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ON STRATIFICATION AND DOMINATION IN GRAPHS

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Abstract

A graph G is 2-stratified if its vertex set is partitioned into two classes (each of which is a stratum or a color class), where the vertices in one class are colored red and those in the other class are colored blue. Let F be a 2-stratified graph rooted at some blue vertex v. An F-coloring of a graph is a red-blue coloring of the vertices of G in which every blue vertex v belongs to a copy of F rooted at v. The F-domination number $\gamma_F(G)$ is the minimum number of red vertices in an F-coloring of G. In this paper, we study F-domination, where F is a 2-stratified red-blue-blue path of order 3 rooted at a blue end-vertex. We present characterizations of connected graphs of order n with F-domination number n or 1 and establish several realization results on F-domination number and other domination parameters.

Keywords: stratified graph, F-domination, domination.

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1. Introduction

Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. The vertices of a graph can be divided into cut-vertices and non-cut-vertices. Equivalently, the vertices of a tree are divided into its leaves and non-leaves. The set of vertices of a graph is partitioned according to the degrees of its vertices. When studying distance, the vertices of a connected graph are partitioned according to their eccentricities. Also, in a connected rooted graph, the vertices are partitioned according to their distance from the root. Perhaps the best known example of this process, however, is graph coloring, where the vertex set of a graph is partitioned into classes each of which is independent in the graph.

A typical Very Large Scale Integrated (VLSI) Circuit chip consists of millions of transistors assembled through layering of various materials in a silicon base. In recent years, advances in VLSI fabrication technology have made it possible to use more than two routing layers for interconnection. In fact, the most popular processors on the market today use three or more layers. In the design of algorithms to solve the multilayer routing problems encountered in this process, it is desirable to use graphs in which the vertices are partitioned into classes. In VLSI design, the design of computer chips often yields a division of the nodes into several layers each of which must induce a planar subgraph. So here too the vertex set of a graph is divided into classes. Motivated by these observations, Rashidi [10] defined a graph G to be a stratified graph if its vertex set is partitioned into classes. He studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [2, 5].

Formally then, a graph G whose vertex set has been partitioned is called a *stratified graph*. If $V(G)$ is partitioned into k subsets, then G is a *k-stratified graph*. The k subsets are called the *strata* or *color classes* of G. Suppose that the vertex set of a k-stratified graph G is partitioned into k subsets $V_1, V_2, \ldots, V_k$. Unlike vertex coloring, no condition is placed on the subsets V_i, $1 \leq i \leq k$. If G is 2-stratified, then we commonly color the vertices of one color class red and color the vertices of the other color class blue. The 2-stratified graphs were first studied from the point of view of domination by Chartrand, Haynes, Henning, and Zhang in [3, 4]. We refer to the book [6] for graph-theoretic notation and terminology not described in this paper.

Let F be a 2-stratified graph. So each vertex of F is colored red or blue and there is at least one vertex of each color. Designate a blue vertex v of F as the "root" of F. Then F is said to *be rooted at* v. For example, two distinct 2-stratified graphs F and F' rooted at a blue vertex v are shown in Figure 1, where the shaded vertices are red vertices and the non-shaded vertices are blue vertices in each graph.

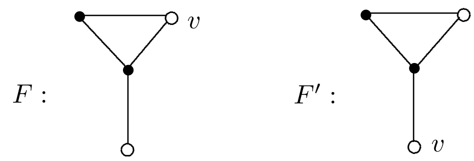

Figure 1. Two 2-stratified graphs F and F'.

For a connected graph G, a *red-blue coloring* of G is a coloring of G in which every vertex is colored red or blue. It is acceptable if all vertices of G are assigned the same color. If there is at least one vertex of each color, then the red-blue coloring produces a 2-stratification of G. By an F*-coloring* of a graph G, we mean a red-blue coloring of G such that for every blue vertex w of G, there is a copy of F in G with v at w. That is, for every blue vertex w of G, there exists a 2-stratified subgraph F' of G containing w and a color-preserving isomorphism α from F to F' such that $\alpha(v) = w$. The red-blue coloring of G in which every vertex is colored red is vacuously an F-coloring for every 2-stratified rooted graph F. In an F-coloring of a graph G, if a blue vertex w of G belongs to a copy F' of F rooted at w and u is a red vertex in F', then w is said to be F*-dominated* by u. If c is an F-coloring of G, then the set R_c of all red vertices of G is called an F*-dominating set* of G.

To illustrate these concepts, consider the 2-stratified graph F and the graph G of Figure 2. The red-blue coloring of G given in Figure 2 is an F-coloring of G since every blue vertex of G belongs to a copy of F rooted at that vertex. For example, the blue vertex w of G belongs to a copy F' of F rooted at w, where $V(F') = \{w, x, y, z\}$. Since x and y are red vertices in F' and F' is rooted at w, it follows that w is F-dominated by each of x and y.

For every 2-stratified graph F and every graph G, the red-blue coloring of G in which every vertex of G is colored red is an F-coloring of G and so it is always possible to give an F-coloring of G. The F*-domination number*

$\gamma_F(G)$ of G was introduced in [3] as the minimum number of red vertices of G in an F-coloring of G. By a *minimum F-coloring* of G, we mean an F-coloring having a minimum number of red vertices, that is, $\gamma_F(G)$ red vertices. In fact, the F-coloring of the graph G in Figure 2 is a minimum F-coloring. Therefore, $\gamma_F(G) = 4$ for the graph G of Figure 2. The F-domination number was introduced and studied in [3, 4] for 2-stratified graphs.

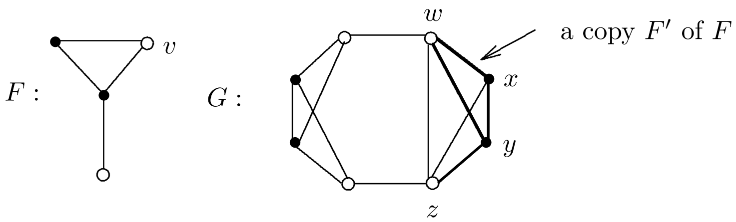

Figure 2. An F-coloring of a graph.

Another closely related concept concerns domination in graphs. A vertex is said to *dominate* itself and each vertex adjacent to it. A set S of vertices in a graph G is called a *dominating set* for G if every vertex of G is dominated by some vertex in S. The *domination number* $\gamma(G)$ of the graph G is the minimum number of vertices in a dominating set for G. A dominating set of cardinality $\gamma(G)$ is called a *minimum dominating set*. The following result appeared in [3].

Theorem A. *If F is a 2-stratified K_2, then $\gamma_F(G) = \gamma(G)$ for every graph G.*

Thus domination can be interpreted as a restricted 2-stratification or 2-coloring, with the red vertices forming the dominating set. In fact, F-domination generalizes not only ordinary domination but other types of domination that have been previously studied as we describe next. Let F be a 2-stratified P_3 rooted at a blue vertex v. The five possible choices for the graph F are shown in Figure 3.

For the 2-stratified graphs F_1, F_2, F_4, and F_5 of Figure 3, the following results are established in [3].

Theorem B. *If G is a connected graph of order at least 3, then $\gamma_{F_1}(G) = \gamma(G)$.*

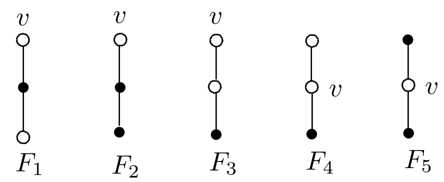

Figure 3. The five 2-Stratified graphs P_3 rooted at v.

A vertex v in a graph G openly dominates each of its neighbors. That is, v dominates the vertices in its neighborhood $N(v)$ but not itself. A set S of vertices in a graph G is an *open dominating set* if every vertex of G is adjacent to at least one vertex of S. In this case, a vertex v in an open dominating set of G is said to *openly dominate* its neighbors but not itself. The minimum cardinality of an open dominating set is the *open domination number* $\gamma_o(G)$ of G. An open dominating set of cardinality $\gamma_o(G)$ is a *minimum open dominating set* or a γ_o-set for G (see [7]).

