{p}$. For $\mu=1$ we proceed as in Case 1 and show that $\tau$ is realizable in $T$. Suppose then $\mu \geq 2$. If $\nu \geq 2$ we proceed as in Case 1 and we show that $\tau$ is realizable in $T$. If $\nu=1$ (the essential difference between Case 1 and Case 2), then $q=a+1$ and $n=(\lambda+1)(a+1)+\mu a$, a contradiction. This finishes the proof of the theorem.

Corollary 7. Let $a, b, 3 \leq a \leq b$ be two integers and $T=S(2,2, a, b)$ a star-like tree on $n$ vertices. Then $T$ is avd if and only if the following conditions hold:

$$
\begin{aligned}
& 1^{\prime}(a, b)=1 ; \\
& 2^{\prime}(a+1, b)=1 ; \\
& 3^{\prime}(a, b+1)=1 ; \\
& 4^{\prime}(a+1, b+1)=2 ; \\
& 5^{\prime} \quad n \neq \alpha \cdot a+\beta \cdot(a+1) \text { for } \alpha, \beta \in \mathbf{N} .
\end{aligned}
$$

Proof. Necessity. Assume that $T$ is avd. Hence, from Proposition 3, $S(3, a, b)$ is avd, $a, b$ are odd, and $a \neq 2(\bmod 3)$ or $b \neq 2(\bmod 3)$.

Therefore, the odd numbers $a$ and $b$ satisfy the conditions $1^{0}-5^{0}$ of Theorem 6, hence also the conditions $1^{\prime}$ and $5^{\prime}$ of our theorem. Since $a$ and $b$ are odd, it follows by $1^{0}, 2^{0}$ and $3^{0}$ that $(a, b)=1,(a+1, b)=1$ and $(a, b+1)=1$. So $a$ and $b$ satisfy $1^{\prime}, 2^{\prime}$ and $3^{\prime}$. By $4^{0},(a+1, b+1) \in\{2,3\}$ and since $a \neq 2(\bmod 3)$ or $b \neq 2(\bmod 3)$, we have $(a+1, b+1) \neq 3$ and the condition $4^{\prime}$ holds.

Sufficiency. If $a$ and $b$ verify the conditions $1^{\prime}-5^{\prime}$ then the conditions $1^{0}-5^{0}$ of Theorem 6 are satisfied. Thus $S(3, a, b)$ is avd and, by $1^{\prime}-3^{\prime}, a$ and $b$ are odd.

Suppose that $a \equiv 2(\bmod 3)$ and $b \equiv 2(\bmod 3)$. Then $a+1 \equiv 0(\bmod 3)$ and $b+1 \equiv 0(\bmod 3)$, so $(a+1, b+1) \geq 3$, a contradiction. This implies that $a \neq 2(\bmod 3)$ or $b \neq 2(\bmod 3)$, and, by Proposition $3, T$ is avd. This finishes the proof.

Corollary 8. There are infinitely many arbitrarily vertex decomposable starlike trees $S(3, a, b)$ and $S(2,2, a, b)$.

Proof. Let $a \geq 5$ be a prime and $b=a+2$. It can be easily seen that $a$ and $b$ satisfy the conditions $1^{\prime}-5^{\prime}$ (and also $1^{0}-5^{0}$ ) for $n=2 a+3$.

## 5. Caterpillars with One Double and One Single Leg

Every caterpillar with one double and one single leg attached at $x$ and $y$ can be constructed in the following way. Take a path $P=x_{1}, \ldots, x_{n-3}$ where $x=x_{a}$ and $y=x_{j}(a<j)$ are two internal vertices of $P$, add three vertices $u, v$ and $z$ and join $u$ and $v$ to $x$ and $v$ to $y$ (see Figure 4).

Let $L_{x}=\left\{x_{1}, x_{2}, \ldots, x\right\}, R_{x}=\left\{x, x_{a+1}, \ldots, x_{n-3}\right\} \cup\{z\}, L_{y}=\left\{x_{1}\right.$, $\left.x_{2}, \ldots, y\right\} \cup\{u, v\}, R_{y}=\left\{y, x_{j+1}, \ldots, x_{n-3}\right\}$ and let $l_{x}=\left|L_{x}\right|, r_{x}=\left|R_{x}\right|$, $l_{y}=\left|L_{y}\right|$ and $r_{y}=\left|R_{y}\right|$.


