

## ON STRATIFICATION AND DOMINATION IN GRAPHS

RALUCCA GERA

*Department of Applied Mathematics*  
*Naval Postgraduate School*  
*Monterey, CA 93943–5216, USA*

**e-mail:** RGera@nps.edu

AND

PING ZHANG

*Department of Mathematics*  
*Western Michigan University*  
*Kalamazoo, MI 49008, USA*

**e-mail:** ping.zhang@wmich.edu

### Abstract

A graph  $G$  is 2-stratified if its vertex set is partitioned into two classes (each of which is a stratum or a color class), where the vertices in one class are colored red and those in the other class are colored blue. Let  $F$  be a 2-stratified graph rooted at some blue vertex  $v$ . An  $F$ -coloring of a graph is a red-blue coloring of the vertices of  $G$  in which every blue vertex  $v$  belongs to a copy of  $F$  rooted at  $v$ . The  $F$ -domination number  $\gamma_F(G)$  is the minimum number of red vertices in an  $F$ -coloring of  $G$ . In this paper, we study  $F$ -domination, where  $F$  is a 2-stratified red-blue-blue path of order 3 rooted at a blue end-vertex. We present characterizations of connected graphs of order  $n$  with  $F$ -domination number  $n$  or 1 and establish several realization results on  $F$ -domination number and other domination parameters.

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## 1. Introduction

Dividing the vertex set of a graph into classes according to some prescribed rule is a fundamental process in graph theory. The vertices of a graph can be divided into cut-vertices and non-cut-vertices. Equivalently, the vertices of a tree are divided into its leaves and non-leaves. The set of vertices of a graph is partitioned according to the degrees of its vertices. When studying distance, the vertices of a connected graph are partitioned according to their eccentricities. Also, in a connected rooted graph, the vertices are partitioned according to their distance from the root. Perhaps the best known example of this process, however, is graph coloring, where the vertex set of a graph is partitioned into classes each of which is independent in the graph.

A typical Very Large Scale Integrated (VLSI) Circuit chip consists of millions of transistors assembled through layering of various materials in a silicon base. In recent years, advances in VLSI fabrication technology have made it possible to use more than two routing layers for interconnection. In fact, the most popular processors on the market today use three or more layers. In the design of algorithms to solve the multilayer routing problems encountered in this process, it is desirable to use graphs in which the vertices are partitioned into classes. In VLSI design, the design of computer chips often yields a division of the nodes into several layers each of which must induce a planar subgraph. So here too the vertex set of a graph is divided into classes. Motivated by these observations, Rashidi [10] defined a graph  $G$  to be a stratified graph if its vertex set is partitioned into classes. He studied a number of problems involving stratified graphs; while distance in stratified graphs was investigated in [2, 5].

Formally then, a graph  $G$  whose vertex set has been partitioned is called a *stratified graph*. If  $V(G)$  is partitioned into  $k$  subsets, then  $G$  is a *k-stratified graph*. The  $k$  subsets are called the *strata* or *color classes* of  $G$ . Suppose that the vertex set of a  $k$ -stratified graph  $G$  is partitioned into  $k$  subsets  $V_1, V_2, \dots, V_k$ . Unlike vertex coloring, no condition is placed on the subsets  $V_i$ ,  $1 \leq i \leq k$ . If  $G$  is 2-stratified, then we commonly color the vertices of one color class red and color the vertices of the other color class blue. The 2-stratified graphs were first studied from the point of view of domination by Chartrand, Haynes, Henning, and Zhang in [3, 4]. We refer to the book [6] for graph-theoretic notation and terminology not described in this paper.

Let  $F$  be a 2-stratified graph. So each vertex of  $F$  is colored red or blue and there is at least one vertex of each color. Designate a blue vertex  $v$  of  $F$  as the “root” of  $F$ . Then  $F$  is said to be *rooted at  $v$* . For example, two distinct 2-stratified graphs  $F$  and  $F'$  rooted at a blue vertex  $v$  are shown in Figure 1, where the shaded vertices are red vertices and the non-shaded vertices are blue vertices in each graph.

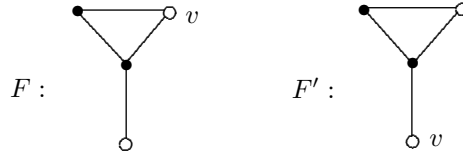


Figure 1. Two 2-stratified graphs  $F$  and  $F'$ .

For a connected graph  $G$ , a *red-blue coloring* of  $G$  is a coloring of  $G$  in which every vertex is colored red or blue. It is acceptable if all vertices of  $G$  are assigned the same color. If there is at least one vertex of each color, then the red-blue coloring produces a 2-stratification of  $G$ . By an  *$F$ -coloring* of a graph  $G$ , we mean a red-blue coloring of  $G$  such that for every blue vertex  $w$  of  $G$ , there is a copy of  $F$  in  $G$  with  $v$  at  $w$ . That is, for every blue vertex  $w$  of  $G$ , there exists a 2-stratified subgraph  $F'$  of  $G$  containing  $w$  and a color-preserving isomorphism  $\alpha$  from  $F$  to  $F'$  such that  $\alpha(v) = w$ . The red-blue coloring of  $G$  in which every vertex is colored red is vacuously an  $F$ -coloring for every 2-stratified rooted graph  $F$ . In an  $F$ -coloring of a graph  $G$ , if a blue vertex  $w$  of  $G$  belongs to a copy  $F'$  of  $F$  rooted at  $w$  and  $u$  is a red vertex in  $F'$ , then  $w$  is said to be  *$F$ -dominated* by  $u$ . If  $c$  is an  $F$ -coloring of  $G$ , then the set  $R_c$  of all red vertices of  $G$  is called an  *$F$ -dominating set* of  $G$ .

To illustrate these concepts, consider the 2-stratified graph  $F$  and the graph  $G$  of Figure 2. The red-blue coloring of  $G$  given in Figure 2 is an  $F$ -coloring of  $G$  since every blue vertex of  $G$  belongs to a copy of  $F$  rooted at that vertex. For example, the blue vertex  $w$  of  $G$  belongs to a copy  $F'$  of  $F$  rooted at  $w$ , where  $V(F') = \{w, x, y, z\}$ . Since  $x$  and  $y$  are red vertices in  $F'$  and  $F'$  is rooted at  $w$ , it follows that  $w$  is  $F$ -dominated by each of  $x$  and  $y$ .

For every 2-stratified graph  $F$  and every graph  $G$ , the red-blue coloring of  $G$  in which every vertex of  $G$  is colored red is an  $F$ -coloring of  $G$  and so it is always possible to give an  $F$ -coloring of  $G$ . The  *$F$ -domination number*

$\gamma_F(G)$  of  $G$  was introduced in [3] as the minimum number of red vertices of  $G$  in an  $F$ -coloring of  $G$ . By a *minimum  $F$ -coloring* of  $G$ , we mean an  $F$ -coloring having a minimum number of red vertices, that is,  $\gamma_F(G)$  red vertices. In fact, the  $F$ -coloring of the graph  $G$  in Figure 2 is a minimum  $F$ -coloring. Therefore,  $\gamma_F(G) = 4$  for the graph  $G$  of Figure 2. The  $F$ -domination number was introduced and studied in [3, 4] for 2-stratified graphs.

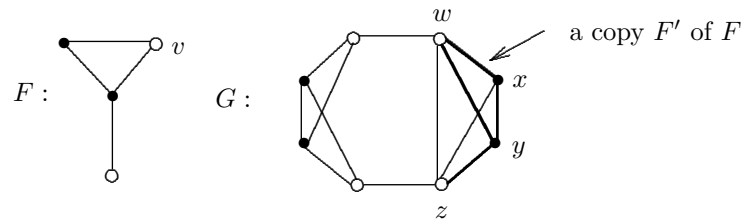


Figure 2. An  $F$ -coloring of a graph.

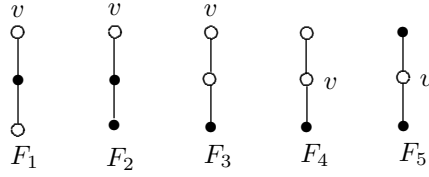
Another closely related concept concerns domination in graphs. A vertex is said to *dominate* itself and each vertex adjacent to it. A set  $S$  of vertices in a graph  $G$  is called a *dominating set* for  $G$  if every vertex of  $G$  is dominated by some vertex in  $S$ . The *domination number*  $\gamma(G)$  of the graph  $G$  is the minimum number of vertices in a dominating set for  $G$ . A dominating set of cardinality  $\gamma(G)$  is called a *minimum dominating set*. The following result appeared in [3].

**Theorem A.** *If  $F$  is a 2-stratified  $K_2$ , then  $\gamma_F(G) = \gamma(G)$  for every graph  $G$ .*

Thus domination can be interpreted as a restricted 2-stratification or 2-coloring, with the red vertices forming the dominating set. In fact,  $F$ -domination generalizes not only ordinary domination but other types of domination that have been previously studied as we describe next. Let  $F$  be a 2-stratified  $P_3$  rooted at a blue vertex  $v$ . The five possible choices for the graph  $F$  are shown in Figure 3.

For the 2-stratified graphs  $F_1$ ,  $F_2$ ,  $F_4$ , and  $F_5$  of Figure 3, the following results are established in [3].

**Theorem B.** *If  $G$  is a connected graph of order at least 3, then  $\gamma_{F_1}(G) = \gamma(G)$ .*

Figure 3. The five 2-Stratified graphs  $P_3$  rooted at  $v$ .