Theorem C. *If G is a graph without isolated vertices, then $\gamma_{F_2}(G) = \gamma_o(G)$.*

An F_4-coloring of G requires that every blue vertex of G is adjacent to both a red and a blue vertex, while $\gamma_{F_4}(G)$ is the minimum number of red vertices required in such a 2-stratification of G. Thus, $\gamma_{F_4}(G)$ is the known domination parameter called the *restrained domination number* $\gamma_r(G)$. A set $S \subseteq V(G)$ is a *restrained dominating set* if every vertex not in S is adjacent to a vertex in S and to a vertex in $V(G) - S$. Every graph has a restrained dominating set since $V(G)$ is such a set. The *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G (see [8]).

Theorem D. *For every graph G, $\gamma_{F_4}(G) = \gamma_r(G)$.*

An F_5-coloring of G requires that every blue vertex of G is adjacent to (at least) two red vertices, while $\gamma_{F_5}(G)$ is the minimum number of red vertices required in such a 2-stratification of G. Thus, $\gamma_{F_5}(G)$ is the well-known domination parameter called the 2-*domination number* $\gamma_2(G)$ (see Jacobson [9]). A set $S \subseteq V(G)$ is a *k-dominating set* if every vertex not in S is adjacent to at least k vertices in S. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality of a k-dominating set of G.

Theorem E. *For every graph G, $\gamma_{F_5}(G) = \gamma_2(G)$.*

While the parameters $\gamma_{F_1}(G)$, $\gamma_{F_2}(G)$, $\gamma_{F_4}(G)$, and $\gamma_{F_5}(G)$ are well-known domination parameters, the parameter $\gamma_{F_3}(G)$ is new, according to the discussion in [3]. Thus, we investigate F_3-domination. To simplify notation, we denote F_3 by F in this paper.

2. Graphs with Prescribed Order and F-domination Number

Since the 2-stratified graph $F = F_3$ contains exactly one red vertex,

$$1 \leq \gamma_F(G) \leq n \tag{1}$$

for every connected graph G of order n. In this section, we first present characterizations of connected graphs of order n with F-domination number n or 1. In order to do this, we present two lemmas, whose routine proofs are omitted.

Lemma 2.1. *Let v be an end-vertex of a connected graph G. Suppose that v is adjacent to the vertex u in G.*

(a) *If $\deg u = 2$, then v is colored red by any F-coloring of G.*

(b) *If u is colored red by an F-coloring c, then v is also colored red by c.*

Lemma 2.2. *Let G be a connected graph that has an F-coloring. If a blue vertex v is F-dominated by a red vertex u that is adjacent to v, then v belongs to a triangle in G that contains u. Consequently, in a triangle-free G, each blue vertex v can only be F-dominated by a red vertex that is not adjacent to v.*

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. The *diameter* $\mathrm{diam}(G)$ of G is the largest distance between two vertices of G.

Theorem 2.3. *Let G be a connected graph of order $n \geq 3$. Then*

(a) *$\gamma_F(G) = n$ if and only if $G = K_{1,n-1}$,*

(b) $\gamma_F(G) = 1$ *if and only if G contains a vertex u such that $N(u)$ is an open dominating set of G. In this case, the red-blue coloring of G defined by assigning red to u and blue to the remaining vertices of G is an F-coloring of G.*

Proof. We first verify (a). Let $G = K_{1,n-1}$ and let x be the central vertex of G. Suppose that c is an arbitrary F-coloring of G. Since there is no $x - y$ path of length 2 in G for any vertex y in G, the vertex x must be colored red by c. However, then every end-vertex of G must also be colored red by c by Lemma 2.1. Hence every vertex of G is colored red by c and so $\gamma_F(G) = n$.

For the converse, let $G \neq K_{1,n-1}$ be a connected graph of order $n \geq 3$. If G is a tree, then $\mathrm{diam}(G) \geq 3$ and so G contains a path $P : v_1, v_2, v_3, v_4$ of length 3. Observe that the red-blue coloring defined by assigning blue to v_2 and v_3 and red to the remaining vertices of G is an F-coloring of G with $n-2$ red vertices. Thus $\gamma_F(G) \leq n-2$. If G is not a tree, then G contains a k-cycle $C : v_1, v_2, \cdots, v_k, v_1$, where $k \geq 3$. Note that the red-blue coloring defined by assigning blue to v_1 and v_2 and red to the remaining vertices of G is an F-coloring of G with $n-2$ red vertices. Thus $\gamma_F(G) \leq n-2$. Therefore, (a) holds.

Next, we verify (b). First assume that $\gamma_F(G) = 1$. Then there is an F-coloring c of G with exactly one red vertex, say u. We show that $N(u)$ is an open dominating set of G. Let $v \in V(G)$. Since u is openly dominated by any vertex in $N(u)$, we may assume that $v \neq u$. Because c is an F-coloring of G, the blue vertex v belongs to a copy of F rooted at v, that is, v is adjacent to a blue vertex w and w is adjacent to u. Thus v is openly dominated by $w \in N(u)$ and so $N(u)$ is an open dominating set of G.

For the converse, assume that G contains a vertex u such that $N(u)$ is an open dominating set. We show that $\gamma_F(G) = 1$ by showing the red-blue coloring c^* defined by assigning red to u and blue to the remaining vertices of G is an F-coloring of G. Let $v \in V(G) - \{u\}$ be a blue vertex of G. Since $N(u)$ is an open dominating set of G, it follows that v is openly dominated by some vertex $w \in N(u)$. Thus v is adjacent to a blue vertex w and w is adjacent to the red vertex u, implying that the blue vertex v belongs to a copy of F rooted at v. Therefore, c^* is an F-coloring of G with exactly one red vertex and so $\gamma_F(G) = 1$. ■

By Theorem 2.3, if G is a nontrivial bipartite graph, then $\gamma_F(G) \geq 2$. In particular, if T is a tree of order $n \geq 3$, then

$$(2) \qquad 2 \le \gamma_F(T) \le n,$$

and by Theorem 2.3, $\gamma_F(T) = n$ if and only if T is a star. Next, we characterize all trees of order at least 3 with F-domination number 2. A *double star* T is a tree of diameter 3. Recall that for an F-coloring c of a graph, we let R_c denote the set of all red vertices of c.

Theorem 2.4. *A tree T of order at least 3 has $\gamma_F(T) = 2$ if and only if T is a double star.*

Proof. Suppose that T is a double star. Let s and t be two vertices of T with $d(s,t) = \mathrm{diam}(G)$. The red-blue coloring of T defined by assigning red to s and t and blue to the remaining vertices of T is an F-coloring of G with exactly two red vertices. Thus $\gamma_F(T) = 2$ by (2).

For the converse, assume that T is a tree of order $n \ge 3$ that is not a double star. Then $\mathrm{diam}\, T \ne 3$. If $\mathrm{diam}\, T \le 2$, then T is a star and so $\gamma_F(T) = n \ge 3$. Thus, we may assume that $\mathrm{diam}\, T \ge 4$. We show that $\gamma_F(T) \ge 3$. Assume, to the contrary, that $\gamma_F(T) \le 2$. Then $\gamma_F(T) = 2$ by (2). Let c be a minimum F-coloring of T with $R_c = \{x, y\}$. Let $d(x, y) = k$ and let $P : x = x_0, x_1, x_2, \ldots, x_k = y$ be an $x - y$ path of length k in T. We consider four cases.

Case 1. $k = 1$. Then x and y are adjacent. Since $T \ne P_2$, at least one of x and y is not an end-vertex of T. Assume, without loss of generality, that x is not an end-vertex of T, and so x is also adjacent to a blue vertex v. Since T is a tree, T is triangle-free. It then follows by Lemma 2.2 that v cannot be F-dominated by x. Hence v is F-dominated by y. Thus v is adjacent to a blue vertex v' that is adjacent to y. However then x, v, v', y, x is a 4-cycle in the tree T, which is impossible.