Figure 4. A caterpillar with one double and one single leg.
Proposition 9. Let $T$ be a caterpillar of order $n$ with one double and one single leg attached at $x$ and $y$ resp. Let $a=l_{x}$ and $b=r_{x}$. If $a \equiv 1(\bmod 6)$, $b \equiv 0(\bmod 3), 7 \leq a<b,(a-3, b)=1, n-1 \neq \alpha a(\alpha \in \mathbf{N}), r_{y}=3$ and $a$ and $b$ satisfy the conditions $1^{\prime}-5^{\prime}$ of Corollary 7 , then $T$ is avd.
Proof. Let $u$ and $v$ denote two vertices of degree one adjacent to $x$ and let $z$ be the vertex of degree one adjacent to $y$ (see Figure 4). It follows from our assumptions that $n=a+b+1 \equiv 2(\bmod 3)$. Let $\tau=\left(a_{1}, \ldots, a_{k}\right)$ be an admissible sequence for the tree $T$. We will show that it suffices to consider the case where $a_{t} \geq 2$ for all $t$. Indeed, the caterpillar $T^{\prime}=T-v$ with two single legs satisfies $l_{x}^{\prime}=a, r_{x}^{\prime}=b, l_{y}^{\prime}=a+b-3=n-4 \equiv 1(\bmod 3)$, $r_{y}^{\prime}=3$, so the conditions $1^{0}-3^{0}$ of Theorem 5 are satisfied. We also have $\left(l_{y}^{\prime}, r_{x}^{\prime}\right)=(a+b-3, b)=(a-3, b)=1<l_{y}^{\prime}-l_{x}^{\prime}=b-3$, so the condition
$4^{0}$ holds. Furthermore, if $\alpha l_{x}^{\prime}+\beta l_{y}^{\prime}=\alpha a+(n-4) \beta=n-1$, for some $\alpha, \beta \in \mathbf{N}$, then, since $a \geq 7$, we have $\beta=0$, which is a contradiction. Assume $n-1=\alpha r_{x}^{\prime}+\beta r_{y}^{\prime}=\alpha b+3 \beta \equiv 0(\bmod 3)(\alpha, \beta \in \mathbf{N})$. But $n-1=a+b \equiv 1(\bmod 3)$, and we get a contradiction. So also $5^{0}-6^{0}$ of Theorem 5 hold. Now, if $a_{1}=1$, we can put $V_{1}=\{v\}$ and the existence of $T$-realization of $\tau$ is obvious. Therefore, we may assume $a_{t} \geq 2$ for all $t$.

Notice that, by Corollary 7, the star-like tree $\hat{T}=S(2,2, a, b)$ obtained by deleting the edge $z y$ and adding $z x_{n-3}$ is avd. Let $\hat{M}=\left(V_{1}, \ldots, V_{k}\right)$ be a $\hat{T}$-realization of $\tau$ such that $V_{p}\left(V_{s}\right.$, resp.) induces a primary tree (a primary tree or a subpath, resp.) of $\hat{T}$ containing $x$ ( $y$, resp.). Observe that if $V_{s}$ contains $x_{n-4}$ (the vertex that follows $y$ in the path $x_{1}, \ldots, x_{n-3}$ ) then $\tau$ is $T$-realizable. Indeed, if $z \in V_{s}$, then $\hat{M}$ is also a $T$-realization of $\tau$ and if $z \in V_{s^{\prime}}$ for some $s^{\prime} \neq s$, then $V_{s^{\prime}}=\left\{x_{n-3}, z\right\}$ and replacing in $\hat{M} V_{s}$ and $V_{s^{\prime}}$ by the sets $\left(V_{s} \backslash\left\{x_{n-4}\right\}\right) \cup\{z\}$ and $\left(V_{s^{\prime}} \backslash\{z\}\right) \cup\left\{x_{n-4}\right\}$, we get a $T$-realization of $\tau$.