A vertex  $v$  in a graph  $G$  openly dominates each of its neighbors. That is,  $v$  dominates the vertices in its neighborhood  $N(v)$  but not itself. A set  $S$  of vertices in a graph  $G$  is an *open dominating set* if every vertex of  $G$  is adjacent to at least one vertex of  $S$ . In this case, a vertex  $v$  in an open dominating set of  $G$  is said to *openly dominate* its neighbors but not itself. The minimum cardinality of an open dominating set is the *open domination number*  $\gamma_o(G)$  of  $G$ . An open dominating set of cardinality  $\gamma_o(G)$  is a *minimum open dominating set* or a  $\gamma_o$ -set for  $G$  (see [7]).

**Theorem C.** *If  $G$  is a graph without isolated vertices, then  $\gamma_{F_2}(G) = \gamma_o(G)$ .*

An  $F_4$ -coloring of  $G$  requires that every blue vertex of  $G$  is adjacent to both a red and a blue vertex, while  $\gamma_{F_4}(G)$  is the minimum number of red vertices required in such a 2-stratification of  $G$ . Thus,  $\gamma_{F_4}(G)$  is the known domination parameter called the *restrained domination number*  $\gamma_r(G)$ . A set  $S \subseteq V(G)$  is a *restrained dominating set* if every vertex not in  $S$  is adjacent to a vertex in  $S$  and to a vertex in  $V(G) - S$ . Every graph has a restrained dominating set since  $V(G)$  is such a set. The *restrained domination number*  $\gamma_r(G)$  is the minimum cardinality of a restrained dominating set of  $G$  (see [8]).

**Theorem D.** *For every graph  $G$ ,  $\gamma_{F_4}(G) = \gamma_r(G)$ .*

An  $F_5$ -coloring of  $G$  requires that every blue vertex of  $G$  is adjacent to (at least) two red vertices, while  $\gamma_{F_5}(G)$  is the minimum number of red vertices required in such a 2-stratification of  $G$ . Thus,  $\gamma_{F_5}(G)$  is the well-known domination parameter called the *2-domination number*  $\gamma_2(G)$  (see Jacobson [9]). A set  $S \subseteq V(G)$  is a *k-dominating set* if every vertex not in  $S$  is adjacent to at least  $k$  vertices in  $S$ . The *k-domination number* of  $G$ , denoted by  $\gamma_k(G)$ , is the minimum cardinality of a  $k$ -dominating set of  $G$ .

**Theorem E.** *For every graph  $G$ ,  $\gamma_{F_5}(G) = \gamma_2(G)$ .*

While the parameters  $\gamma_{F_1}(G)$ ,  $\gamma_{F_2}(G)$ ,  $\gamma_{F_4}(G)$ , and  $\gamma_{F_5}(G)$  are well-known domination parameters, the parameter  $\gamma_{F_3}(G)$  is new, according to the discussion in [3]. Thus, we investigate  $F_3$ -domination. To simplify notation, we denote  $F_3$  by  $F$  in this paper.

## 2. Graphs with Prescribed Order and $F$ -domination Number

Since the 2-stratified graph  $F = F_3$  contains exactly one red vertex,

$$(1) \quad 1 \leq \gamma_F(G) \leq n$$

for every connected graph  $G$  of order  $n$ . In this section, we first present characterizations of connected graphs of order  $n$  with  $F$ -domination number  $n$  or 1. In order to do this, we present two lemmas, whose routine proofs are omitted.

**Lemma 2.1.** *Let  $v$  be an end-vertex of a connected graph  $G$ . Suppose that  $v$  is adjacent to the vertex  $u$  in  $G$ .*

- (a) *If  $\deg u = 2$ , then  $v$  is colored red by any  $F$ -coloring of  $G$ .*
- (b) *If  $u$  is colored red by an  $F$ -coloring  $c$ , then  $v$  is also colored red by  $c$ .*

**Lemma 2.2.** *Let  $G$  be a connected graph that has an  $F$ -coloring. If a blue vertex  $v$  is  $F$ -dominated by a red vertex  $u$  that is adjacent to  $v$ , then  $v$  belongs to a triangle in  $G$  that contains  $u$ . Consequently, in a triangle-free  $G$ , each blue vertex  $v$  can only be  $F$ -dominated by a red vertex that is not adjacent to  $v$ .*

The distance  $d(u, v)$  between two vertices  $u$  and  $v$  in a connected graph  $G$  is the length of a shortest  $u - v$  path in  $G$ . The diameter  $\text{diam}(G)$  of  $G$  is the largest distance between two vertices of  $G$ .

**Theorem 2.3.** *Let  $G$  be a connected graph of order  $n \geq 3$ . Then*

- (a)  $\gamma_F(G) = n$  if and only if  $G = K_{1, n-1}$ ,

- (b)  $\gamma_F(G) = 1$  if and only if  $G$  contains a vertex  $u$  such that  $N(u)$  is an open dominating set of  $G$ . In this case, the red-blue coloring of  $G$  defined by assigning red to  $u$  and blue to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$ .

**Proof.** We first verify (a). Let  $G = K_{1,n-1}$  and let  $x$  be the central vertex of  $G$ . Suppose that  $c$  is an arbitrary  $F$ -coloring of  $G$ . Since there is no  $x - y$  path of length 2 in  $G$  for any vertex  $y$  in  $G$ , the vertex  $x$  must be colored red by  $c$ . However, then every end-vertex of  $G$  must also be colored red by  $c$  by Lemma 2.1. Hence every vertex of  $G$  is colored red by  $c$  and so  $\gamma_F(G) = n$ .

For the converse, let  $G \neq K_{1,n-1}$  be a connected graph of order  $n \geq 3$ . If  $G$  is a tree, then  $\text{diam}(G) \geq 3$  and so  $G$  contains a path  $P : v_1, v_2, v_3, v_4$  of length 3. Observe that the red-blue coloring defined by assigning blue to  $v_2$  and  $v_3$  and red to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$  with  $n - 2$  red vertices. Thus  $\gamma_F(G) \leq n - 2$ . If  $G$  is not a tree, then  $G$  contains a  $k$ -cycle  $C : v_1, v_2, \dots, v_k, v_1$ , where  $k \geq 3$ . Note that the red-blue coloring defined by assigning blue to  $v_1$  and  $v_2$  and red to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$  with  $n - 2$  red vertices. Thus  $\gamma_F(G) \leq n - 2$ . Therefore, (a) holds.

Next, we verify (b). First assume that  $\gamma_F(G) = 1$ . Then there is an  $F$ -coloring  $c$  of  $G$  with exactly one red vertex, say  $u$ . We show that  $N(u)$  is an open dominating set of  $G$ . Let  $v \in V(G)$ . Since  $u$  is openly dominated by any vertex in  $N(u)$ , we may assume that  $v \neq u$ . Because  $c$  is an  $F$ -coloring of  $G$ , the blue vertex  $v$  belongs to a copy of  $F$  rooted at  $v$ , that is,  $v$  is adjacent to a blue vertex  $w$  and  $w$  is adjacent to  $u$ . Thus  $v$  is openly dominated by  $w \in N(u)$  and so  $N(u)$  is an open dominating set of  $G$ .

For the converse, assume that  $G$  contains a vertex  $u$  such that  $N(u)$  is an open dominating set. We show that  $\gamma_F(G) = 1$  by showing the red-blue coloring  $c^*$  defined by assigning red to  $u$  and blue to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$ . Let  $v \in V(G) - \{u\}$  be a blue vertex of  $G$ . Since  $N(u)$  is an open dominating set of  $G$ , it follows that  $v$  is openly dominated by some vertex  $w \in N(u)$ . Thus  $v$  is adjacent to a blue vertex  $w$  and  $w$  is adjacent to the red vertex  $u$ , implying that the blue vertex  $v$  belongs to a copy of  $F$  rooted at  $v$ . Therefore,  $c^*$  is an  $F$ -coloring of  $G$  with exactly one red vertex and so  $\gamma_F(G) = 1$ . ■

By Theorem 2.3, if  $G$  is a nontrivial bipartite graph, then  $\gamma_F(G) \geq 2$ . In particular, if  $T$  is a tree of order  $n \geq 3$ , then

$$(2) \quad 2 \leq \gamma_F(T) \leq n,$$

and by Theorem 2.3,  $\gamma_F(T) = n$  if and only if  $T$  is a star. Next, we characterize all trees of order at least 3 with  $F$ -domination number 2. A *double star*  $T$  is a tree of diameter 3. Recall that for an  $F$ -coloring  $c$  of a graph, we let  $R_c$  denote the set of all red vertices of  $c$ .

**Theorem 2.4.** *A tree  $T$  of order at least 3 has  $\gamma_F(T) = 2$  if and only if  $T$  is a double star.*

**Proof.** Suppose that  $T$  is a double star. Let  $s$  and  $t$  be two vertices of  $T$  with  $d(s, t) = \text{diam}(G)$ . The red-blue coloring of  $T$  defined by assigning red to  $s$  and  $t$  and blue to the remaining vertices of  $T$  is an  $F$ -coloring of  $G$  with exactly two red vertices. Thus  $\gamma_F(T) = 2$  by (2).