Case 2. $k = 2$. Then the blue vertex x_1 in P is adjacent to both x and y. Since x and y are the only red vertices in G, it follows that x_1 is F-dominated by a red vertex that is adjacent to x_1, which contradicts Lemma 2.2.

Case 3. $k = 3$. Then $P : x = x_0, x_1, x_2, x_3 = y$ is a path of length 3. Since $\mathrm{diam}\, T \ge 4$, it follows that $V(T) - V(P) \ne \emptyset$. We claim that every vertex in $V(T) - V(P)$ is adjacent to either x_1 or x_2. Assume, to the contrary, that there is $v \in V(T) - V(P)$ such that v is adjacent to neither x_1 nor x_2. Suppose, without loss of generality, that v is F-dominated by x. Then v is adjacent to a blue vertex $v' \notin \{x_1, x_2\}$ that is adjacent x. Since v' is adjacent to x, it follows by Lemma 2.2 that v' cannot be F-dominated by x and so v' is F-dominated by y. Thus v' is adjacent to a blue vertex v'' that is adjacent to y. This implies that $x, v', v'', y, x_2, x_1, x$ is a cycle in the tree T, which is impossible. Therefore, as claimed, every vertex in $V(T) - V(P)$ is adjacent to either x_1 or x_2. Moreover, x is adjacent to x_1 and y is adjacent to x_2. Therefore, every vertex in $V(T) - \{x_1, x_2\}$ is adjacent to either x_1 or x_2, implying that T is a double star with central vertices x_1 and x_2. This contradicts our assumption that T is not a double star.

Case 4. $k \geq 4$. Then $d(x_1, y) \geq 3$. One the other hand, since x_1 is adjacent to x, it follows by Lemma 2.2 that x_1 cannot be F-dominated by x and so x_1 is F-dominated by y. This implies that $d(x_1, y) \leq 2$, which is a contradiction. ■

We have seen in (1) that if G is a connected graph of order n, then $1 \leq \gamma_F(G) \leq n$. Next, we determine which pairs k, n of positive integers with $1 \leq k \leq n$ can be realized as the F-domination number and the order, respectively, for some connected graph. Certainly, since F contains two blue vertices, not every such pair is realizable, as we see next.

Observation 2.5. There is no connected graph G of order $n \geq 3$ with $\gamma_F(G) = n - 1$.

On the other hand, for each pair k, n of integers, where $1 \leq k \leq n$, $k \neq n-1$, and $3 \leq n \leq 6$, there exists a connected graph G of order n with $\gamma_F(G) = k$, as shown in Figure 4. Moreover, if G is a connected graph of order $n = 1$ or $n = 2$, then G does not contain F as a subgraph and so $\gamma_F(G) = n$.

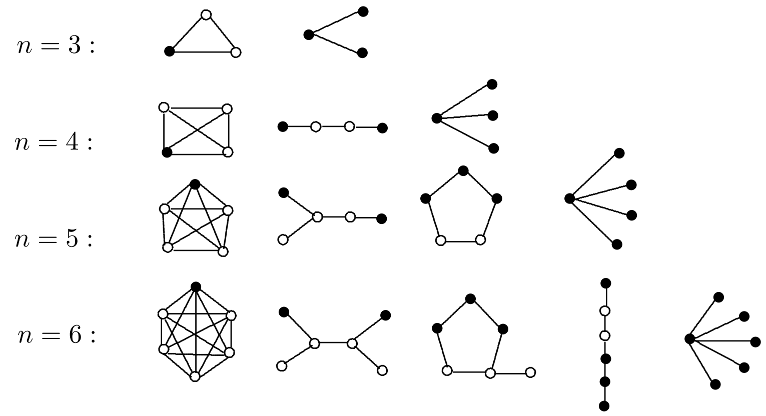

Figure 4. Realizable pairs k, n for $3 \leq n \leq 6$.

Thus, if $1 \leq n \leq 6$, then, for each pair k, n of integers with $1 \leq k \leq n$ and $k \neq n-1$, there exists a connected graph G of order n with $\gamma_F(G) = k$. However, if $n \geq 7$, then $\gamma_F(G) \neq n-2$ for every connected graph G of order n, as we show next.

Theorem 2.6. *There is no connected graph G of order $n \geq 7$ with $\gamma_F(G) = n-2$.*

Proof. Assume, to the contrary, that there exists a connected graph G of order $n \geq 7$ such that $\gamma_F(G) = n-2$. Let c be a minimum F-coloring of G and let x and y be the two blue vertices of G. Necessarily, x and y are adjacent in G. Suppose that x is F-dominated by a red vertex u and y is F-dominated by a red vertex v.

First, we claim that $u \neq v$. If $u = v$, then u, x, y, u is a triangle in G. Since G is connected and the order of G is at least 7, there is at least one vertex, say w, such that $uw \in E(G)$. Then the red-blue coloring that assigns blue to u, w, y and red to the remaining vertices of G is an F-coloring of G with $n-3$ red vertices. Thus $\gamma_F(G) \leq n-3$, which is a contradiction. Therefore, $u \neq v$ and there is a path $P : u, y, x, v$ in G, where $u, v \in R_c$.

Next, we claim that each vertex in $R_c - \{u, v\}$ is adjacent to neither x nor y. For otherwise, assume that there exists $w \in R_c - \{u, v\}$ such that w is adjacent to x or y, say the former. Then the red-blue coloring defined by assigning blue to x, y, w and red the remaining vertices of G is an F-coloring with $n-3$ red vertices, and so $\gamma_F(G) \leq n-3$, which is a contradiction. Therefore, as claimed, each vertex in $R_c - \{u, v\}$ is adjacent to neither x nor y.

Since G is a connected graph of order $n \geq 7$, it can be shown that each vertex in $R_c - \{u, v\}$ is adjacent to either u or v. We consider two cases.

Case 1. *Every vertex in $R_c - \{u, v\}$ is adjacent to u or every vertex in $R_c - \{u, v\}$ is adjacent to v, say the former.* Let $w \in R_c - \{u, v\}$. Then the red-blue coloring that assigns red to v, w, x and blue to the remaining vertices of G is an F-coloring of G with three red vertices. Thus $\gamma_F(G) \leq 3 < 4 \leq n - 3$.

Case 2. *Case* 1 *does not occur.* Let W_1 be the set of vertices in $R_c - \{u, v\}$ that are adjacent to u and let W_2 be the set of vertices in $R_c - \{u, v\}$ that are adjacent to v. Then $W_1 \neq \emptyset$ and $W_2 \neq \emptyset$. Since $n \geq 7$, at least one of W_1 and W_2 contains at least two vertices, say $|W_1| \geq 2$. Let $w \in W_1$. Then the red-blue coloring defined by assigning blue to each vertex in $\{u, w, y\}$ and red to the remaining vertices of G is an F-coloring of G with $n - 3$ red vertices. Therefore, $\gamma_F(G) \leq n - 3$. ■

Next, we show that certain pairs k, n with $1 \leq k \leq n$ can be realized as the F-domination number and the order of some connected graph.

Theorem 2.7. *Let $n \geq 7$. If k is an integer with $1 \leq k \leq \lfloor n/2 \rfloor + 1$ or $k = n$, then there exists a connected graph G of order n with $\gamma_F(G) = k$.*

Proof. For each pair k, n of integers with $1 \leq k \leq \lfloor n/2 \rfloor + 1$ or $k = n$, we construct a graph $G_{k,n}$ of order n with $\gamma_F(G_{k,n}) = k$. Let $G_{1,n} = K_n$ and $G_{n,n} = K_{1,n-1}$. Thus, we may assume that $2 \leq k \leq \lfloor n/2 \rfloor + 1$. We consider three cases.

