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ON UNIQUELY PARTITIONABLE RELATIONAL STRUCTURES AND OBJECT SYSTEMS

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Abstract

We introduce *object systems* as a common generalization of graphs, hypergraphs, digraphs and relational structures. Let **C** be a concrete category, a *simple object system* over **C** is an ordered pair $S = (V, E)$,

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where $E = \{A_1, A_2, \ldots, A_m\}$ is a finite set of the objects of $\mathbf{C}$, such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements and $V \supseteq \bigcup_{i=1}^{m} V(A_i)$. To generalize the results on graph colourings to simple object systems we define, analogously as for graphs, that an additive induced-hereditary property of simple object systems over a category $\mathbf{C}$ is any class of systems closed under isomorphism, induced-subsystems and disjoint union of systems, respectively. We present a survey of recent results and conditions for object systems to be uniquely partitionable into subsystems of given properties.

Keywords: graph, digraph, hypergraph, vertex colouring, uniquely partitionable system.

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1. Introduction

A graph G is said to be uniquely n-colorable if any two proper n-colorings of G induce the same partition of the vertex set $V(G)$. The concept of uniquely n-colorable graphs was introduced in [12]. Different generalizations of this concept may be found e.g. in [1, 2, 5, 16, 20, 22]. Let us recall the basic notions which are needed to summarize the basic results. Let $\mathcal{I}$ denotes the class of all simple graphs. A graph property $\mathcal{P}$ is any nonempty proper isomorphism closed subclass of $\mathcal{I}$. A graph property is said to be induced-hereditary if it is closed under taking induced subgraphs and additive if it is closed under disjoint union of graphs (see [4]). Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be graph properties, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a graph G is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(G)$ such that each partition class V_i induces a subgraph $G[V_i]$ of property $\mathcal{P}_i, i = 1, 2, \ldots, n$. A graph G is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if G has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs and by $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)$ the set of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs. In the case $\mathcal{P}_1 = \mathcal{P}_2 = \cdots = \mathcal{P}_n = \mathcal{P}$, we write $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n = \mathcal{P}^n$, we say that $G \in \mathcal{P}^n$ is $(\mathcal{P}, n)$-partitionable and the graphs belonging to $\boldsymbol{U}(\mathcal{P}^n)$ are called (see e.g. [2, 16]) uniquely $(\mathcal{P}, n)$-partitionable. A property $\mathcal{P}$ is said to be *reducible* if there exist properties $\mathcal{P}_1, \mathcal{P}_2$ such that $\mathcal{P} = \mathcal{P}_1 \circ \mathcal{P}_2$. Otherwise $\mathcal{P}$ is called *irreducible*. For example, the property $\mathcal{O}$ — "to be an edgeless graph" related to regular colouring is irreducible and the smallest additive induced-hereditary reducible property is the class $\mathcal{O}^2$ of all bipartite graphs. The notion of reducible property have been introduced in [16], where it was proved that for any reducible property $\mathcal{P}$ there are no uniquely $(\mathcal{P}, n)$-partitionable graphs. D. Achlioptas, J.I. Brown, D.G. Corneil, M.S.O. Molloy in [1] proved the existence of uniquely $(n, -G)$-partitionable graphs for $n \geq 2$, where $-G$ denotes the property "do not contain G as an induced subgraph". Using the results of [18], the problem of the existence of uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable graphs for additive induced-hereditary graph properties was solved in [6] as follows:

Theorem 1. *Let $\mathcal{P}$ be an additive induced-hereditary graph property. Then for $n \geq 2, \boldsymbol{U}(\mathcal{P}^n) \neq \emptyset$ if and only if $\mathcal{P}$ is irreducible.*

Let us recall, that the requirement for additivity is substantial. A counterexample for nonadditive induced-hereditary properties is described in [17, 19]. The proof of the Theorem 1 is based on the Unique Factorization Theorem for additive induced-hereditary graph property (see [7, 8, 9, 14, 18, 19]). Some results on graph properties can be generalized and proved analogously for digraphs, hypergraphs, posets and other structures (see e.g. [13]). The extension of the Unique Factorization Theorem and the existence of uniquely partitionable digraphs have been presented in [23].