For the converse, assume that  $T$  is a tree of order  $n \geq 3$  that is not a double star. Then  $\text{diam } T \neq 3$ . If  $\text{diam } T \leq 2$ , then  $T$  is a star and so  $\gamma_F(T) = n \geq 3$ . Thus, we may assume that  $\text{diam } T \geq 4$ . We show that  $\gamma_F(T) \geq 3$ . Assume, to the contrary, that  $\gamma_F(T) \leq 2$ . Then  $\gamma_F(T) = 2$  by (2). Let  $c$  be a minimum  $F$ -coloring of  $T$  with  $R_c = \{x, y\}$ . Let  $d(x, y) = k$  and let  $P : x = x_0, x_1, x_2, \dots, x_k = y$  be an  $x - y$  path of length  $k$  in  $T$ . We consider four cases.

*Case 1.*  $k = 1$ . Then  $x$  and  $y$  are adjacent. Since  $T \neq P_2$ , at least one of  $x$  and  $y$  is not an end-vertex of  $T$ . Assume, without loss of generality, that  $x$  is not an end-vertex of  $T$ , and so  $x$  is also adjacent to a blue vertex  $v$ . Since  $T$  is a tree,  $T$  is triangle-free. It then follows by Lemma 2.2 that  $v$  cannot be  $F$ -dominated by  $x$ . Hence  $v$  is  $F$ -dominated by  $y$ . Thus  $v$  is adjacent to a blue vertex  $v'$  that is adjacent to  $y$ . However then  $x, v, v', y, x$  is a 4-cycle in the tree  $T$ , which is impossible.

*Case 2.*  $k = 2$ . Then the blue vertex  $x_1$  in  $P$  is adjacent to both  $x$  and  $y$ . Since  $x$  and  $y$  are the only red vertices in  $G$ , it follows that  $x_1$  is  $F$ -dominated by a red vertex that is adjacent to  $x_1$ , which contradicts Lemma 2.2.

*Case 3.*  $k = 3$ . Then  $P : x = x_0, x_1, x_2, x_3 = y$  is a path of length 3. Since  $\text{diam } T \geq 4$ , it follows that  $V(T) - V(P) \neq \emptyset$ . We claim that every vertex in  $V(T) - V(P)$  is adjacent to either  $x_1$  or  $x_2$ . Assume, to the contrary,



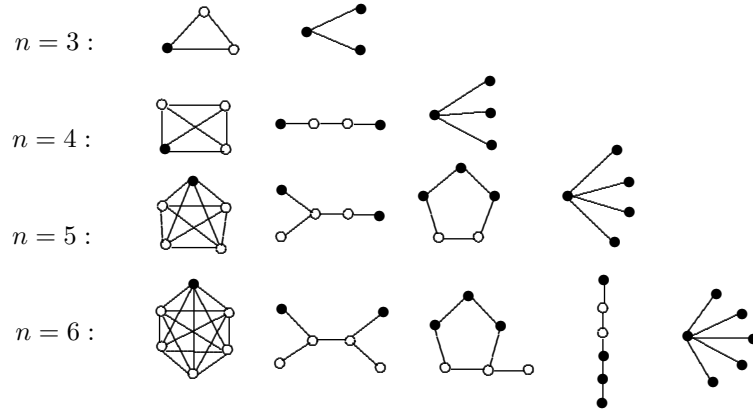
that there is  $v \in V(T) - V(P)$  such that  $v$  is adjacent to neither  $x_1$  nor  $x_2$ . Suppose, without loss of generality, that  $v$  is  $F$ -dominated by  $x$ . Then  $v$  is adjacent to a blue vertex  $v' \notin \{x_1, x_2\}$  that is adjacent  $x$ . Since  $v'$  is adjacent to  $x$ , it follows by Lemma 2.2 that  $v'$  cannot be  $F$ -dominated by  $x$  and so  $v'$  is  $F$ -dominated by  $y$ . Thus  $v'$  is adjacent to a blue vertex  $v''$  that is adjacent to  $y$ . This implies that  $x, v', v'', y, x_2, x_1, x$  is a cycle in the tree  $T$ , which is impossible. Therefore, as claimed, every vertex in  $V(T) - V(P)$  is adjacent to either  $x_1$  or  $x_2$ . Moreover,  $x$  is adjacent to  $x_1$  and  $y$  is adjacent to  $x_2$ . Therefore, every vertex in  $V(T) - \{x_1, x_2\}$  is adjacent to either  $x_1$  or  $x_2$ , implying that  $T$  is a double star with central vertices  $x_1$  and  $x_2$ . This contradicts our assumption that  $T$  is not a double star.

*Case 4.*  $k \geq 4$ . Then  $d(x_1, y) \geq 3$ . On the other hand, since  $x_1$  is adjacent to  $x$ , it follows by Lemma 2.2 that  $x_1$  cannot be  $F$ -dominated by  $x$  and so  $x_1$  is  $F$ -dominated by  $y$ . This implies that  $d(x_1, y) \leq 2$ , which is a contradiction. ■

We have seen in (1) that if  $G$  is a connected graph of order  $n$ , then  $1 \leq \gamma_F(G) \leq n$ . Next, we determine which pairs  $k, n$  of positive integers with  $1 \leq k \leq n$  can be realized as the  $F$ -domination number and the order, respectively, for some connected graph. Certainly, since  $F$  contains two blue vertices, not every such pair is realizable, as we see next.

**Observation 2.5.** There is no connected graph  $G$  of order  $n \geq 3$  with  $\gamma_F(G) = n - 1$ .

On the other hand, for each pair  $k, n$  of integers, where  $1 \leq k \leq n$ ,  $k \neq n - 1$ , and  $3 \leq n \leq 6$ , there exists a connected graph  $G$  of order  $n$  with  $\gamma_F(G) = k$ , as shown in Figure 4. Moreover, if  $G$  is a connected graph of order  $n = 1$  or  $n = 2$ , then  $G$  does not contain  $F$  as a subgraph and so  $\gamma_F(G) = n$ .

Figure 4. Realizable pairs  $k, n$  for  $3 \leq n \leq 6$ .

Thus, if  $1 \leq n \leq 6$ , then, for each pair  $k, n$  of integers with  $1 \leq k \leq n$  and  $k \neq n - 1$ , there exists a connected graph  $G$  of order  $n$  with  $\gamma_F(G) = k$ . However, if  $n \geq 7$ , then  $\gamma_F(G) \neq n - 2$  for every connected graph  $G$  of order  $n$ , as we show next.

**Theorem 2.6.** *There is no connected graph  $G$  of order  $n \geq 7$  with  $\gamma_F(G) = n - 2$ .*

**Proof.** Assume, to the contrary, that there exists a connected graph  $G$  of order  $n \geq 7$  such that  $\gamma_F(G) = n - 2$ . Let  $c$  be a minimum  $F$ -coloring of  $G$  and let  $x$  and  $y$  be the two blue vertices of  $G$ . Necessarily,  $x$  and  $y$  are adjacent in  $G$ . Suppose that  $x$  is  $F$ -dominated by a red vertex  $u$  and  $y$  is  $F$ -dominated by a red vertex  $v$ .

First, we claim that  $u \neq v$ . If  $u = v$ , then  $u, x, y, u$  is a triangle in  $G$ . Since  $G$  is connected and the order of  $G$  is at least 7, there is at least one vertex, say  $w$ , such that  $uw \in E(G)$ . Then the red-blue coloring that assigns blue to  $u, w, y$  and red to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$  with  $n - 3$  red vertices. Thus  $\gamma_F(G) \leq n - 3$ , which is a contradiction. Therefore,  $u \neq v$  and there is a path  $P : u, y, x, v$  in  $G$ , where  $u, v \in R_c$ .

Next, we claim that each vertex in  $R_c - \{u, v\}$  is adjacent to neither  $x$  nor  $y$ . For otherwise, assume that there exists  $w \in R_c - \{u, v\}$  such that  $w$  is adjacent to  $x$  or  $y$ , say the former. Then the red-blue coloring defined by assigning blue to  $x, y, w$  and red the remaining vertices of  $G$  is an  $F$ -coloring with  $n - 3$  red vertices, and so  $\gamma_F(G) \leq n - 3$ , which is a

contradiction. Therefore, as claimed, each vertex in  $R_c - \{u, v\}$  is adjacent to neither  $x$  nor  $y$ .

Since  $G$  is a connected graph of order  $n \geq 7$ , it can be shown that each vertex in  $R_c - \{u, v\}$  is adjacent to either  $u$  or  $v$ . We consider two cases.

*Case 1. Every vertex in  $R_c - \{u, v\}$  is adjacent to  $u$  or every vertex in  $R_c - \{u, v\}$  is adjacent to  $v$ , say the former.* Let  $w \in R_c - \{u, v\}$ . Then the red-blue coloring that assigns red to  $v, w, x$  and blue to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$  with three red vertices. Thus  $\gamma_F(G) \leq 3 < 4 \leq n - 3$ .

*Case 2. Case 1 does not occur.* Let  $W_1$  be the set of vertices in  $R_c - \{u, v\}$  that are adjacent to  $u$  and let  $W_2$  be the set of vertices in  $R_c - \{u, v\}$  that are adjacent to  $v$ . Then  $W_1 \neq \emptyset$  and  $W_2 \neq \emptyset$ . Since  $n \geq 7$ , at least one of  $W_1$  and  $W_2$  contains at least two vertices, say  $|W_1| \geq 2$ . Let  $w \in W_1$ . Then the red-blue coloring defined by assigning blue to each vertex in  $\{u, w, y\}$  and red to the remaining vertices of  $G$  is an  $F$ -coloring of  $G$  with  $n - 3$  red vertices. Therefore,  $\gamma_F(G) \leq n - 3$ . ■

Next, we show that certain pairs  $k, n$  with  $1 \leq k \leq n$  can be realized as the  $F$ -domination number and the order of some connected graph.