Case 1. $k = 2$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a cycle of order 4 and let $G_{2,n}$ be obtained from C_4 by (1) adding $n - 4$ new vertices $x_1, x_2, \ldots, x_{n-4}$ and joining each vertex x_i $(1 \leq i \leq n - 4)$ to v_1. Define a red-blue coloring c^* of $G_{2,n}$ by assigning red to v_2 and v_3 and blue to the remaining vertices of $G_{2,n}$. Observe that (i) the blue vertex v_1 is F-dominated by the red vertex v_3, (ii) the blue vertex v_4 is F-dominated by the red vertex v_2, and (iii) the blue vertex x_i $(1 \leq i \leq n - 4)$ is F-dominated by the red vertex v_2. Thus, every blue vertex v of G belongs to a copy of F rooted at v, implying that c^* is a an F-coloring of $G_{2,n}$. Therefore, $\gamma_F(G_{2,n}) \leq |R_{c^*}| = 2$.

Next, we show that $\gamma_F(G_{2,n}) \geq 2$. Assume, to the contrary, that $\gamma_F(G_{2,n}) = 1$. Let there be given an F-coloring c of $G_{2,n}$ with exactly one red vertex. Since v_3 is only F-dominated by v_1 and v_1 is only F-dominated by v_3, it follows that at least one of v_1 and v_3 belongs to R_c. By Lemma 2.1, we have $R_c = \{v_3\}$. However, none of the blue vertices x_i $(1 \leq i \leq n-4)$ is F-dominated by v_3 and so c is not an F-coloring of G, which is a contradiction. Therefore, $\gamma_F(G_{2,n}) = 2$.

Case 2. $3 \leq k \leq \lfloor n/2 \rfloor$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a cycle of order 4. Then the graph $G_{k,n}$ is obtained from the graph C_4 by (1) adding $k-2 > 0$ new vertices $x_1, x_2, \ldots, x_{k-2}$ and joining each vertex x_i $(1 \leq i \leq k-2)$ to v_1 and (2) adding $n-2-k > 0$ new vertices $y_1, y_2, \ldots, y_{n-2-k}$ and joining each vertex y_j $(1 \leq j \leq n-2-k)$ to v_3. Then the order of G is $4 + (k-2) + (n-2-k) = n$. We show that $\gamma_F(G_{k,n}) = k$.
We first show that $\gamma_F(G_{n,k}) \leq k$. Let $S = \{v_1, v_2\} \cup \{x_i : 1 \leq i \leq k-2\}$. Define a red-blue coloring c^* of $G_{n,k}$ by assigning red to each vertex in S and blue to the remaining vertices of $G_{n,k}$. Observe that (i) the blue vertex v_3 is F-dominated by the red vertex v_1, (ii) the blue vertex v_4 is F-dominated by the red vertex v_2, and (iii) the blue vertex y_j $(1 \leq j \leq n-2-k)$ is F-dominated by the red vertex v_2. Thus, every blue vertex v of G belongs to a copy of F rooted at v, implying that c^* is a an F-coloring of $G_{n,k}$. Therefore, $\gamma_F(G_{n,k}) \leq |R_{c^*}| = |S| = k$.

Next, we show that $\gamma_F(G_{n,k}) \geq k$. Assume, to the contrary, that $\gamma_F(G_{n,k}) \leq k-1$. Let there be given a minimum F-coloring c of $G_{n,k}$. Observe that v_3 is only F-dominated by v_1 and v_1 is only F-dominated by v_3. Thus at least one of v_1 and v_3 belongs to R_c. By Lemma 2.1, if $v_1 \in R_c$, then $x_i \in R_c$ for $1 \leq i \leq k-2$. Similarly, if $v_3 \in R_c$, then $y_j \in R_c$ for $1 \leq j \leq n-2-k$. Thus $|R_c| \geq \min\{k-1, n-1-k\}$. Since $k \leq \lfloor n/2 \rfloor$, it follows that $k-1 \leq n-1-k$. This implies that $\gamma_F(G_{n,k}) \geq k-1$. Therefore, $\gamma_F(G_{n,k}) = k-1$ and we may assume that $R_c = \{v_1, x_1, x_2, \ldots, x_{k-2}\}$. However then, the blue vertex v_2 does not belong to a copy of F rooted at v_2, which is a contradiction. Therefore, $\gamma_F(G_{n,k}) = k$.

Case 3. $k = \lfloor n/2 \rfloor + 1$. Since $n \geq 7$, it follows that $k \geq 4$. We consider two subcases.

Subcase 3.1. *n is even.* Then $n = 2\ell$ for some integer $\ell \geq 4$ and $k = \ell + 1 \geq 5$. Let $P_4 : v_1, v_2, v_3, v_4$ be a path of order 4. Then the graph $G_{k,n}$ is obtained from P_4 by (1) adding $k-3$ new vertices $x_1, x_2, \ldots, x_{k-3}$

and joining each vertex x_i $(1 \le i \le k-3)$ to v_1 and (2) adding $k-3$ new vertices $y_1, y_2, \ldots, y_{k-3}$ and joining each vertex y_j $(1 \le j \le k-3)$ to v_4. Then the order of G is $4 + 2(k-3) = 4 + 2(\ell - 2) = 2\ell = n$.

We first show that $\gamma_F(G_{n,k}) \le k$. Let $S_0 = \{x_1, v_3, v_4, y_1, y_2, \ldots, y_{k-3}\}$. Define a red-blue coloring c^* of $G_{n,k}$ by assigning red to each vertex in S_0 and blue to the remaining vertices of $G_{n,k}$. Observe that (i) the blue vertex v_1 is F-dominated by the red vertex v_3, (ii) the blue vertex v_2 is F-dominated by the red vertex x_1, and (iii) the blue vertex x_i $(2 \le i \le k-3)$ is F-dominated by the red vertex x_1. Thus, every blue vertex v of G belongs to a copy of F rooted at v, implying that c^* is a an F-coloring of $G_{n,k}$. Therefore, $\gamma_F(G_{n,k}) \le |R_{c^*}| = |S_0| = k$.

Next we show that $\gamma_F(G_{n,k}) \ge k$. Let $X = \{x_1,\ x_2,\ \ldots,\ x_{k-3}\}$ and $Y = \{y_1,\ y_2,\ \ldots,\ y_{k-3}\}$. Suppose that c is a minimum F-coloring of $G_{n,k}$. We consider three subcases.

Subcase 3.1.1. *$v_1, v_4 \in R_c$.* By Lemma 2.1, $X \cup Y \subseteq R_c$. This implies that

$$\gamma_F(G_{n,k}) = |R_c| \geq |X \cup Y \cup \{v_1, v_4\}| = 2k - 4 > k,$$

as $k \geq 5$ (note that $n \geq 7$ is even), which is a contradiction.

Subcase 3.1.2. *$v_1 \notin R_c$ and $v_4 \notin R_c$.* Since c is an F-coloring, v_1 is F-dominated by some red vertex. Since v_1 is only F-dominated by v_3, it follows that v_2 is blue and v_3 is red. However then, v_4 cannot be F-dominated by any red vertex as v_4 is only F-dominated by v_2, which is a contradiction.

Subcase 3.1.3. *Exactly one of v_1 and v_4 belongs to R_c, say $v_1 \notin R_c$ and $v_4 \in R_c$.* Since $v_4 \in R_c$, it follows that $Y \subseteq R_c$ by Lemma 2.1. Since v_1 is only F-dominated v_3, it follows that v_2 is blue and v_3 is red. Note that each vertex in X is only F-dominated by v_2 or some vertex in X. Since v_2 is blue, at least one vertex in X must be colored red. Hence $Y \cup \{v_3, v_4, x\} \subseteq R_c$, where $x \in X$. Therefore, $\gamma_F(G_{n,k}) = |R_c| \geq 3 + (k - 3) = k$.

