

## ON A PERFECT PROBLEM

IGOR E. ZVEROVICH

*RUTCOR — Rutgers Center for Operations Research, Rutgers*  
*The State University of New Jersey*  
*640 Bartholomew Road, Piscataway, NJ 08854–8003, USA*  
**e-mail:** igor@rutcor.rutgers.edu

### Abstract

We solve Open Problem (xvi) from *Perfect Problems* of Chvátal [1] available at <ftp://dimacs.rutgers.edu/pub/perfect/problems.tex>:

Is there a class  $\mathcal{C}$  of perfect graphs such that

- (a)  $\mathcal{C}$  does not include all perfect graphs and
- (b) every perfect graph contains a vertex whose neighbors induce a subgraph that belongs to  $\mathcal{C}$ ?

A class  $\mathcal{P}$  is called locally reducible if there exists a proper subclass  $\mathcal{C}$  of  $\mathcal{P}$  such that every graph in  $\mathcal{P}$  contains a local subgraph belonging to  $\mathcal{C}$ . We characterize locally reducible hereditary classes. It implies that there are infinitely many solutions to Open Problem (xvi). However, it is impossible to find a hereditary class  $\mathcal{C}$  of perfect graphs satisfying both (a) and (b).

**Keywords:** hereditary classes, perfect graphs.

**2000 Mathematics Subject Classification:** 05C17 (Perfect graphs).

## 1. Locally Reducible Classes

A class of graphs  $\mathcal{P}$  is *hereditary* if  $H \in \mathcal{P}$  for each induced subgraph  $H$  of every graph  $G \in \mathcal{P}$ . As usual,  $N(u) = N_G(u)$  is the neighborhood of a vertex  $u$  in a graph  $G$ . A *local subgraph* in a graph  $G$  is a subgraph induced by  $N(u)$ , where  $u$  is a vertex of  $G$ . If  $u$  is an isolated vertex [i.e.,  $N(u) = \emptyset$ ], then the corresponding local subgraph is  $K_0$ , the vertexless graph. Let  $\mathcal{P}$  be a hereditary class of graphs. If there is a proper subclass  $\mathcal{C}$  of  $\mathcal{P}$  such

that every graph in  $\mathcal{P}$  with at least one vertex contains a local subgraph belonging to  $\mathcal{C}$ , then  $\mathcal{P}$  is called a *locally reducible class*.

**Problem 1.** Characterize locally reducible hereditary classes.

Not all hereditary classes are locally reducible. For example, let us consider the class  $\mathcal{K} = \{K_n : n \geq 0\}$ , of all complete graphs. Let  $\mathcal{C}$  be an arbitrary proper subclass of  $\mathcal{K}$ . Since  $\mathcal{C} \neq \mathcal{K}$ , there exists  $m$  such that  $K_m \notin \mathcal{C}$ . The graph  $K_{m+1}$  belongs to  $\mathcal{K}$ . However, all local subgraphs in  $K_{m+1}$  are  $K_m$ , and therefore they are not in  $\mathcal{C}$ . By definition,  $\mathcal{K}$  is not locally reducible.

**Theorem 1.** *A non-empty hereditary class  $\mathcal{P}$  is locally reducible if and only if  $\mathcal{P} \neq \mathcal{K}$ .*

**Proof.** Necessity was shown above.

*Sufficiency.* As usual, the star  $K_{1,n}$  has  $n + 1$  vertices  $v_0, v_1, \dots, v_n$  and  $n$  edges  $v_0v_1, v_0v_2, \dots, v_0v_n$ , the vertex  $v_0$  being the *center* of the star.

**Claim 1.** For a fixed  $n \geq 2$ , there is no graph  $G$  such that the neighborhood of each vertex of  $G$  induces  $K_{1,n}$ .

**Proof.** Suppose that there exists a graph  $G$  such that the neighborhood of each vertex induces  $K_{1,n}$ . We consider an arbitrary vertex  $u$  of  $G$ . Its neighborhood induces the subgraph  $H$  isomorphic to  $K_{1,n}$ . We denote  $V(H) = \{v_0, v_1, \dots, v_n\}$ , where  $v_0$  is the center, see Figure 1.