**Theorem 2.7.** *Let  $n \geq 7$ . If  $k$  is an integer with  $1 \leq k \leq \lfloor n/2 \rfloor + 1$  or  $k = n$ , then there exists a connected graph  $G$  of order  $n$  with  $\gamma_F(G) = k$ .*

**Proof.** For each pair  $k, n$  of integers with  $1 \leq k \leq \lfloor n/2 \rfloor + 1$  or  $k = n$ , we construct a graph  $G_{k,n}$  of order  $n$  with  $\gamma_F(G_{k,n}) = k$ . Let  $G_{1,n} = K_n$  and  $G_{n,n} = K_{1,n-1}$ . Thus, we may assume that  $2 \leq k \leq \lfloor n/2 \rfloor + 1$ . We consider three cases.

*Case 1.  $k = 2$ .* Let  $C_4 : v_1, v_2, v_3, v_4, v_1$  be a cycle of order 4 and let  $G_{2,n}$  be obtained from  $C_4$  by (1) adding  $n - 4$  new vertices  $x_1, x_2, \dots, x_{n-4}$  and joining each vertex  $x_i$  ( $1 \leq i \leq n - 4$ ) to  $v_1$ . Define a red-blue coloring  $c^*$  of  $G_{2,n}$  by assigning red to  $v_2$  and  $v_3$  and blue to the remaining vertices of  $G_{2,n}$ . Observe that (i) the blue vertex  $v_1$  is  $F$ -dominated by the red vertex  $v_3$ , (ii) the blue vertex  $v_4$  is  $F$ -dominated by the red vertex  $v_2$ , and (iii) the blue vertex  $x_i$  ( $1 \leq i \leq n - 4$ ) is  $F$ -dominated by the red vertex  $v_2$ . Thus, every blue vertex  $v$  of  $G$  belongs to a copy of  $F$  rooted at  $v$ , implying that  $c^*$  is an  $F$ -coloring of  $G_{2,n}$ . Therefore,  $\gamma_F(G_{2,n}) \leq |R_{c^*}| = 2$ .

Next, we show that  $\gamma_F(G_{2,n}) \geq 2$ . Assume, to the contrary, that  $\gamma_F(G_{2,n}) = 1$ . Let there be given an  $F$ -coloring  $c$  of  $G_{2,n}$  with exactly one red vertex. Since  $v_3$  is only  $F$ -dominated by  $v_1$  and  $v_1$  is only  $F$ -dominated by  $v_3$ , it follows that at least one of  $v_1$  and  $v_3$  belongs to  $R_c$ . By Lemma 2.1, we have  $R_c = \{v_3\}$ . However, none of the blue vertices  $x_i$  ( $1 \leq i \leq n-4$ ) is  $F$ -dominated by  $v_3$  and so  $c$  is not an  $F$ -coloring of  $G$ , which is a contradiction. Therefore,  $\gamma_F(G_{2,n}) = 2$ .

*Case 2.*  $3 \leq k \leq \lfloor n/2 \rfloor$ . Let  $C_4 : v_1, v_2, v_3, v_4, v_1$  be a cycle of order 4. Then the graph  $G_{k,n}$  is obtained from the graph  $C_4$  by (1) adding  $k-2 > 0$  new vertices  $x_1, x_2, \dots, x_{k-2}$  and joining each vertex  $x_i$  ( $1 \leq i \leq k-2$ ) to  $v_1$  and (2) adding  $n-2-k > 0$  new vertices  $y_1, y_2, \dots, y_{n-2-k}$  and joining each vertex  $y_j$  ( $1 \leq j \leq n-2-k$ ) to  $v_3$ . Then the order of  $G$  is  $4 + (k-2) + (n-2-k) = n$ . We show that  $\gamma_F(G_{k,n}) = k$ .

We first show that  $\gamma_F(G_{n,k}) \leq k$ . Let  $S = \{v_1, v_2\} \cup \{x_i : 1 \leq i \leq k-2\}$ . Define a red-blue coloring  $c^*$  of  $G_{n,k}$  by assigning red to each vertex in  $S$  and blue to the remaining vertices of  $G_{n,k}$ . Observe that (i) the blue vertex  $v_3$  is  $F$ -dominated by the red vertex  $v_1$ , (ii) the blue vertex  $v_4$  is  $F$ -dominated by the red vertex  $v_2$ , and (iii) the blue vertex  $y_j$  ( $1 \leq j \leq n-2-k$ ) is  $F$ -dominated by the red vertex  $v_2$ . Thus, every blue vertex  $v$  of  $G$  belongs to a copy of  $F$  rooted at  $v$ , implying that  $c^*$  is an  $F$ -coloring of  $G_{n,k}$ . Therefore,  $\gamma_F(G_{n,k}) \leq |R_{c^*}| = |S| = k$ .

Next, we show that  $\gamma_F(G_{n,k}) \geq k$ . Assume, to the contrary, that  $\gamma_F(G_{n,k}) \leq k-1$ . Let there be given a minimum  $F$ -coloring  $c$  of  $G_{n,k}$ . Observe that  $v_3$  is only  $F$ -dominated by  $v_1$  and  $v_1$  is only  $F$ -dominated by  $v_3$ . Thus at least one of  $v_1$  and  $v_3$  belongs to  $R_c$ . By Lemma 2.1, if  $v_1 \in R_c$ , then  $x_i \in R_c$  for  $1 \leq i \leq k-2$ . Similarly, if  $v_3 \in R_c$ , then  $y_j \in R_c$  for  $1 \leq j \leq n-2-k$ . Thus  $|R_c| \geq \min\{k-1, n-1-k\}$ . Since  $k \leq \lfloor n/2 \rfloor$ , it follows that  $k-1 \leq n-1-k$ . This implies that  $\gamma_F(G_{n,k}) \geq k-1$ . Therefore,  $\gamma_F(G_{n,k}) = k-1$  and we may assume that  $R_c = \{v_1, x_1, x_2, \dots, x_{k-2}\}$ . However then, the blue vertex  $v_2$  does not belong to a copy of  $F$  rooted at  $v_2$ , which is a contradiction. Therefore,  $\gamma_F(G_{n,k}) = k$ .

*Case 3.*  $k = \lfloor n/2 \rfloor + 1$ . Since  $n \geq 7$ , it follows that  $k \geq 4$ . We consider two subcases.

*Subcase 3.1.*  $n$  is even. Then  $n = 2\ell$  for some integer  $\ell \geq 4$  and  $k = \ell + 1 \geq 5$ . Let  $P_4 : v_1, v_2, v_3, v_4$  be a path of order 4. Then the graph  $G_{k,n}$  is obtained from  $P_4$  by (1) adding  $k-3$  new vertices  $x_1, x_2, \dots, x_{k-3}$

and joining each vertex  $x_i$  ( $1 \leq i \leq k-3$ ) to  $v_1$  and (2) adding  $k-3$  new vertices  $y_1, y_2, \dots, y_{k-3}$  and joining each vertex  $y_j$  ( $1 \leq j \leq k-3$ ) to  $v_4$ . Then the order of  $G$  is  $4 + 2(k-3) = 4 + 2(\ell-2) = 2\ell = n$ .

We first show that  $\gamma_F(G_{n,k}) \leq k$ . Let  $S_0 = \{x_1, v_3, v_4, y_1, y_2, \dots, y_{k-3}\}$ . Define a red-blue coloring  $c^*$  of  $G_{n,k}$  by assigning red to each vertex in  $S_0$  and blue to the remaining vertices of  $G_{n,k}$ . Observe that (i) the blue vertex  $v_1$  is  $F$ -dominated by the red vertex  $v_3$ , (ii) the blue vertex  $v_2$  is  $F$ -dominated by the red vertex  $x_1$ , and (iii) the blue vertex  $x_i$  ( $2 \leq i \leq k-3$ ) is  $F$ -dominated by the red vertex  $x_1$ . Thus, every blue vertex  $v$  of  $G$  belongs to a copy of  $F$  rooted at  $v$ , implying that  $c^*$  is an  $F$ -coloring of  $G_{n,k}$ . Therefore,  $\gamma_F(G_{n,k}) \leq |R_{c^*}| = |S_0| = k$ .

Next we show that  $\gamma_F(G_{n,k}) \geq k$ . Let  $X = \{x_1, x_2, \dots, x_{k-3}\}$  and  $Y = \{y_1, y_2, \dots, y_{k-3}\}$ . Suppose that  $c$  is a minimum  $F$ -coloring of  $G_{n,k}$ . We consider three subcases.

*Subcase 3.1.1.*  $v_1, v_4 \in R_c$ . By Lemma 2.1,  $X \cup Y \subseteq R_c$ . This implies that

$$\gamma_F(G_{n,k}) = |R_c| \geq |X \cup Y \cup \{v_1, v_4\}| = 2k - 4 > k,$$

as  $k \geq 5$  (note that  $n \geq 7$  is even), which is a contradiction.

*Subcase 3.1.2.*  $v_1 \notin R_c$  and  $v_4 \notin R_c$ . Since  $c$  is an  $F$ -coloring,  $v_1$  is  $F$ -dominated by some red vertex. Since  $v_1$  is only  $F$ -dominated by  $v_3$ , it follows that  $v_2$  is blue and  $v_3$  is red. However then,  $v_4$  cannot be  $F$ -dominated by any red vertex as  $v_4$  is only  $F$ -dominated by  $v_2$ , which is a contradiction.