The main aim of this paper is to introduce a general notion of *object systems* and to show, how the presented results on uniquely partitionable graphs can be generalized for object systems over a given concrete category. We present a survey of results on uniquely partitionable systems of objects and describe the main ideas, which help to prove such general statements. In the first paper on uniquely colorable graphs [12] it has been proved that every k-colorable graph is an induced subgraph of a uniquely k-colorable graph. For additive induced-hereditary graph properties the generalization of this fact was presented in [6]. We will present, that the same is true for object systems.

2. Systems of Objects of a Concrete Category

We use the basic elementary notions of category theory (see [21]). A concrete category **C** is a collection of *objects* and *arrows* called *morphisms*. An object in a concrete category **C** is *a set with structure*. We will denote the *ground-set* of the object A by $V(A)$. The morphism between two objects is a *structure preserving mapping*. Obviously, the morphisms of **C** have to satisfy the axioms of the category theory. The natural examples of concrete categories are: **Set** of sets, **FinSet** of finite sets, **Graph** of graphs,

Poset of partially ordered sets with structure preserving mappings, called homomorphisms of corresponding structures.

In our investigations we will consider concrete categories **C** with *isomorphisms* i.e., structure preserving bijections between the ground-sets of objects only. Moreover, we will assume that by renaming (relabeling) the elements of the object A, we obtain an object A^* in **C** isomorphic to A in each considered concrete category **C**. To avoid formal difficulties, we will assume in what follows, that the elements $V(A)$ of each objects A of **C** are labelled by labels taken from a given infinite set $\mathcal{V}$ of cardinality α. Hence, any considered concrete category **C** will be small and the objects of **C** are "labelled" objects. We also assume, that for each object A of **C**, after relabelling of the elements of $V(A)$ by labels from a set $V^* \subset \mathcal{V}$ the obtained "new" object A^* with $V(A^*) = V^*$ belongs to **C**) too and it is always isomorphic to A. This requirements are natural, they are satisfied e.g. if the concrete category **C** is any small full subcategory of the above mentioned categories **Set**, **FinSet**, **Graph**, **Poset**, etc. Let us call such categories *wide*.

For example, a simple finite graph is a finite system of two element sets, a simple finite hypergraph $H = (V, E)$ can be considered as a system of its hyperedges $E = \{e_1, e_2, \ldots, e_m\}$, where edges are finite sets and the set of its vertices $V(H)$ is a superset of the union of hyperedges, i.e., $V \supseteq \bigcup_{i=1}^{m} e_i$. The following definition gives a natural generalization of graphs and hypergraphs.

Let **C** be a wide concrete category. A *simple system of objects* of **C** is an ordered pair $S = (V, E)$, where $E = \{A_1, A_2, \ldots, A_m\}$ is a finite set of the objects of **C**, such that the ground-set $V(A_i)$ of each object $A_i \in E$ is a finite set with at least two elements (i.e., there are no loops) and $V \supseteq \bigcup_{i=1}^{m} V(A_i)$. The class of all simple systems of objects of **C** wil be denoted by $\mathcal{I}(\mathbf{C})$. The symbols K_0 and K_1 denotes the null system $K_0 = (\emptyset, \emptyset)$ and system consisting of one isolated element, respectively.

For example, graphs can be viewed as systems of objects of a concrete category of two-element sets with bijections as arrows, digraphs are systems of objects of the category of ordered pairs, hypergraphs are finite set systems, etc. Let us remark, that the relational L-structures generalizing graphs, digraphs and k-uniform hypergraphs introduced by Fraïssé in [10] see e.g. [3, 11], are systems of objects on category of relations given by the signature L.

To generalize the results on generalized colourings of graphs to arbitrary simple systems of objects we need to define *isomorphism of systems, disjoint* *union of systems and induced-subsystems*, respectively. We can do this in a natural way.