Subcase 3.2. *n is odd.* Then $n = 2\ell + 1$ for some integer $\ell \geq 3$ and $k = \ell + 1 \geq 4$. Let $P_4 : v_1, v_2, v_3, v_4$ be a path of order 4. So the graph $G_{k,n}$ is obtained from P_4 by (1) adding $k - 2$ new vertices $x_1, x_2, \ldots, x_{k-2}$ and joining each vertex x_i $(1 \leq i \leq k-2)$ to v_1 and (2) adding $k-3$ new vertices $y_1, y_2, \ldots, y_{k-3}$ and joining each vertex y_j $(1 \leq j \leq k - 3)$ to v_4. Then the order of G is $4 + (k - 2) + (k - 3) = 4 + (\ell - 1) + (\ell - 2) = 2\ell + 1 = n$. We show that $\gamma_F(G_{k,n}) = k$.

To show that $\gamma_F(G_{n,k}) \leq k$, let $S_1 = \{x_1, v_3, v_4, y_1, y_2, \ldots, y_{k-3}\}$. Define a red-blue coloring c^* of $G_{n,k}$ by assigning red to each vertex in S_1 and blue to the remaining vertices of $G_{n,k}$. An argument similar to the one used in Subcase 3.1 shows that c^* is an F-coloring of $G_{n,k}$ and so $\gamma_F(G_{n,k}) \leq |R_{c^*}| = |S_1| = k$.

Next, we show that $\gamma_F(G_{n,k}) \geq k$. Suppose that c is a minimum F-coloring of $G_{n,k}$. An argument similar to the one in Subcases 3.1.1 and 3.1.2 shows that exactly one of v_1 and v_4 belongs to R_c. If $v_1 \in R_c$ and $v_4 \notin R_c$, then $X \subseteq R_c$ by Lemma 2.1. Since v_4 is only F-dominated by v_2, it follows that v_2 is red and v_3 is blue. Moreover, each vertex in Y is only F-dominated by v_3 or by some vertex in Y. Since v_3 is blue, at least one vertex in Y must be colored red. Hence $X \cup \{v_1, v_2, y\} \subseteq R_c$, where $y \in Y$. However then, $\gamma_F(G_{n,k}) = |R_c| \geq 3 + (k - 2) = k + 1 > k$, which is impossible. Similarly, if $v_1 \notin R_c$ and $v_4 \in R_c$, then R_c contains at least k red vertices and so $\gamma_F(G_{n,k}) = |R_c| \geq k$. ■

Although it can be shown that there are infinitely many pairs k, n of integers with $\lfloor n/2 \rfloor + 1 < k < n - 2$ and $n \geq 7$ for which there exists a connected graph G of order n with $\gamma_F(G) = k$, we conclude this section with the following question.

Problem 2.8. *For which pairs k, n of integers with $\lfloor n/2 \rfloor + 1 < k < n - 2$ and $n \geq 7$, does there exist a connected graph G of order n with $\gamma_F(G) = k$?.*

3. Realization Results on Three Domination Parameters

For a nontrivial connected graph G without isolated vertices, there are three possibilities related to the three parameters $\gamma(G)$, $\gamma_o(G)$, and $\gamma_F(G)$, namely

(1) $\gamma(G) \leq \gamma_F(G) \leq \gamma_o(G)$,

(2) $\gamma(G) \leq \gamma_o(G) \leq \gamma_F(G)$,

(3) $\gamma_F(G) \leq \gamma(G) \leq \gamma_o(G)$.

First, we show that it is possible for these three parameters to be equal.

Proposition 3.1. *For each integer $k \geq 2$, there exists a connected graph G such that*

$$\gamma(G) = \gamma_F(G) = \gamma_o(G) = k.$$

Proof. For $k = 2$, let G be the double star and so $\gamma(G) = \gamma_F(G) = \gamma_o(G) = 2$. Let $k \geq 3$. For each integer i with $1 \leq i \leq k-1$, let $F_i : v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{1i}$ be a copy of C_4. The graph G is obtained from the graphs F_i $(1 \leq i \leq k-1)$ by identifying vertices v_{4i} $(1 \leq i \leq k-1)$ and labeling the identified vertex by v. Then it can be verified that $\gamma(G) = \gamma_F(G) = \gamma_o(G) = k$. ■

Next result shows that every pair a, b of positive integers can be realizable as the domination and F-domination of some connected graph.

Theorem 3.2. *Let a and b be positive integers with $a \leq b$.*

(i) *There exists a connected graph G such that $\gamma(G) = a$ and $\gamma_F(G) = b$;*

(ii) *There exists a connected graph H such that $\gamma_F(H) = a$ and $\gamma(H) = b$.*

Proof. First, we verify (i). Suppose that $a = 1$. If $b = 1$, let $G = K_3$; while if $b \geq 2$, let $G = K_{1,b-1}$. Then $\gamma(G) = 1$ and $\gamma_F(G) = b$ for each integer $b \geq 1$. Hence the result holds for $a = 1$. Thus, we may assume that $a \geq 2$. By Proposition 3.1, the result holds if $a = b$. Therefore we may assume that $a < b$ and so $b - a > 0$. There are two cases.

Case 1. $a = 2$. Let G be the graph in the proof of Case 2 for Theorem 2.7, where $k = b$. Then $\gamma(G) = 2$ and $\gamma_F(G) = k = b$.

Case 2. $a \geq 3$. We begin with a double star T whose central vertices are u and v. Let $U = \{u_1, u_2, \ldots, u_{b-a}\}$ and $V = \{v_1, v_2, \ldots, v_{a+b-1}\}$ be the sets of vertices of T such that u is adjacent to every vertex in U and v is adjacent to every vertex in V. Then the graph G is obtained from T by (1) subdividing the edge uv with a new vertex x and (2) adding $a - 2$ new vertices $w_1, w_2, \ldots, w_{a-2}$ and joining each w_i to v_i for $1 \leq i \leq a - 2$. Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$.

First, we show that $\gamma(G) = a$. Since $\{u, v, v_1, v_2, \ldots, v_{a-2}\}$ is a dominating set of G, it follows that $\gamma(G) \leq a$. Next, we show that $\gamma(G) \geq a$. Let S be a minimum dominating set of G. Since each vertex w_i $(1 \leq i \leq a-2)$ is only dominated by itself or by v_i, it follows that S must contain at least one vertex from each set $\{v_i, w_i\}$ for $1 \leq i \leq a-2$. Also, each vertex u_i $(1 \leq i \leq b-a)$ is only dominated by u_i or by u. Thus, either $u_i \in S$ for all i with $1 \leq i \leq b-a$ or $u \in S$. Furthermore, each vertex v_j $(a-1 \leq j \leq a+b-1)$ is only dominated by v_j or by v. Thus either $v_j \in S$ for each j with $a-1 \leq j \leq a+b-1$ or $v \in S$. This implies that $\gamma(G) = |S| \geq (a-2) + 2 = a$.

Next, we show that $\gamma_F(G) = b$. Let $S_0 = U \cup \{u, v_{a+b-1}\} \cup W$. Then $|S_0| = (b-a) + 2 + (a-2) = b$. Since the red-blue coloring that assigns red to each vertex in S_0 and blue to the remaining vertices of G is an F-coloring with b red vertices, $\gamma_F(G) \leq b$. To show that $\gamma_F(G) \geq b$, let c be a minimum F-coloring of G. We make four observations: (1) By Lemma 2.1, $W \subset R_c$. (2) If v is colored red by c, then v_i is colored red by c for $1 \leq i \leq a+b-1$ and so $\gamma_F(G) \geq a+b-1 > b$, a contradiction. Thus v must be colored blue by c. (3) If u is colored blue by c, then u is not F-dominated by any red vertex in R_c since u is only F-dominated by v and v is colored blue by c as shown in (2). Thus, u must be colored red by c. (4) By (3), each vertex $u_i \in R_c$ for $1 \leq i \leq b-a$. It then follows by (1)-(4) that $U \cup \{u\} \cup W \subseteq R_c$ and so $|R_c| \geq (b-a) + 1 + (a-2) = b-1$. Assume, to the contrary, that $\gamma_F(G) = b - 1$. Then $R_c = U \cup \{u\} \cup W$. However then v_{a+b-1} is not F-dominated by any vertex in R_c, which is a contradiction. Therefore, $\gamma_F(G) \geq b$ and so $\gamma_F(G) = b$.