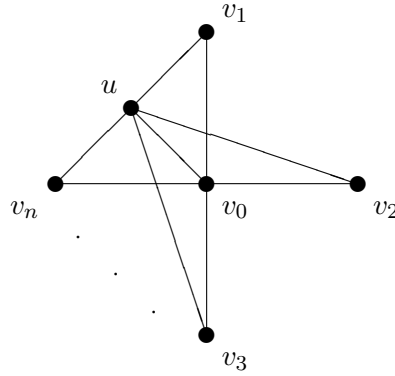


Figure 1. An illustration

The set  $N_G(v_0) = \{u, v_1, v_2, \dots, v_n\}$  induces  $K_{1,n}$  centered at  $u$ . The vertex  $v_1$  is adjacent to both  $u$  and  $v_0$ , and  $v_1$  is non-adjacent to all the vertices  $v_2, v_3, \dots, v_n$ . It follows that  $\{u, v_0\}$  is a connected component of the induced subgraph  $G(N(v_1))$ . Since  $n \geq 2$ ,  $N(v_1)$  cannot induce  $K_{1,n}$ , a contradiction. ■

First suppose that the path  $P_3$  belongs to  $\mathcal{P}$ . Then  $\mathcal{C} = \mathcal{P} \setminus \{P_3\}$  is a proper subclass of  $\mathcal{P}$ . We consider an arbitrary graph  $G \in \mathcal{P}$ . Claim 1 implies that there exists a vertex  $x \in V(G)$  such that  $N_G(x)$  does not induce  $P_3 \cong K_{1,2}$ . By the definition of  $\mathcal{C}$ ,  $G(N(x)) \in \mathcal{C}$ , as required.

It remains to consider the case, where  $P_3 \notin \mathcal{P}$ . Since  $P_3$  is a forbidden induced subgraph, each graph  $G \in \mathcal{P}$  is a disjoint union of complete subgraphs. Clearly, all local subgraphs of  $G$  are complete graphs.

Suppose that  $\mathcal{P}$  contains  $O_2$ , the graph with two non-adjacent vertices. Clearly, we can define  $\mathcal{C} = \mathcal{P} \setminus \{O_2\}$ . If  $\mathcal{P}$  does not contain  $O_2$ , then  $\mathcal{P}$  consists of complete graphs only. According to the condition,  $\mathcal{P} \neq \mathcal{K}$ , i.e., there exists  $m$  such that  $K_m \notin \mathcal{P}$ . Note that the class  $\mathcal{P}$  is not empty implying that  $m \geq 1$ . We may assume that  $K_{m-1} \in \mathcal{P}$ . Since  $\mathcal{P}$  is a hereditary class,  $\mathcal{P} = \{K_0, K_1, \dots, K_{m-1}\}$ . We may set  $\mathcal{C} = \mathcal{P} \setminus \{K_{m-1}\}$ , thus completing the proof. ■

Recall that a graph  $G$  is called *perfect* if  $\omega(H) = \chi(H)$  for each induced subgraph  $H$  of  $G$ , where  $\omega(H)$  is the clique number of  $H$  – the size of the largest complete subgraph in  $H$ , and  $\chi(H)$  is the chromatic number of  $H$  – the minimum number of colors in proper vertex colorings of  $H$ , see [3]. If  $\mathcal{P} = \mathcal{PERF}$  is the class of all perfect graphs, Problem 1 coincides with Open Problem (xvi) in Chvátal's list [1]. Theorem 1 gives a solution to this problem. Since all stars are perfect graphs, Claim 1 implies a more general fact.

**Corollary 1.** *There are infinitely many proper subclasses  $\mathcal{C}$  of  $\mathcal{PERF}$  such that every perfect graph contains a local subgraph belonging to  $\mathcal{C}$ .*

**Proof.** We define  $\mathcal{C}_n = \mathcal{PERF} \setminus \{K_{1,n}\}$  for each  $n \geq 2$  and apply Claim 1. ■

A *Zykov graph*  $H$  is defined by the property that there exists a graph  $G$  such that neighborhood of each vertex  $u \in V(G)$  induces  $H$ , see the *Neighborhood Problem* in Zykov [4]. In our proof we used the fact that all stars  $K_{1,n}$  with  $n \geq 2$  are not Zykov graphs.