Let $S_1 = (V_1, E_1)$ and $S_2 = (V_2, E_2)$ be two simple systems of objects of a given concrete category $\mathbf{C}$. The systems S_1 and S_2 are said to be isomorphic if there is a pair of bijection:

$$\phi : V_1 \longleftrightarrow V_2, \qquad\qquad \psi : E_1 \longleftrightarrow E_2,$$

such that if $\psi(A_{1i}) = A_{2j}$ then $\phi/V(A_{1i}) : V(A_{1i}) \longleftrightarrow V(A_{2j})$ is an isomorphism of the objects $A_{1i} \in E_1$ and $A_{2j} \in E_2$ in the category $\mathbf{C}$. The homomorphism of the systems can be defined in a similar way.

The disjoint union of the systems S_1 and S_2 is the system $S_1 \cup S_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where we assume that $V_1 \cap V_2 = \emptyset$. A system is said to be connected if it cannot be expressed as a disjoint union of two systems.

The subsystem of S_1 induced by the set $U \subseteq V(S_1)$ is $S_1[U]$, with objects $E(S_1[U]) := \{A_{1i} \in E(S_1) | V(A_{1i}) \subseteq U\}$. S_2 is an induced-subsystem of S_1 if it is isomorphic to $S_1[U]$ for some $U \subseteq V(S_1)$.

Using these definitions we can define, analogously as for graphs, that an additive induced-hereditary property of simple systems of objects over a category $\mathbf{C}$ is any class of systems closed under isomorphism, induced-subsystems and disjoint union of systems. Let us denote by $\mathbb{M}^a(\mathbf{C})$ the set of all additive induced-hereditary properties of simple systems of objects of a category $\mathbf{C}$. In the following we will call simple systems of objects of a category $\mathbf{C}$ shortly *systems*.

In [15] it is proved, that the set $\mathbb{M}^a(\mathbf{C})$ of all additive induced-hereditary properties of simple object-systems over $\mathbf{C}$ partially ordered by set inclusion forms a complete and distributive lattice and the structure of the lattice $\mathbb{M}^a(\mathbf{C})$ is investigated.

3. Reducible Properties and Unique Factorization Theorem for Systems

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of systems, a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition of a system S is a partition $\{V_1, V_2, \ldots, V_n\}$ of $V(S)$ such that each partition class V_i induces a subsystem $S[V_i]$ of property $\mathcal{P}_i, i = 1, 2, \ldots, n$.. A system S is said to be uniquely $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable if S has exactly one (unordered) vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. Let us denote by $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ the set of all vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partitionable systems and by $\boldsymbol{U}(\mathcal{P}_1\circ\mathcal{P}_2\circ\ldots\circ\mathcal{P}_n)$ the set of uniquely $(\mathcal{P}_1,\mathcal{P}_2,\ldots,\mathcal{P}_n)$-partitionable systems. A property $\mathcal{P}$ is said to be *reducible* in $\mathbb{M}^a(\mathbf{C})$ if there exist properties $\mathcal{P}_1,\mathcal{P}_2$ in $\mathbb{M}^a(\mathbf{C})$ such that $\mathcal{P}=\mathcal{P}_1\circ\mathcal{P}_2$. Otherwise $\mathcal{P}$ is called *irreducible.*

Following the same way as for graph properties the Unique Factorization Theorem for reducible properties of systems can be proved:

Theorem 2. *Any reducible additive induced-hereditary property of object systems is uniquely factorizable into irreducible factors* (*up to the order of factors*).

Let us recall the main steps of the proof, the details can be proved analogously as in [7, 8, 19, 23]. We follow the idea of the proof from [18].

We define the set $\mathcal{S}\subseteq\mathcal{I}(\mathbf{C})$ to be a *generating set of* $\mathcal{P}\in\mathbb{M}^a(\mathbf{C})$ if $S\in\mathcal{P}$ if and only if S is an induced subsystem of some system from $\mathcal{S}$. The fact that $\mathcal{S}$ is a generating set of $\mathcal{P}$ will be written as $[\mathcal{S}]=\mathcal{P}$. The members of $\mathcal{S}$ are called *generators* of $\mathcal{P}$.