Next, we verify (ii). First, suppose that $a = 1$. If $b = 1$, let $H = K_3$ and so $\gamma_F(H) = \gamma(H) = 1$. Thus we may assume that $b \geq 2$.

For the integer $b \geq 2$ and $1 \leq i \leq b$ let F_i be a copy of $K_4 - e$ with $V(F_i) = \{u_i, v_i, x_i, y_i\}$ such that $\deg u_i = \deg v_i = 2$ and $\deg x_i = \deg y_i = 3$. The graph H is obtained from the graphs F_i $(1 \leq i \leq b)$ by identifying all the vertices u_i and calling the new vertex u. We first show that $\gamma(H) = b$. Since $S_0 = \{x_i : 1 \leq i \leq b\}$ is a dominating set of H, it follows that $\gamma(H) \leq |S_0| = b$. On the other hand, since v_i is only adjacent to x_i and y_i for each i $(1 \leq i \leq b)$, at least one vertex in each set $\{x_i, y_i, v_i\}$ must belong to any dominating set of H and so $\gamma(H) \geq b$. Next we show that $\gamma_F(H) = 1$. Since the red-blue coloring that assigns red to vertex u and blue to the remaining vertices of H is an F-coloring, by (1) it follows that $\gamma_F(H) = 1$.

Now let $a \geq 2$. By Proposition 3.1, the result holds if $a = b$. Thus we may assume that $a < b$. Thus $b - a > 0$ and so $b - a + 2 \geq 3$. We start with the graph $W_{b-a+2} = C_{b-a+2} + K_1$, where $C_{b-a+2} : y_1, y_2, \ldots, y_{b-a+2}, y_1$, and x is the vertex of degree $b - a + 2$ in W_{b-a+2}. For each i with $1 \leq i \leq a - 1$, let $F_i : s_i, t_i$ be a copy of P_2. Then the graph H is obtained from the graphs F_i $(1 \leq i \leq a-1)$ and W_{b-a+2} by (1) adding $a-1$ new edges $s_i y_1$ $(1 \leq i \leq a - 1)$ and (2) adding $b - a + 1$ new vertices $z_2, z_3, \ldots, z_{b-a+2}$ and joining each vertex z_j to y_j for $2 \leq j \leq b - a + 2$. Let $T = \{t_1, t_2, \ldots, t_{a-1}\}$.

We first show that $\gamma_F(H) = a$. Since the red-blue coloring that assigns red to each vertex in $\{x\} \cup T$ and blue to the remaining vertices of H is an F-coloring with a red vertices, $\gamma_F(H) \leq a$. To show that $\gamma_F(H) \geq a$, let c be a minimum F-coloring of H. By Lemma 2.1, we have $T \subseteq R_c$ and so $\gamma_F(H) \geq a - 1$. Assume, to the contrary, that $\gamma_F(H) = a - 1$. Hence $R_c = T$. However then, no blue vertex (different from y_1) is F-dominated by any vertex in R_c, which is a contradiction. Thus $\gamma_F(H) = a$.

Next we show that $\gamma(H) = b$. Since $S_0 = \{s_1, s_2, \ldots, s_{a-1}, y_2, \ldots, y_{b-a+2}\}$ is a dominating set of H, it follows that $\gamma(H) \leq |S_0| = b$. To show that $\gamma(H) \geq b$, let S be a minimum dominating set of H. Then S contains at least one vertex from each set $\{s_i, t_i\}$ for $1 \leq i \leq a - 1$ and at least one vertex from each set $\{y_j, z_j\}$ for $2 \leq j \leq b - a + 2$. Thus $\gamma(H) = |S| \geq (a - 1) + (b - a + 1) = b$. Therefore, $\gamma(H) = b$. ∎

Next, we show that every pair a, b of positive integers can be realizable as the open domination number and F-domination number of some connected graph.

Theorem 3.3. *Let a and b be positive integers with $a \leq b$.*

(i) *For $a \geq 2$, there exists a connected graph G such that $\gamma_o(G) = a$ and $\gamma_F(G) = b$;*

(ii) *For $b \geq 2$, there exists a connected graph H such that $\gamma_F(H) = a$ and $\gamma_o(H) = b$.*

Proof. By Proposition 3.1, the result holds if $a = b$. Thus we may assume that $a < b$. We first verify (i). We consider three cases.

Case 1. $a = 2$. Let $G = K_{1,b-1}$. Then $\gamma_o(G) = 2$ and $\gamma_F(G) = b$.

Case 2. $a = 3$. Let G be the graph in the Proof of Case 2 for Theorem 2.7, where $k = b$. Then $\gamma_o(G) = 3$ and $\gamma_F(G) = k = b$.

Case 3. $a \geq 4$. We begin with a double star T whose central vertices are u and v. Let $U = \{u_1, u_2, \ldots, u_{b-a+1}\}$ and $V = \{v_1, v_2, \ldots, v_{a+b-1}\}$ be the sets of vertices of T such that u is adjacent to every vertex in U and v is adjacent to every vertex in V. Then the graph G is obtained from T by (1) subdividing the edge uv with a new vertex x and (2) adding $a-3$ new vertices $w_1, w_2, \ldots, w_{a-3}$ and joining each w_i to v_i for $1 \leq i \leq a-3$. Let $W = \{w_1, w_2, \ldots, w_{a-3}\}$.

First, we show that $\gamma_o(G) = a$. Since $S_0 = \{u, x, v, v_1, v_2, \ldots, v_{a-3}\}$ is an open dominating set of G, it follows that $\gamma_o(G) \leq |S_0| = a$. To show that $\gamma_o(G) \geq a$, let S be a minimum open dominating set of G. Since each w_i $(1 \leq i \leq a-3)$ is only openly dominated by v_i $(1 \leq i \leq a-3)$ and each u_i $(1 \leq i \leq b-a+1)$ is only openly dominated by u, we have $v_i \in S$ for $1 \leq i \leq a-3$ and $u \in S$. Similarly, since each v_j $(a-2 \leq j \leq a+b-1)$ is only openly dominated by v, we have $v \in S$. Thus $\gamma_o(G) = |S| \geq a-1$. Assume, to the contrary, that $\gamma_o(G) = a-1$. However then, $S = \{u, v, v_1, v_2, \ldots, v_{a-3}\}$ and u is not openly dominated by any vertex in S, which is a contradiction. Therefore, $\gamma_o(G) \geq a$ and so $\gamma_o(G) = a$.

Next, we show that $\gamma_F(G) = b$. Let $S' = U \cup \{u\} \cup W \cup \{v_{a+b-1}\}$. By the proof of Theorem 3.2(i), the red-blue coloring that assigns red to each vertex of S' and blue to the remaining vertices of G is a minimum F-coloring with b red vertices. Therefore, $\gamma_F(G) = b$.

Next, we verify (ii). First, suppose that $a = 1$ and $b \geq 2$. Let H be the graph in Theorem 3.2(ii). So for the integer $b \geq 2$ and $1 \leq i \leq b-1$, let F_i be a copy of $K_4 - e$ with $V(F_i) = \{u_i, v_i, x_i, y_i\}$ such that $\deg u_i = \deg v_i = 2$ and $\deg x_i = \deg y_i = 3$. The graph H is obtained from the graphs F_i $(1 \leq i \leq b-1)$ by identifying all the vertices u_i and calling the new vertex u.