*Subcase 3.1.3.* Exactly one of  $v_1$  and  $v_4$  belongs to  $R_c$ , say  $v_1 \notin R_c$  and  $v_4 \in R_c$ . Since  $v_4 \in R_c$ , it follows that  $Y \subseteq R_c$  by Lemma 2.1. Since  $v_1$  is only  $F$ -dominated by  $v_3$ , it follows that  $v_2$  is blue and  $v_3$  is red. Note that each vertex in  $X$  is only  $F$ -dominated by  $v_2$  or some vertex in  $X$ . Since  $v_2$  is blue, at least one vertex in  $X$  must be colored red. Hence  $Y \cup \{v_3, v_4, x\} \subseteq R_c$ , where  $x \in X$ . Therefore,  $\gamma_F(G_{n,k}) = |R_c| \geq 3 + (k - 3) = k$ .

*Subcase 3.2.*  $n$  is odd. Then  $n = 2\ell + 1$  for some integer  $\ell \geq 3$  and  $k = \ell + 1 \geq 4$ . Let  $P_4 : v_1, v_2, v_3, v_4$  be a path of order 4. So the graph  $G_{k,n}$  is obtained from  $P_4$  by (1) adding  $k - 2$  new vertices  $x_1, x_2, \dots, x_{k-2}$  and joining each vertex  $x_i$  ( $1 \leq i \leq k - 2$ ) to  $v_1$  and (2) adding  $k - 3$  new vertices  $y_1, y_2, \dots, y_{k-3}$  and joining each vertex  $y_j$  ( $1 \leq j \leq k - 3$ ) to  $v_4$ . Then the order of  $G$  is  $4 + (k - 2) + (k - 3) = 4 + (\ell - 1) + (\ell - 2) = 2\ell + 1 = n$ . We show that  $\gamma_F(G_{k,n}) = k$ .

To show that  $\gamma_F(G_{n,k}) \leq k$ , let  $S_1 = \{x_1, v_3, v_4, y_1, y_2, \dots, y_{k-3}\}$ . Define a red-blue coloring  $c^*$  of  $G_{n,k}$  by assigning red to each vertex in  $S_1$  and blue to the remaining vertices of  $G_{n,k}$ . An argument similar to the one used in Subcase 3.1 shows that  $c^*$  is an  $F$ -coloring of  $G_{n,k}$  and so  $\gamma_F(G_{n,k}) \leq |R_{c^*}| = |S_1| = k$ .

Next, we show that  $\gamma_F(G_{n,k}) \geq k$ . Suppose that  $c$  is a minimum  $F$ -coloring of  $G_{n,k}$ . An argument similar to the one in Subcases 3.1.1 and 3.1.2 shows that exactly one of  $v_1$  and  $v_4$  belongs to  $R_c$ . If  $v_1 \in R_c$  and  $v_4 \notin R_c$ , then  $X \subseteq R_c$  by Lemma 2.1. Since  $v_4$  is only  $F$ -dominated by  $v_2$ , it follows that  $v_2$  is red and  $v_3$  is blue. Moreover, each vertex in  $Y$  is only  $F$ -dominated by  $v_3$  or by some vertex in  $Y$ . Since  $v_3$  is blue, at least one vertex in  $Y$  must be colored red. Hence  $X \cup \{v_1, v_2, y\} \subseteq R_c$ , where  $y \in Y$ . However then,  $\gamma_F(G_{n,k}) = |R_c| \geq 3 + (k - 2) = k + 1 > k$ , which is

impossible. Similarly, if  $v_1 \notin R_c$  and  $v_4 \in R_c$ , then  $R_c$  contains at least  $k$  red vertices and so  $\gamma_F(G_{n,k}) = |R_c| \geq k$ . ■

Although it can be shown that there are infinitely many pairs  $k, n$  of integers with  $\lfloor n/2 \rfloor + 1 < k < n - 2$  and  $n \geq 7$  for which there exists a connected graph  $G$  of order  $n$  with  $\gamma_F(G) = k$ , we conclude this section with the following question.

**Problem 2.8.** *For which pairs  $k, n$  of integers with  $\lfloor n/2 \rfloor + 1 < k < n - 2$  and  $n \geq 7$ , does there exist a connected graph  $G$  of order  $n$  with  $\gamma_F(G) = k$ ?*

### 3. Realization Results on Three Domination Parameters

For a nontrivial connected graph  $G$  without isolated vertices, there are three possibilities related to the three parameters  $\gamma(G)$ ,  $\gamma_o(G)$ , and  $\gamma_F(G)$ , namely

- (1)  $\gamma(G) \leq \gamma_F(G) \leq \gamma_o(G)$ ,
- (2)  $\gamma(G) \leq \gamma_o(G) \leq \gamma_F(G)$ ,
- (3)  $\gamma_F(G) \leq \gamma(G) \leq \gamma_o(G)$ .

First, we show that it is possible for these three parameters to be equal.

**Proposition 3.1.** *For each integer  $k \geq 2$ , there exists a connected graph  $G$  such that*

$$\gamma(G) = \gamma_F(G) = \gamma_o(G) = k.$$

**Proof.** For  $k = 2$ , let  $G$  be the double star and so  $\gamma(G) = \gamma_F(G) = \gamma_o(G) = 2$ . Let  $k \geq 3$ . For each integer  $i$  with  $1 \leq i \leq k - 1$ , let  $F_i : v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{1i}$  be a copy of  $C_4$ . The graph  $G$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq k - 1$ ) by identifying vertices  $v_{4i}$  ( $1 \leq i \leq k - 1$ ) and labeling the identified vertex by  $v$ . Then it can be verified that  $\gamma(G) = \gamma_F(G) = \gamma_o(G) = k$ . ■

Next result shows that every pair  $a, b$  of positive integers can be realizable as the domination and  $F$ -domination of some connected graph.

**Theorem 3.2.** *Let  $a$  and  $b$  be positive integers with  $a \leq b$ .*

- (i) *There exists a connected graph  $G$  such that  $\gamma(G) = a$  and  $\gamma_F(G) = b$ ;*

(ii) *There exists a connected graph  $H$  such that  $\gamma_F(H) = a$  and  $\gamma(H) = b$ .*

**Proof.** First, we verify (i). Suppose that  $a = 1$ . If  $b = 1$ , let  $G = K_3$ ; while if  $b \geq 2$ , let  $G = K_{1,b-1}$ . Then  $\gamma(G) = 1$  and  $\gamma_F(G) = b$  for each integer  $b \geq 1$ . Hence the result holds for  $a = 1$ . Thus, we may assume that  $a \geq 2$ . By Proposition 3.1, the result holds if  $a = b$ . Therefore we may assume that  $a < b$  and so  $b - a > 0$ . There are two cases.

*Case 1.*  $a = 2$ . Let  $G$  be the graph in the proof of Case 2 for Theorem 2.7, where  $k = b$ . Then  $\gamma(G) = 2$  and  $\gamma_F(G) = k = b$ .

*Case 2.*  $a \geq 3$ . We begin with a double star  $T$  whose central vertices are  $u$  and  $v$ . Let  $U = \{u_1, u_2, \dots, u_{b-a}\}$  and  $V = \{v_1, v_2, \dots, v_{a+b-1}\}$  be the sets of vertices of  $T$  such that  $u$  is adjacent to every vertex in  $U$  and  $v$  is adjacent to every vertex in  $V$ . Then the graph  $G$  is obtained from  $T$  by (1) subdividing the edge  $uv$  with a new vertex  $x$  and (2) adding  $a - 2$  new vertices  $w_1, w_2, \dots, w_{a-2}$  and joining each  $w_i$  to  $v_i$  for  $1 \leq i \leq a - 2$ . Let  $W = \{w_1, w_2, \dots, w_{a-2}\}$ .

First, we show that  $\gamma(G) = a$ . Since  $\{u, v, v_1, v_2, \dots, v_{a-2}\}$  is a dominating set of  $G$ , it follows that  $\gamma(G) \leq a$ . Next, we show that  $\gamma(G) \geq a$ . Let  $S$  be a minimum dominating set of  $G$ . Since each vertex  $w_i$  ( $1 \leq i \leq a - 2$ ) is only dominated by itself or by  $v_i$ , it follows that  $S$  must contain at least one vertex from each set  $\{v_i, w_i\}$  for  $1 \leq i \leq a - 2$ . Also, each vertex  $u_i$  ( $1 \leq i \leq b - a$ ) is only dominated by  $u_i$  or by  $u$ . Thus, either  $u_i \in S$  for all  $i$  with  $1 \leq i \leq b - a$  or  $u \in S$ . Furthermore, each vertex  $v_j$  ( $a - 1 \leq j \leq a + b - 1$ ) is only dominated by  $v_j$  or by  $v$ . Thus either  $v_j \in S$  for each  $j$  with  $a - 1 \leq j \leq a + b - 1$  or  $v \in S$ . This implies that  $\gamma(G) = |S| \geq (a - 2) + 2 = a$ .