**Corollary 2.** *Let  $\mathcal{P}$  be a class of graphs closed under taking local subgraphs. If  $\mathcal{P}$  contains a graph  $H$  which is not a Zykov graph, then  $\mathcal{P}$  is locally reducible.*

**Proof.** We define  $\mathcal{C} = \mathcal{P} \setminus \{H\}$ . Since  $H$  is not a Zykov graph, an arbitrary graph  $G \in \mathcal{P}$  has a local subgraph  $L \not\cong H$ . According to the condition,  $L \in \mathcal{P}$ . Thus,  $L \in \mathcal{P} \setminus \{H\} = \mathcal{C}$ . ■

## 2. Hereditary Subclasses

Now we consider a more complicated problem. A hereditary class  $\mathcal{P}$  of graphs is called *locally h-reducible* if there exists a proper hereditary subclass  $\mathcal{C}$  of  $\mathcal{P}$  such that every graph in  $\mathcal{P}$  with at least one vertex contains a local subgraph belonging to  $\mathcal{C}$ .

**Problem 2.** Characterize locally h-reducible hereditary classes.

*Join* of graphs  $G$  and  $H$ , denoted by  $G + H$ , is obtained from vertex-disjoint copies of  $G$  and  $H$  by adding all edges between  $V(G)$  and  $V(H)$ . A class  $\mathcal{P}$  of graphs is called *join-closed* if  $G + H \in \mathcal{P}$  whenever  $G, H \in \mathcal{P}$ .

**Claim 2.** Each join-closed hereditary class  $\mathcal{P}$  having a graph  $H$  with at least one vertex is not locally h-reducible.

**Proof.** Suppose that  $\mathcal{P}$  is a locally h-reducible class, i.e., there exists a proper hereditary subclass  $\mathcal{C}$  of  $\mathcal{P}$  such that every graph in  $\mathcal{P}$  with at least one vertex contains a local subgraph belonging to  $\mathcal{C}$ . There exists a graph  $H \in \mathcal{P} \setminus \mathcal{C}$ . Since the class  $\mathcal{C}$  is hereditary, each graph in  $\mathcal{C}$  is  $H$ -free. We consider the graph  $G = H + H \in \mathcal{P}$ . We see that each local subgraph  $L$  in  $G$  contains  $H$  as an induced subgraph. It implies that  $L \notin \mathcal{C}$ , a contradiction to the assumption that  $\mathcal{P}$  is a locally h-reducible class. ■

Claim 2 shows that the class  $\mathcal{PERF}$  is not locally h-reducible. Indeed, join of perfect graphs  $G$  and  $H$  always produces a perfect graph:  $\omega(G + H) = \omega(G) + \omega(H)$  and  $\chi(G + H) = \chi(G) + \chi(H)$ . Thus, it is impossible to strengthen Corollary 1 requiring that  $\mathcal{C}$  is a hereditary class.

A graph is *chordal* if it does not contain the cycles  $C_n$  with  $n \geq 4$  as induced subgraphs. Claim 2 does not hold for the class  $\mathcal{P} = \mathcal{CHORD}$  of all chordal graphs. Indeed, according to Dirac [2] each chordal graph

$G \neq K_0$  has a *simplicial* vertex — a vertex whose neighborhood induces a complete subgraph. It shows that we can choose  $\mathcal{C} = \mathcal{K}$  as a hereditary proper subclass of all chordal graphs. The reason is that the class  $\mathcal{CHORD}$  is not join-closed:  $C_4 = O_2 + O_2$  is not a chordal graph, while  $O_2$  is. Thus, Problem 2 remains open for all hereditary classes which are not join-closed.

### Acknowledgment

We thank the anonymous referees, whose suggestions helped to improve the presentation of the paper.

### References

- [1] V. Chvátal, Perfect Problems, available at <http://www.cs.concordia.ca/~chvatal/perfect/problems.html>
- [2] G.A. Dirac, *On rigid circuit graphs*, Abh. Math. Semin. Univ. Hamburg **25** (1961) 71–76.
- [3] Perfect graphs, Edited by J.L. Ramírez Alfonsín and B.A. Reed, Wiley-Interscience Series in Discrete Mathematics and Optimization (John Wiley & Sons, Ltd., Chichester, 2001) xxii+362 pp.
- [4] A.A. Zykov, Fundamentals of Graph Theory (Nauka, Moscow, 1987) 382 pp. (in Russian), translated and edited by L. Boron, C. Christenson and B. Smith (BCS Associates, Moscow, ID, 1990) vi+365 pp.

Received 6 September 2005

Revised 15 March 2006