We first show that $\gamma_o(H) = b$. Since $S_0 = \{u\} \cup \{x_i : 1 \leq i \leq b-1\}$ is an open dominating set in H, it follows that $\gamma_o(H) \leq |S_0| = b-1$. On the other hand, since v_i is only adjacent to x_i and y_i for each i $(1 \leq i \leq b-1)$, at least one vertex in each set $\{x_i, y_i, v_i\}$ must belong to any open dominating set of H and so $\gamma_o(H) \geq b-1$. Assume, to the contrary, that $\gamma_o(H) = b-1$. Then

$$S = \{w_i : 1 \leq i \leq b-1\} \subseteq \{x_i, y_i, v_i : 1 \leq i \leq b-1\},$$

where $w_i \in \{x_i, y_i, v_i\}$ for each i with $1 \leq i \leq b-1$. However then, w_i is not openly dominated by any vertex in S_o, which is a contradiction. Thus $\gamma_o(H) \geq b$. Next we show that $\gamma_F(H) = 1$. Since the red-blue coloring that assigns red to vertex u and blue to the remaining vertices of H is an F-coloring, it follows by (1) that $\gamma_F(H) = 1$.

Now let $2 \leq a < b$. We consider two cases.

Case 1. $b = a+1$. For each integer i with $1 \leq i \leq a-1$, let $F_i : u_i, v_i, w_i$ be a copy of the path P_3 and let $C_3 : x, y, z, x$ be a copy of a 3-cycle. Then the graph H is obtained from the graphs F_i $(1 \leq i \leq a-1)$ and C_3 by (1) identifying the vertices u_i $(1 \leq i \leq a-1)$ and calling the new vertex u and (2) joining the vertex u to x.

We first show that $\gamma_F(H) = a$. Let $S_0 = \{x\} \cup \{w_i : 1 \leq i \leq a-1\}$. Since the red-blue coloring that assigns red to each vertex in S_0 and blue to the remaining vertices of H is an F-coloring with a red vertices, $\gamma_F(H) \leq a$.

To show that $\gamma_F(H) \geq a$, let c be a minimum F-coloring of H. By Lemma 2.1, each end-vertex in w_i $(1 \leq i \leq a-1)$ must be colored red by c and so $\gamma_F(H) \geq a-1$. Assume to the contrary that $\gamma_F(H) = a-1$. Then $R_c = \{w_i : 1 \leq i \leq a-1\}$. However then, y is not F-dominated by any vertex in R_c, which is a contradiction. Thus $\gamma_F(H) \geq a$.

Next we show that $\gamma_o(H) = a+1 = b$. Since $S_1 = \{u, x\} \cup \{v_i : 1 \leq i \leq a-1\}$ is an open dominating set in H, it follows that $\gamma_o(H) \leq |S_1| = a+1$. On the other hand, since w_i $(1 \leq i \leq a-1)$ is only openly dominated by v_i, it follows that $v_i \in S$ for all $1 \leq i \leq a-1$ and so $\gamma_o(H) \geq a-1$.

Also, since each v_i is only openly dominated by w_i or by u, it follows that either $w_i \in S$ $(1 \le i \le a-1)$ or $u \in S$. This implies that $\gamma_o(H) \ge (a-1) + 1 = a$. Assume, to the contrary, that $\gamma_o(H) = a$. Let S be a minimum open dominating set of H. Then $S \subset (\{u\} \cup \{v_i, w_i : 1 \le i \le a-1\})$. However then, y is not openly dominated by any vertex of S, which is a contradiction. Hence $\gamma_o(H) \ge a+1$.

Case 2. $b \ge a+2$. Then $b-a+1 \ge 3$. We start with the graph $W_{b-a+1} = C_{b-a+1} + K_1$, where $C_{b-a+1} : y_1, y_2, \ldots, y_{b-a+1}, y_1$ and x is the vertex of degree $b-a+1$ in W_{b-a+1}. For each i with $1 \le i \le a-1$, let $F_i : s_i, t_i$ be a copy of P_2. Then the graph H is obtained from the graphs F_i $(1 \le i \le a-1)$ and W_{b-a+1} by (1) adding $a-1$ new edges $s_i y_1$ $(1 \le i \le a-1)$ and (2) adding $b-a$ new vertices z_2, z_3, $\ldots$, z_{b-a+1} and joining each z_i $(2 \le i \le b-a+1)$ with y_i. Let $T = \{t_1, t_2, \ldots, t_{a-1}\}$, $S = \{s_1, s_2, \ldots, s_{a-1}\}$, and $Y = \{y_1, y_2, y_3, \ldots, y_{b-a+1}\}$. The red-blue coloring that assigns red to each vertex of the set $\{x\} \cup T$ and blue to the remaining vertices of H is a minimum F-coloring with a red vertices. Therefore, $\gamma_F(H) = a$.

Next we show that $\gamma_o(H) = b$. Since $S \cup Y$ is an open dominating set of H, it follows that $\gamma_o(H) \le |S \cup Y| = (a-1) + (b-a+1) = b$, To show that $\gamma_o(H) \ge b$, observe that every open dominating set of H contains $S \cup (Y - \{y_1\})$. On the other hand, s_1 is not openly dominated by any vertex in $S \cup (Y - \{y_1\})$ and so $S \cup (Y - \{y_1\})$ is not an open dominating set of H. Therefore, $\gamma_o(H) \ge |S \cup (Y - \{y_1\})| + 1 = (a-1) + (b-a) + 1 = b$. Therefore, $\gamma_o(H) = b$. ■

Recall that for a graph G without isolated vertices, the following are possible:

(1) $\gamma(G) \le \gamma_F(G) \le \gamma_o(G)$,

(2) $\gamma(G) \le \gamma_o(G) \le \gamma_F(G)$,

(3) $\gamma_F(G) \le \gamma(G) \le \gamma_o(G)$.

Also, we have seen that every pair a, b of positive integers can be realizable as the domination number and F-domination number of some connected graph or the open domination number and F-domination number of some connected graph. This gives rise to the following natural question.

Problem 3.4. For which triples a, b, c of positive integers with $a \le b \le 2a$ and $b \ge 2$, does there exist a connected graph G such that $\gamma(G) = a$, $\gamma_o(G) = b$, and $\gamma_F(G) = c$?

Although it is not known whether *every* triple a, b, c in Problem 3.4 is realizable as the domination, open domination, and F-domination number of some connected graph, it can be shown that there are infinitely many such realizable triples. As an example, we present the following.

Theorem 3.5. *For each pair a, b of integers with $1 \le a \le b \le 2a$ and $b \ge 2$, there exists a connected graph G with $\gamma_F(G) = 1$ such that $\gamma(G) = a$ and $\gamma_o(G) = b$.*

Proof. For $a = 1$ and $b = 2$, let $G = K_3$ and so $\gamma_F(G) = \gamma(G) = 1$ and $\gamma_o(G) = 2$. Thus we may assume that $a \ge 2$. We consider two cases, according to whether $a = b$ or $a \ne b$.

Case 1. $a = b$. If $a = b = 2$, let G be the graph obtained from the graph $K_4 - e$ by adding a new vertex and joining this new vertex to a vertex of degree 2 in $K_4 - e$. Then $\gamma_F(G) = 1$ and $\gamma(G) = \gamma_o(G) = 2$. Now let $a = b \ge 3$. Let $s > a$ be an integer and consider the graph $P_s + K_1$, where $P_s : u_1, u_2, \cdots, u_s$ and u is the vertex in $P_s + K_1$ with $\deg u = s$. Then the graph G is obtained from $P_s + K_1$ by adding $a - 1$ new vertices $v_1, v_2, \cdots, v_{a-1}$ and joining each v_i to u_i for $1 \le i \le a - 1$. Since $N(u)$ is an open dominating set of G, it follows by Theorem 2.3 that $\gamma_F(G) = 1$. Since $\{u, u_1, u_2, \cdots, u_{a-1}\}$ is a minimum dominating and minimum open dominating set of G, it follows that $\gamma(G) = \gamma_o(G) = a$.

Case 2. $a < b \le 2a$. We consider three subcases.

Subcase 2.1. $a < b \le 2a - 2$. Let $b = a + k$, where $k \ge 1$, and let $\ell = a - k - 1$. Since $b = a + k \le 2a - 2$, it follows that $\ell \ge 1$. Consider the graphs H_1 and H_2 in Figure 5.