Therefore, we shall assume that $V_{s}$ does not contain $x_{n-4}$. Hence, because $a_{r} \geq 2$ for all $r$, there is $g$ with $V_{g}=\left\{x_{n-4}, x_{n-3}, z\right\}$ (see Figure 5).


Figure 5. Case 1.1

Notice that for every $r$ such that $V_{r} \subset R_{x}$ we have $\left|V_{r}\right|=3$, for otherwise $g \neq r$ and assuming $V_{r}$ and $V_{g}$ are neighboring in $R_{x}$ we could transpose $V_{r}$ and $V_{g}$ into $V_{r}^{\prime}$ and $V_{g}^{\prime}$, in such a way that $V_{r}^{\prime}$ or $V_{g}^{\prime}$ contains the set $\left\{y, x_{n-4}\right\}$.

Now, since $\left|R_{x}\right|=b \equiv 0(\bmod 3)$, we have $\left|V_{p} \cap\left(R_{x} \backslash\{x\}\right)\right| \equiv 2(\bmod 3)$, hence $\left|V_{p} \cap\left(R_{x} \backslash\{x\}\right)\right| \geq 2$. Furthermore, since $a_{r} \geq 2$ for all $r$, we have $u, v \in V_{p}$ and $\left|V_{p}\right| \geq 5$.

Case 1. There is $h$ such that $V_{h} \subset L_{x}$ and $\left|V_{h}\right| \neq 3$. Obviously, we may suppose that $V_{h}$ and $V_{p}$ are neighboring in $L_{x}$.

Case 1.1. $\left|V_{h}\right| \leq\left|V_{p} \cap\left(R_{x} \backslash\{x\}\right)\right|$ (see Figure 5). Now we can transpose $V_{p}$ and $V_{h}$ into $V_{p}^{\prime}$ and $V_{h}^{\prime}$ with $V_{h}^{\prime} \subset R_{x}$. Using the same argument as above, we easily find a $T$-realization $\left(V_{1}^{\prime}, \ldots, V_{k}^{\prime}\right)$ of $\tau$.

Case 1.2. $\left|V_{h}\right|>\left|V_{p} \cap\left(R_{x} \backslash\{x\}\right)\right|$. Let $b=3 q$ and $\left|V_{h}\right|=3 w+r$, where $q, w, r$ are three integers such that $3 \leq q, 1 \leq w$ and $r \in\{0,1,2\}$. We have by assumption $3<\left|V_{h}\right|=3 w+r<a<b=3 q$, so setting:
$V_{h}^{\prime}=\left\{x_{n-3 w-r-1}, x_{n-3 w-r}, \ldots, x_{n-3}, z\right\}$, $V_{p}^{\prime}=\left\{x_{t}, x_{t+1}, \ldots, x_{a}, \ldots, x_{a+2-r}\right\} \cup\{u, v\}$, where $\left|V_{p}^{\prime}\right|=\left|V_{p}\right|=a_{p}$, we can cover the remaining vertices of $R_{x}$ by $q-w-$ $1 \geq 0$ sets of cardinality 3 and the existence of a $T$-realization $\left(V_{1}^{\prime}, \ldots, V_{k}^{\prime}\right)$ of $\tau$ is obvious.

Case 2. $\tau=\left(3,3, \ldots, 3,\left|V_{p}\right|\right)$. Because $a-1 \equiv 0(\bmod 3)$ and $\left|V_{p}\right|>3$, we can place the set of cardinality $\left|V_{p}\right|$ at the end of the path $x_{1}, x_{2}, \ldots$, $x_{n-3}, z$ and easily construct a realization of $\tau$ in $T$.

Theorem 10. The number of avd caterpillars with one double and one single leg is infinite.

Proof. Take $a$ such that $b=a+2=3 p$, where $p$ is a prime number greater than five. Therefore, $a \equiv 1(\bmod 6),(b, a-3)=1, n=2 a+3, n-1=2 a+2$ and it is easy to check that the assumptions $1^{\prime}-5^{\prime}$ of Corollary 7 hold. By Proposition 9 our caterpillar is avd.

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