Next, we show that  $\gamma_F(G) = b$ . Let  $S_0 = U \cup \{u, v_{a+b-1}\} \cup W$ . Then  $|S_0| = (b - a) + 2 + (a - 2) = b$ . Since the red-blue coloring that assigns red to each vertex in  $S_0$  and blue to the remaining vertices of  $G$  is an  $F$ -coloring with  $b$  red vertices,  $\gamma_F(G) \leq b$ . To show that  $\gamma_F(G) \geq b$ , let  $c$  be a minimum  $F$ -coloring of  $G$ . We make four observations: (1) By Lemma 2.1,  $W \subset R_c$ . (2) If  $v$  is colored red by  $c$ , then  $v_i$  is colored red by  $c$  for  $1 \leq i \leq a + b - 1$  and so  $\gamma_F(G) \geq a + b - 1 > b$ , a contradiction. Thus  $v$  must be colored blue by  $c$ . (3) If  $u$  is colored blue by  $c$ , then  $u$  is not  $F$ -dominated by any red vertex in  $R_c$  since  $u$  is only  $F$ -dominated by  $v$  and  $v$  is colored blue by  $c$  as shown in (2). Thus,  $u$  must be colored red by  $c$ . (4) By (3), each vertex  $u_i \in R_c$  for  $1 \leq i \leq b - a$ . It then follows by (1)-(4) that  $U \cup \{u\} \cup W \subseteq R_c$  and so  $|R_c| \geq (b - a) + 1 + (a - 2) = b - 1$ . Assume, to the contrary,



that  $\gamma_F(G) = b - 1$ . Then  $R_c = U \cup \{u\} \cup W$ . However then  $v_{a+b-1}$  is not  $F$ -dominated by any vertex in  $R_c$ , which is a contradiction. Therefore,  $\gamma_F(G) \geq b$  and so  $\gamma_F(G) = b$ .

Next, we verify (ii). First, suppose that  $a = 1$ . If  $b = 1$ , let  $H = K_3$  and so  $\gamma_F(H) = \gamma(H) = 1$ . Thus we may assume that  $b \geq 2$ .

For the integer  $b \geq 2$  and  $1 \leq i \leq b$  let  $F_i$  be a copy of  $K_4 - e$  with  $V(F_i) = \{u_i, v_i, x_i, y_i\}$  such that  $\deg u_i = \deg v_i = 2$  and  $\deg x_i = \deg y_i = 3$ . The graph  $H$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq b$ ) by identifying all the vertices  $u_i$  and calling the new vertex  $u$ . We first show that  $\gamma(H) = b$ . Since  $S_0 = \{x_i : 1 \leq i \leq b\}$  is a dominating set of  $H$ , it follows that  $\gamma(H) \leq |S_0| = b$ . On the other hand, since  $v_i$  is only adjacent to  $x_i$  and  $y_i$  for each  $i$  ( $1 \leq i \leq b$ ), at least one vertex in each set  $\{x_i, y_i, v_i\}$  must belong to any dominating set of  $H$  and so  $\gamma(H) \geq b$ . Next we show that  $\gamma_F(H) = 1$ . Since the red-blue coloring that assigns red to vertex  $u$  and blue to the remaining vertices of  $H$  is an  $F$ -coloring, by (1) it follows that  $\gamma_F(H) = 1$ .

Now let  $a \geq 2$ . By Proposition 3.1, the result holds if  $a = b$ . Thus we may assume that  $a < b$ . Thus  $b - a > 0$  and so  $b - a + 2 \geq 3$ . We start with the graph  $W_{b-a+2} = C_{b-a+2} + K_1$ , where  $C_{b-a+2} : y_1, y_2, \dots, y_{b-a+2}$ ,  $y_1$ , and  $x$  is the vertex of degree  $b - a + 2$  in  $W_{b-a+2}$ . For each  $i$  with  $1 \leq i \leq a - 1$ , let  $F_i : s_i, t_i$  be a copy of  $P_2$ . Then the graph  $H$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq a - 1$ ) and  $W_{b-a+2}$  by (1) adding  $a - 1$  new edges  $s_i y_1$  ( $1 \leq i \leq a - 1$ ) and (2) adding  $b - a + 1$  new vertices  $z_2, z_3, \dots, z_{b-a+2}$  and joining each vertex  $z_j$  to  $y_j$  for  $2 \leq j \leq b - a + 2$ . Let  $T = \{t_1, t_2, \dots, t_{a-1}\}$ .

We first show that  $\gamma_F(H) = a$ . Since the red-blue coloring that assigns red to each vertex in  $\{x\} \cup T$  and blue to the remaining vertices of  $H$  is an  $F$ -coloring with  $a$  red vertices,  $\gamma_F(H) \leq a$ . To show that  $\gamma_F(H) \geq a$ , let  $c$  be a minimum  $F$ -coloring of  $H$ . By Lemma 2.1, we have  $T \subseteq R_c$  and so  $\gamma_F(H) \geq a - 1$ . Assume, to the contrary, that  $\gamma_F(H) = a - 1$ . Hence  $R_c = T$ . However then, no blue vertex (different from  $y_1$ ) is  $F$ -dominated by any vertex in  $R_c$ , which is a contradiction. Thus  $\gamma_F(H) = a$ .

Next we show that  $\gamma(H) = b$ . Since  $S_0 = \{s_1, s_2, \dots, s_{a-1}, y_2, \dots, y_{b-a+2}\}$  is a dominating set of  $H$ , it follows that  $\gamma(H) \leq |S_0| = b$ . To show that  $\gamma(H) \geq b$ , let  $S$  be a minimum dominating set of  $H$ . Then  $S$  contains at least one vertex from each set  $\{s_i, t_i\}$  for  $1 \leq i \leq a - 1$  and at least one vertex from each set  $\{y_j, z_j\}$  for  $2 \leq j \leq b - a + 2$ . Thus  $\gamma(H) = |S| \geq (a - 1) + (b - a + 1) = b$ . Therefore,  $\gamma(H) = b$ . ■

Next, we show that every pair  $a, b$  of positive integers can be realizable as the open domination number and  $F$ -domination number of some connected graph.

**Theorem 3.3.** *Let  $a$  and  $b$  be positive integers with  $a \leq b$ .*

- (i) *For  $a \geq 2$ , there exists a connected graph  $G$  such that  $\gamma_o(G) = a$  and  $\gamma_F(G) = b$ ;*
- (ii) *For  $b \geq 2$ , there exists a connected graph  $H$  such that  $\gamma_F(H) = a$  and  $\gamma_o(H) = b$ .*

**Proof.** By Proposition 3.1, the result holds if  $a = b$ . Thus we may assume that  $a < b$ . We first verify (i). We consider three cases.

*Case 1.*  $a = 2$ . Let  $G = K_{1,b-1}$ . Then  $\gamma_o(G) = 2$  and  $\gamma_F(G) = b$ .

*Case 2.*  $a = 3$ . Let  $G$  be the graph in the Proof of Case 2 for Theorem 2.7, where  $k = b$ . Then  $\gamma_o(G) = 3$  and  $\gamma_F(G) = k = b$ .

*Case 3.*  $a \geq 4$ . We begin with a double star  $T$  whose central vertices are  $u$  and  $v$ . Let  $U = \{u_1, u_2, \dots, u_{b-a+1}\}$  and  $V = \{v_1, v_2, \dots, v_{a+b-1}\}$  be the sets of vertices of  $T$  such that  $u$  is adjacent to every vertex in  $U$  and  $v$  is adjacent to every vertex in  $V$ . Then the graph  $G$  is obtained from  $T$  by (1) subdividing the edge  $uv$  with a new vertex  $x$  and (2) adding  $a - 3$  new vertices  $w_1, w_2, \dots, w_{a-3}$  and joining each  $w_i$  to  $v_i$  for  $1 \leq i \leq a - 3$ . Let  $W = \{w_1, w_2, \dots, w_{a-3}\}$ .

First, we show that  $\gamma_o(G) = a$ . Since  $S_0 = \{u, x, v, v_1, v_2, \dots, v_{a-3}\}$  is an open dominating set of  $G$ , it follows that  $\gamma_o(G) \leq |S_0| = a$ . To show that  $\gamma_o(G) \geq a$ , let  $S$  be a minimum open dominating set of  $G$ . Since each  $w_i$  ( $1 \leq i \leq a - 3$ ) is only openly dominated by  $v_i$  ( $1 \leq i \leq a - 3$ ) and each  $u_i$  ( $1 \leq i \leq b - a + 1$ ) is only openly dominated by  $u$ , we have  $v_i \in S$  for  $1 \leq i \leq a - 3$  and  $u \in S$ . Similarly, since each  $v_j$  ( $a - 2 \leq j \leq a + b - 1$ ) is only openly dominated by  $v$ , we have  $v \in S$ . Thus  $\gamma_o(G) = |S| \geq a - 1$ . Assume, to the contrary, that  $\gamma_o(G) = a - 1$ . However then,  $S = \{u, v, v_1, v_2, \dots, v_{a-3}\}$  and  $u$  is not openly dominated by any vertex in  $S$ , which is a contradiction. Therefore,  $\gamma_o(G) \geq a$  and so  $\gamma_o(G) = a$ .

Next, we show that  $\gamma_F(G) = b$ . Let  $S' = U \cup \{u\} \cup W \cup \{v_{a+b-1}\}$ . By the proof of Theorem 3.2(i), the red-blue coloring that assigns red to each vertex of  $S'$  and blue to the remaining vertices of  $G$  is a minimum  $F$ -coloring with  $b$  red vertices. Therefore,  $\gamma_F(G) = b$ .

Next, we verify (ii). First, suppose that  $a = 1$  and  $b \geq 2$ . Let  $H$  be the graph in Theorem 3.2(ii). So for the integer  $b \geq 2$  and  $1 \leq i \leq b-1$ , let  $F_i$  be a copy of  $K_4 - e$  with  $V(F_i) = \{u_i, v_i, x_i, y_i\}$  such that  $\deg u_i = \deg v_i = 2$  and  $\deg x_i = \deg y_i = 3$ . The graph  $H$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq b-1$ ) by identifying all the vertices  $u_i$  and calling the new vertex  $u$ .