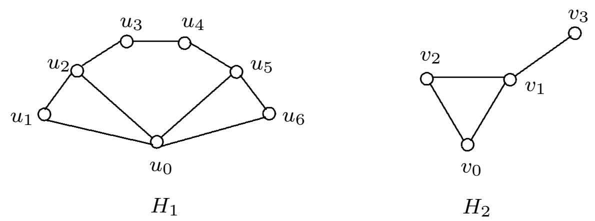

Figure 5. The graphs H_1 and H_2 in Case 2.

For each i with $1 \le i \le k$, let F_i be a copy of H_1, where $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \cdots, u_{i,6}\}$, where $u_{i,p}$ corresponds to u_p in H_1 for $0 \le p \le 6$. For each j with $1 \le j \le \ell$, let G_j be a copy of H_2 with $V(G_j) = \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}$, where $v_{j,q}$ corresponds to v_q in H_2 for $0 \le q \le 3$. The graph G is then obtained from the graphs F_i and G_j for $1 \le i \le k$ and $1 \le j \le \ell$ by identifying all vertices $u_{i,0}$ and $v_{j,0}$ and labeling the identified vertex v. Observe that $N(v)$ is an open dominating set of G. Thus $\gamma_F(G) = 1$ by Theorem 2.3. Also, since

$$S = \{v\} \cup \{u_{i,3} : 1 \le i \le k\} \cup \{v_{j,1} : 1 \le j \le \ell\}$$

is a minimum dominating set of G, it follows that $\gamma(G) = |S| = 1 + k + \ell = 1 + k + (a - k - 1) = a$. Furthermore, the set $S_o = S \cup \{u_{i,2} : 1 \le i \le k\}$ is a minimum open dominating set of G and so $\gamma_o(G) = |S_o| = |S| + k = a + k = b$.

Subcase 2.2. $b = 2a - 1$. For each i with $1 \le i \le a - 1$, let F_i be a copy of H_1 in Figure 5 such that $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \cdots, u_{i,6}\}$, where $u_{i,p}$ corresponds to u_p in H_1 for $0 \le p \le 6$. Then the graph G is obtained from the graphs F_i, $1 \le i \le a-1$, by identifying all vertices $u_{i,0}$ and labeling the identified vertex by v. Again, $\gamma_F(G) = 1$ and $\{v\}$ is the minimum F-dominating set. Since $S = \{v\} \cup \{u_{i,3} : 1 \le i \le a - 1\}$ is a minimum dominating set of G and $S \cup \{u_{i,2} : 1 \le i \le a - 1\}$ is a minimum open dominating set of G, it follows that $\gamma(G) = |S| = a$ and $\gamma_o(G) = |S| + (a - 1) = 2a - 1 = b$.

Subcase 2.3. $b = 2a$. If $a = 1$ and $b = 2$, then the graph H_2 of Figure 5 has the desired property. Thus we may assume that $a \ge 2$. Let $p \ge 2$ be an integer. For each integer i with $1 \le i \le a - 1$, let F_i be the graph obtained from the path u_i, y_i, v_i by adding $2p$ new vertices $r_{i,j}$ $(1 \le j \le 2p)$ and joining (1) each vertex $r_{i,j}$ $(1 \le j \le p)$ to u_i and y_i and (2) each vertex $r_{i,j}$ $(p + 1 \le j \le 2p)$ to y_i and v_i. Then the graph G is obtained from the $a-1$ graphs F_i $(1 \le i \le a-1)$ and the path $P : z, w, x, w'$ of order 4 by (1) adding the edge xz and (2) joining each of the two vertices w and z to each vertex in $\{u_i, v_i\}$ for $1 \le i \le a - 1$. The graph G is shown in Figure 6 for $a = 3$. Since $N(w) = \{x, z\} \cup \{u_i, v_i : 1 \le i \le a - 1\}$ is an open dominating set of G, it follows by Theorem 2.3 that $\gamma_F(G) = 1$. It remains to show that $\gamma(G) = a$ and $\gamma_o(G) = b$.

We first show that $\gamma(G) = a$. Since the set $\{x\} \cup \{y_i : 1 \le i \le a - 1\}$ is a dominating set of G, it follows that $\gamma(G) \le a$. On the other hand, let S

be a minimum dominating set of G. For each integer i with $1 \leq i \leq a-1$, let $R_i = \{r_{i,j} : 1 \leq j \leq p\}$ and $R'_i = \{r_{i,j} : p+1 \leq j \leq 2p\}$ Observe that

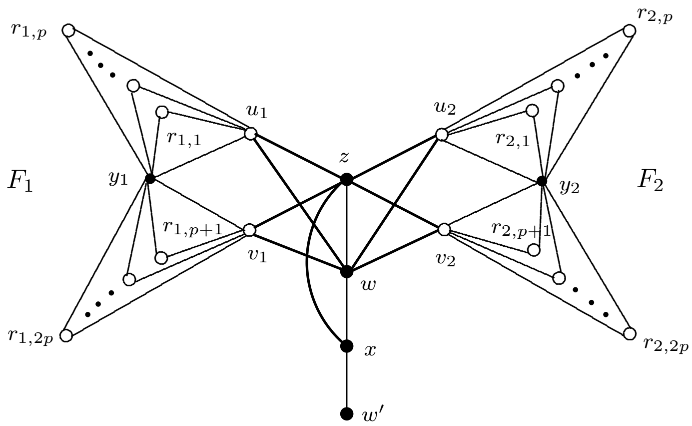

Figure 6. The graph G in Subcase 2.3 for $a = 3$.

(b1) since w' is an end-vertex of G and w' is adjacent to x, the set S contains at least one vertex in $\{w', x\}$, and

(b2) since y_i is only dominated by a vertex in the set

$$A_i = \{u_i, v_i, y_i\} \cup R_i \cup R'_i, \tag{3}$$

the set S must contain at least one vertex in each set A_i.

The a sets $\{w', x\}$ and A_i $(1 \leq i \leq a-1)$ are pairwise disjoint. It then follows by (b1) and (b2) that S contains at least a distinct vertices of G and so $\gamma(G) \geq a$. Therefore, $\gamma(G) = a$.

Next, we show that $\gamma_o(G) = b$. Since $N(w)$ is an open dominating set of G, it follows that $\gamma_o(G) \leq |N(w)| = 2a$. On the other hand, let S_o be a minimum open dominating set of G. First, we verify the following claim.

Claim. For each integer i with $1 \leq i \leq a-1$, the set S_o must contain at least two vertices in each set A_i in (3).

Proof of Claim. Assume, to the contrary, that S_o contains at most one vertex in A_i for some i with $1 \leq i \leq a-1$. Observe that each vertex in R_i is only openly dominated by a vertex in $B_i = \{u_i, y_i\}$ and so S_o must contain at least one vertex in B_i. Similarly, each vertex in R_i' is only openly dominated by a vertex in $C_i = \{v_i, y_i\}$ and so S_o must contain at least one vertex in C_i. Since $B_i \cup C_i = \{u_i, v_i, y_i\} \subseteq A_i$, it follows that S_o contains at least one vertex in A_i. Hence S_o contains exactly one vertex in A_i. Because $B_i \cap C_i = \{y_i\}$, the vertex y_i is the only vertex of A_i that belongs to S_o. However, y_i is only openly dominated by a vertex in $A_i - \{y_i\}$, implying that y_i is not openly dominated by any vertex in S_o, which is a contradiction.

This completes the proof of the claim. Therefore, S_o must contain at least two vertices in each set A_i for $1 \leq i \leq a-1$. Moreover, the end-vertex w' is only openly dominated by x and x is only openly dominated by a vertex in the set $V(P) - \{x\} = \{w, w', z\}$. Thus S_o must contain at least two vertices in $V(P)$. Since the a subsets $V(P)$ and A_i $(1 \leq i \leq a-1)$ of $V(G)$ are pairwise disjoint, S_o contains at least $2a$ distinct vertices of G and so $\gamma_o(G) = |S_o| \geq 2a$. Therefore, $\gamma_o(G) = 2a = b$. ■

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