Let us define *the operation* $*$:

For given vertex disjoint systems $S_1,S_2,\ldots,S_n$, $n\geq 2$, denote by

$$S_1*S_2*\cdots*S_n=\{S\in\mathcal{I}(\mathbf{C}):V(S)=\bigcup_{i=1}^{n}V(S_i)\text{ and }S[V(S_i)]=S_i\}.$$

The operation "$*$" is motivated by the following observation. Let us suppose that $S\in\mathcal{R}=\mathcal{P}\circ\mathcal{Q}$ and let $V(S)=(V_1,V_2)$ be a $(\mathcal{P},\mathcal{Q})$-partition of S. Then by additivity of $\mathcal{P}$ and $\mathcal{Q}$ the class $k.S[V_1]*k.S[V_2]\subseteq\mathcal{R}$ for every positive integer k. Therefore we define, for a given system $S\in\mathcal{S}(\mathcal{R})$, the invariant $dec_{\mathcal{R}}(S)$ as follows:

$dec_{\mathcal{R}}(S)=\max\{n$: there exist a partition $(V_1,V_2,\ldots,V_n)$, $V_i\neq\emptyset$, such that for each $k\geq 1, k.S[V_1]*k.S[V_2]*\ldots*k.S[V_n]\subseteq\mathcal{R}\}$. If $S\notin\mathcal{R}$ we set $dec_{\mathcal{R}}(S)$ to be zero.

A system $S\in\mathcal{P}$ is called *$\mathcal{P}$-strict* if $S*K_1\not\subseteq\mathcal{P}$. A system S is said to be *$\mathcal{R}$-decomposable* if $dec_{\mathcal{R}}\geq 2$ and *$\mathcal{R}$-indecomposable* otherwise. Thus if the property $\mathcal{R}$ is reducible, every system $S\in\mathcal{R}$ with at least two vertices is $\mathcal{R}$-decomposable. For any additive reducible induced-hereditary property the converse assertion is also valid.

An induced-hereditary additive property $\mathcal{R}$ is reducible if and only if all the $\mathcal{R}$-strict systems with at least two vertices are $\mathcal{R}$-decomposable.

In proofs of this fact the number $dec(\mathcal{R})$ by $dec(\mathcal{R})=\min\{dec_{\mathcal{R}}(S):S\in\mathcal{S}(\mathcal{R})\}$ is defined. The main part of the proof is to show that there exists a generating set $\mathcal{S}^* \subseteq \mathcal{S}(\mathcal{R})$ of $\mathcal{R}$ which consists of systems with decomposability number equal to $dec(\mathcal{R})$ and these generators are uniquely $\mathcal{R}$-decomposable. The final step of the proof of Theorem 2 presents the determination of the corresponding irreducible factors.

4. Uniquely Partitionable Systems

Following the proof of Theorem 2 one can see that in fact any reducible additive induced-hereditary property of systems $\mathcal{R}$ is generated by uniquely partitionable systems. More precisely:

Let $\mathcal{S}^*$ be a generating set of a reducible property $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n$, $n \geq 2$ consisting of $\mathcal{R}$-strict systems of decomposability number equal to $dec(\mathcal{R}) = n$. Since the generators of $\mathcal{S}^*$ are uniquely $\mathcal{R}$-decomposable into n indecomposable systems, they have exactly one $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition. This implies also that each object system $S \in \mathcal{R}$ is an induced subsytem of a uniquely partitionable system.

Theorem 3. *Let $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$, $n \geq 2$ be a factorization of a reducible property $\mathcal{R} \in \mathbb{M}^a(\mathbf{C})$ into irreducible factors. Then $\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \cdots \circ \mathcal{P}_n) \neq \emptyset$ and moreover $[\boldsymbol{U}(\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n)] = \mathcal{R}$.*

We can present the previous results in the form of criteria of irreducibility.

Theorem 4. *Let $\mathcal{P}$ be an additive induced-hereditary property of systems. Then the following statements are equivalent:*

1. *the property $\mathcal{P}$ is irreducible,*
2. *for every $n \geq 2$ the property $\mathcal{P}^n$ can be generated by a set of uniquely $(\mathcal{P}, n)$-partitionable systems,*
3. *there exist uniquely $(\mathcal{P}, n)$-partitionable systems for every $n \geq 2$,*
4. *the property $\mathcal{P}$ can be generated by a set of $\mathcal{P}$-indecomposable systems,*
5. *there exists a $\mathcal{P}$-indecomposable $\mathcal{P}$-strict system.*

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