We first show that  $\gamma_o(H) = b$ . Since  $S_0 = \{u\} \cup \{x_i : 1 \leq i \leq b-1\}$  is an open dominating set in  $H$ , it follows that  $\gamma_o(H) \leq |S_0| = b-1$ . On the other hand, since  $v_i$  is only adjacent to  $x_i$  and  $y_i$  for each  $i$  ( $1 \leq i \leq b-1$ ), at least one vertex in each set  $\{x_i, y_i, v_i\}$  must belong to any open dominating set of  $H$  and so  $\gamma_o(H) \geq b-1$ . Assume, to the contrary, that  $\gamma_o(H) = b-1$ . Then

$$S = \{w_i : 1 \leq i \leq b-1\} \subseteq \{x_i, y_i, v_i : 1 \leq i \leq b-1\},$$

where  $w_i \in \{x_i, y_i, v_i\}$  for each  $i$  with  $1 \leq i \leq b-1$ . However then,  $w_i$  is not openly dominated by any vertex in  $S_o$ , which is a contradiction. Thus  $\gamma_o(H) \geq b$ . Next we show that  $\gamma_F(H) = 1$ . Since the red-blue coloring that assigns red to vertex  $u$  and blue to the remaining vertices of  $H$  is an  $F$ -coloring, it follows by (1) that  $\gamma_F(H) = 1$ .

Now let  $2 \leq a < b$ . We consider two cases.

*Case 1.*  $b = a+1$ . For each integer  $i$  with  $1 \leq i \leq a-1$ , let  $F_i : u_i, v_i, w_i$  be a copy of the path  $P_3$  and let  $C_3 : x, y, z, x$  be a copy of a 3-cycle. Then the graph  $H$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq a-1$ ) and  $C_3$  by (1) identifying the vertices  $u_i$  ( $1 \leq i \leq a-1$ ) and calling the new vertex  $u$  and (2) joining the vertex  $u$  to  $x$ .

We first show that  $\gamma_F(H) = a$ . Let  $S_0 = \{x\} \cup \{w_i : 1 \leq i \leq a-1\}$ . Since the red-blue coloring that assigns red to each vertex in  $S_0$  and blue to the remaining vertices of  $H$  is an  $F$ -coloring with  $a$  red vertices,  $\gamma_F(H) \leq a$ .

To show that  $\gamma_F(H) \geq a$ , let  $c$  be a minimum  $F$ -coloring of  $H$ . By Lemma 2.1, each end-vertex in  $w_i$  ( $1 \leq i \leq a-1$ ) must be colored red by  $c$  and so  $\gamma_F(H) \geq a-1$ . Assume to the contrary that  $\gamma_F(H) = a-1$ . Then  $R_c = \{w_i : 1 \leq i \leq a-1\}$ . However then,  $y$  is not  $F$ -dominated by any vertex in  $R_c$ , which is a contradiction. Thus  $\gamma_F(H) \geq a$ .

Next we show that  $\gamma_o(H) = a+1 = b$ . Since  $S_1 = \{u, x\} \cup \{v_i : 1 \leq i \leq a-1\}$  is an open dominating set in  $H$ , it follows that  $\gamma_o(H) \leq |S_1| = a+1$ . On the other hand, since  $w_i$  ( $1 \leq i \leq a-1$ ) is only openly dominated by  $v_i$ , it follows that  $v_i \in S$  for all  $1 \leq i \leq a-1$  and so  $\gamma_o(H) \geq a-1$ .

Also, since each  $v_i$  is only openly dominated by  $w_i$  or by  $u$ , it follows that either  $w_i \in S$  ( $1 \leq i \leq a-1$ ) or  $u \in S$ . This implies that  $\gamma_o(H) \geq (a-1) + 1 = a$ . Assume, to the contrary, that  $\gamma_o(H) = a$ . Let  $S$  be a minimum open dominating set of  $H$ . Then  $S \subset (\{u\} \cup \{v_i, w_i : 1 \leq i \leq a-1\})$ . However then,  $y$  is not openly dominated by any vertex of  $S$ , which is a contradiction. Hence  $\gamma_o(H) \geq a+1$ .

*Case 2.*  $b \geq a+2$ . Then  $b-a+1 \geq 3$ . We start with the graph  $W_{b-a+1} = C_{b-a+1} + K_1$ , where  $C_{b-a+1} : y_1, y_2, \dots, y_{b-a+1}, y_1$  and  $x$  is the vertex of degree  $b-a+1$  in  $W_{b-a+1}$ . For each  $i$  with  $1 \leq i \leq a-1$ , let  $F_i : s_i, t_i$  be a copy of  $P_2$ . Then the graph  $H$  is obtained from the graphs  $F_i$  ( $1 \leq i \leq a-1$ ) and  $W_{b-a+1}$  by (1) adding  $a-1$  new edges  $s_i y_1$  ( $1 \leq i \leq a-1$ ) and (2) adding  $b-a$  new vertices  $z_2, z_3, \dots, z_{b-a+1}$  and joining each  $z_i$  ( $2 \leq i \leq b-a+1$ ) with  $y_i$ . Let  $T = \{t_1, t_2, \dots, t_{a-1}\}$ ,  $S = \{s_1, s_2, \dots, s_{a-1}\}$ , and  $Y = \{y_1, y_2, y_3, \dots, y_{b-a+1}\}$ . The red-blue coloring that assigns red to each vertex of the set  $\{x\} \cup T$  and blue to the remaining vertices of  $H$  is a minimum  $F$ -coloring with  $a$  red vertices. Therefore,  $\gamma_F(H) = a$ .

Next we show that  $\gamma_o(H) = b$ . Since  $S \cup Y$  is an open dominating set of  $H$ , it follows that  $\gamma_o(H) \leq |S \cup Y| = (a-1) + (b-a+1) = b$ . To show that  $\gamma_o(H) \geq b$ , observe that every open dominating set of  $H$  contains  $S \cup (Y - \{y_1\})$ . On the other hand,  $s_1$  is not openly dominated by any vertex in  $S \cup (Y - \{y_1\})$  and so  $S \cup (Y - \{y_1\})$  is not an open dominating set of  $H$ . Therefore,  $\gamma_o(H) \geq |S \cup (Y - \{y_1\})| + 1 = (a-1) + (b-a) + 1 = b$ . Therefore,  $\gamma_o(H) = b$ . ■

Recall that for a graph  $G$  without isolated vertices, the following are possible:

- (1)  $\gamma(G) \leq \gamma_F(G) \leq \gamma_o(G)$ ,
- (2)  $\gamma(G) \leq \gamma_o(G) \leq \gamma_F(G)$ ,
- (3)  $\gamma_F(G) \leq \gamma(G) \leq \gamma_o(G)$ .

Also, we have seen that every pair  $a, b$  of positive integers can be realizable as the domination number and  $F$ -domination number of some connected graph or the open domination number and  $F$ -domination number of some connected graph. This gives rise to the following natural question.

**Problem 3.4.** For which triples  $a, b, c$  of positive integers with  $a \leq b \leq 2a$  and  $b \geq 2$ , does there exist a connected graph  $G$  such that  $\gamma(G) = a$ ,  $\gamma_o(G) = b$ , and  $\gamma_F(G) = c$ ?

Although it is not known whether *every* triple  $a, b, c$  in Problem 3.4 is realizable as the domination, open domination, and  $F$ -domination number of some connected graph, it can be shown that there are infinitely many such realizable triples. As an example, we present the following.

**Theorem 3.5.** *For each pair  $a, b$  of integers with  $1 \leq a \leq b \leq 2a$  and  $b \geq 2$ , there exists a connected graph  $G$  with  $\gamma_F(G) = 1$  such that  $\gamma(G) = a$  and  $\gamma_o(G) = b$ .*

**Proof.** For  $a = 1$  and  $b = 2$ , let  $G = K_3$  and so  $\gamma_F(G) = \gamma(G) = 1$  and  $\gamma_o(G) = 2$ . Thus we may assume that  $a \geq 2$ . We consider two cases, according to whether  $a = b$  or  $a \neq b$ .

*Case 1.  $a = b$ .* If  $a = b = 2$ , let  $G$  be the graph obtained from the graph  $K_4 - e$  by adding a new vertex and joining this new vertex to a vertex of degree 2 in  $K_4 - e$ . Then  $\gamma_F(G) = 1$  and  $\gamma(G) = \gamma_o(G) = 2$ . Now let  $a = b \geq 3$ . Let  $s > a$  be an integer and consider the graph  $P_s + K_1$ , where  $P_s : u_1, u_2, \dots, u_s$  and  $u$  is the vertex in  $P_s + K_1$  with  $\deg u = s$ . Then the graph  $G$  is obtained from  $P_s + K_1$  by adding  $a - 1$  new vertices  $v_1, v_2, \dots, v_{a-1}$  and joining each  $v_i$  to  $u_i$  for  $1 \leq i \leq a - 1$ . Since  $N(u)$  is an open dominating set of  $G$ , it follows by Theorem 2.3 that  $\gamma_F(G) = 1$ . Since  $\{u, u_1, u_2, \dots, u_{a-1}\}$  is a minimum dominating and minimum open dominating set of  $G$ , it follows that  $\gamma(G) = \gamma_o(G) = a$ .

*Case 2.  $a < b \leq 2a$ .* We consider three subcases.

*Subcase 2.1.  $a < b \leq 2a - 2$ .* Let  $b = a + k$ , where  $k \geq 1$ , and let  $\ell = a - k - 1$ . Since  $b = a + k \leq 2a - 2$ , it follows that  $\ell \geq 1$ . Consider the graphs  $H_1$  and  $H_2$  in Figure 5.

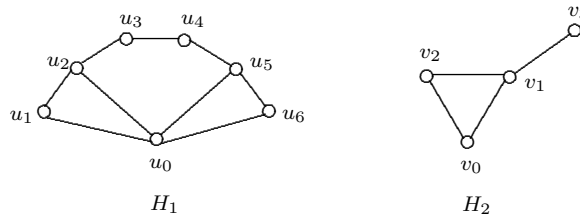


Figure 5. The graphs  $H_1$  and  $H_2$  in Case 2.

For each  $i$  with  $1 \leq i \leq k$ , let  $F_i$  be a copy of  $H_1$ , where  $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \dots, u_{i,6}\}$ , where  $u_{i,p}$  corresponds to  $u_p$  in  $H_1$  for  $0 \leq p \leq 6$ . For each  $j$  with  $1 \leq j \leq \ell$ , let  $G_j$  be a copy of  $H_2$  with  $V(G_j) = \{v_{j,0}, v_{j,1}, v_{j,2}, v_{j,3}\}$ , where  $v_{j,q}$  corresponds to  $v_q$  in  $H_2$  for  $0 \leq q \leq 3$ . The graph  $G$  is then obtained from the graphs  $F_i$  and  $G_j$  for  $1 \leq i \leq k$  and  $1 \leq j \leq \ell$  by identifying all vertices  $u_{i,0}$  and  $v_{j,0}$  and labeling the identified vertex  $v$ . Observe that  $N(v)$  is an open dominating set of  $G$ . Thus  $\gamma_F(G) = 1$  by Theorem 2.3. Also, since

$$S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq k\} \cup \{v_{j,1} : 1 \leq j \leq \ell\}$$

is a minimum dominating set of  $G$ , it follows that  $\gamma(G) = |S| = 1 + k + \ell = 1 + k + (a - k - 1) = a$ . Furthermore, the set  $S_o = S \cup \{u_{i,2} : 1 \leq i \leq k\}$  is a minimum open dominating set of  $G$  and so  $\gamma_o(G) = |S_o| = |S| + k = a + k = b$ .

*Subcase 2.2.*  $b = 2a - 1$ . For each  $i$  with  $1 \leq i \leq a - 1$ , let  $F_i$  be a copy of  $H_1$  in Figure 5 such that  $V(F_i) = \{u_{i,0}, u_{i,1}, u_{i,2}, \dots, u_{i,6}\}$ , where  $u_{i,p}$  corresponds to  $u_p$  in  $H_1$  for  $0 \leq p \leq 6$ . Then the graph  $G$  is obtained from the graphs  $F_i$ ,  $1 \leq i \leq a - 1$ , by identifying all vertices  $u_{i,0}$  and labeling the identified vertex by  $v$ . Again,  $\gamma_F(G) = 1$  and  $\{v\}$  is the minimum  $F$ -dominating set. Since  $S = \{v\} \cup \{u_{i,3} : 1 \leq i \leq a - 1\}$  is a minimum dominating set of  $G$  and  $S \cup \{u_{i,2} : 1 \leq i \leq a - 1\}$  is a minimum open dominating set of  $G$ , it follows that  $\gamma(G) = |S| = a$  and  $\gamma_o(G) = |S| + (a - 1) = 2a - 1 = b$ .

*Subcase 2.3.*  $b = 2a$ . If  $a = 1$  and  $b = 2$ , then the graph  $H_2$  of Figure 5 has the desired property. Thus we may assume that  $a \geq 2$ . Let  $p \geq 2$  be an integer. For each integer  $i$  with  $1 \leq i \leq a - 1$ , let  $F_i$  be the graph obtained from the path  $u_i, y_i, v_i$  by adding  $2p$  new vertices  $r_{i,j}$  ( $1 \leq j \leq 2p$ ) and joining (1) each vertex  $r_{i,j}$  ( $1 \leq j \leq p$ ) to  $u_i$  and  $y_i$  and (2) each vertex  $r_{i,j}$  ( $p + 1 \leq j \leq 2p$ ) to  $y_i$  and  $v_i$ . Then the graph  $G$  is obtained from the  $a - 1$  graphs  $F_i$  ( $1 \leq i \leq a - 1$ ) and the path  $P : z, w, x, w'$  of order 4 by (1) adding the edge  $xz$  and (2) joining each of the two vertices  $w$  and  $z$  to each vertex in  $\{u_i, v_i\}$  for  $1 \leq i \leq a - 1$ . The graph  $G$  is shown in Figure 6 for  $a = 3$ . Since  $N(w) = \{x, z\} \cup \{u_i, v_i : 1 \leq i \leq a - 1\}$  is an open dominating set of  $G$ , it follows by Theorem 2.3 that  $\gamma_F(G) = 1$ . It remains to show that  $\gamma(G) = a$  and  $\gamma_o(G) = b$ .

We first show that  $\gamma(G) = a$ . Since the set  $\{x\} \cup \{y_i : 1 \leq i \leq a - 1\}$  is a dominating set of  $G$ , it follows that  $\gamma(G) \leq a$ . On the other hand, let  $S$

be a minimum dominating set of  $G$ . For each integer  $i$  with  $1 \leq i \leq a - 1$ , let  $R_i = \{r_{i,j} : 1 \leq j \leq p\}$  and  $R'_i = \{r_{i,j} : p + 1 \leq j \leq 2p\}$ . Observe that

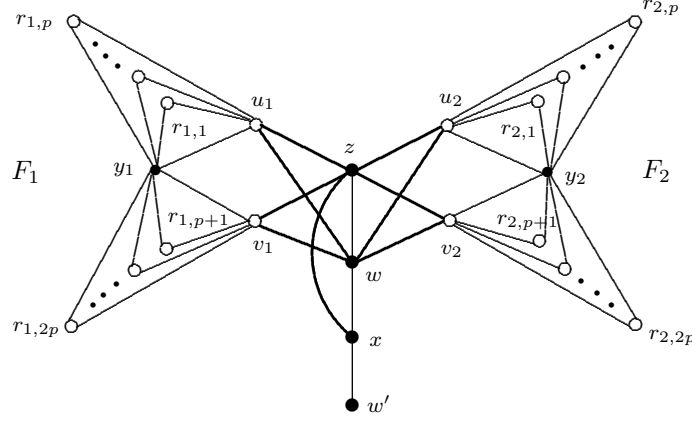


Figure 6. The graph  $G$  in Subcase 2.3 for  $a = 3$ .

- (b1) since  $w'$  is an end-vertex of  $G$  and  $w'$  is adjacent to  $x$ , the set  $S$  contains at least one vertex in  $\{w', x\}$ , and
- (b2) since  $y_i$  is only dominated by a vertex in the set

$$(3) \quad A_i = \{u_i, v_i, y_i\} \cup R_i \cup R'_i,$$

the set  $S$  must contain at least one vertex in each set  $A_i$ .

The  $a$  sets  $\{w', x\}$  and  $A_i$  ( $1 \leq i \leq a - 1$ ) are pairwise disjoint. It then follows by (b1) and (b2) that  $S$  contains at least  $a$  distinct vertices of  $G$  and so  $\gamma(G) \geq a$ . Therefore,  $\gamma(G) = a$ .

Next, we show that  $\gamma_o(G) = b$ . Since  $N(w)$  is an open dominating set of  $G$ , it follows that  $\gamma_o(G) \leq |N(w)| = 2a$ . On the other hand, let  $S_o$  be a minimum open dominating set of  $G$ . First, we verify the following claim.

**Claim.** For each integer  $i$  with  $1 \leq i \leq a - 1$ , the set  $S_o$  must contain at least two vertices in each set  $A_i$  in (3).

**Proof of Claim.** Assume, to the contrary, that  $S_o$  contains at most one vertex in  $A_i$  for some  $i$  with  $1 \leq i \leq a - 1$ . Observe that each vertex in  $R_i$  is only openly dominated by a vertex in  $B_i = \{u_i, y_i\}$  and so  $S_o$  must

contain at least one vertex in  $B_i$ . Similarly, each vertex in  $R'_i$  is only openly dominated by a vertex in  $C_i = \{v_i, y_i\}$  and so  $S_o$  must contain at least one vertex in  $C_i$ . Since  $B_i \cup C_i = \{u_i, v_i, y_i\} \subseteq A_i$ , it follows that  $S_o$  contains at least one vertex in  $A_i$ . Hence  $S_o$  contains exactly one vertex in  $A_i$ . Because  $B_i \cap C_i = \{y_i\}$ , the vertex  $y_i$  is the only vertex of  $A_i$  that belongs to  $S_o$ . However,  $y_i$  is only openly dominated by a vertex in  $A_i - \{y_i\}$ , implying that  $y_i$  is not openly dominated by any vertex in  $S_o$ , which is a contradiction.

This completes the proof of the claim. Therefore,  $S_o$  must contain at least two vertices in each set  $A_i$  for  $1 \leq i \leq a - 1$ . Moreover, the end-vertex  $w'$  is only openly dominated by  $x$  and  $x$  is only openly dominated by a vertex in the set  $V(P) - \{x\} = \{w, w', z\}$ . Thus  $S_o$  must contain at least two vertices in  $V(P)$ . Since the  $a$  subsets  $V(P)$  and  $A_i$  ( $1 \leq i \leq a - 1$ ) of  $V(G)$  are pairwise disjoint,  $S_o$  contains at least  $2a$  distinct vertices of  $G$  and so  $\gamma_o(G) = |S_o| \geq 2a$ . Therefore,  $\gamma_o(G) = 2a = b$ . ■

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