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ISOMORPHIC COMPONENTS OF DIRECT PRODUCTS OF BIPARTITE GRAPHS

Richard Hammack*

Department of Mathematics
Virginia Commonwealth University
Richmond, Virginia 23284–2014, USA

e-mail: rhammack@rmc.edu

Abstract

A standard result states the direct product of two connected bipartite graphs has exactly two components. Jha, Klavžar and Zmazek proved that if one of the factors admits an automorphism that interchanges partite sets, then the components are isomorphic. They conjectured the converse to be true. We prove the converse holds if the factors are square-free. Further, we present a matrix-theoretic conjecture that, if proved, would prove the general case of the converse; if refuted, it would produce a counterexample.

Keywords: direct product, tensor product, Kronecker product, bipartite graph.

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1. Introduction

The *direct product* of two simple graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \times H$ whose vertex set is $V(G) \times V(H)$, and whose edge set is $\{(x, u)(y, v) | xy \in E(G), uv \in E(H)\}$. The direct product is also called the *Kronecker product*, the *categorical product* or the *tensor product*. (See Section 5.3 of [2].) Figures 1-A and 1-B show two direct products of paths (where P_n denotes the path on n vertices). For clarity, the factors are drawn to the left of and below the products.

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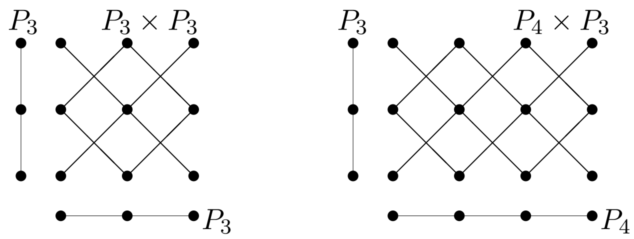

Figure 1-A Figure 1-B

A standard result, first proved by Weichsel [4], states the direct product $G \times H$ of two nontrivial connected graphs is connected if and only if one of the factors G or H contains an odd cycle; moreover, if both factors are bipartite, then $G \times H$ has exactly two components. As examples, the paths P_3 and P_4 are bipartite, and, as is evident in Figures 1-A and 1-B, each of the products $P_3 \times P_3$ and $P_4 \times P_3$ has exactly two components.

Notice that the components of $P_3 \times P_3$ are not isomorphic, but those of $P_4 \times P_3$ are. This article addresses the question of under what circumstances the direct product of two connected bipartite graphs has isomorphic components. Some work on this topic was done by Jha, Klavžar and Zmazek [3]. They define a bipartite graph to have *property* π if it has an automorphism which interchanges its partite sets, and they prove that if at least one of G or H has property π, then then $G \times H$ has isomorphic components. They leave the converse as a conjecture, but present several results supporting its validity.

As examples of this result (and conjecture), neither of the two automorphisms of the path P_3 interchanges its partite sets, as both leave the middle vertex fixed, and as Figure 1-A shows, the components of $P_3 \times P_3$ are not isomorphic. On the other hand, the nontrivial automorphism of P_4 interchanges its partite sets, and Figure 1-B shows the components of $P_4 \times P_3$ are isomprphic.

This article proves a partial converse of the theorem by Jha, Klavžar and Zmazek. We prove that if G and H are connected *square-free* bipartite graphs and $G \times H$ has isomorphic components, then one of G or H has property π. Also, we present a matrix-theoretic conjecture that — if proven — would solve the converse in complete generality.

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2. Matrix Preliminaries

If A and B are matrices, then their *tensor product* $A \otimes B$ is the matrix obtained by replacing each entry a_{ij} of A with the block $a_{ij}B$. It is straightforward to verify that $(A \otimes B)^T = A^T \otimes B^T$ (where T indicates transpose). Also, if $A \neq 0$, then $A \otimes B = A \otimes C$ implies $B = C$. Although it is not generally true that $A \otimes B = B \otimes A$, there are always permutation matrices M and N for which $M(A \otimes B)N = B \otimes A$. These facts are used without further comment.

Suppose G is a bipartite graph with partite sets X and Y. In what follows, it is useful to think of X and Y as ordered pairs. We say that (X, Y) is a *bipartition* of G, and we regard (Y, X) as a different bipartition. If G has c components, all nontrivial, then it has 2^c bipartitions.

If we order the vertices of a bipartition, so that $(X, Y) = (\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$, then relative to this ordering of the vertices, the adjacency matrix of G has form $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$. Because of the built-in redundancy (A^T carries the same information as A), the structure of G is completely determined by A alone, and we call A the *bipartite adjacency matrix of G.*

Definition. If bipartite graph G has an ordered bipartition $(X, Y) = (\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$, then the *bipartite adjacency matrix of G relative to the bipartition* is the $m \times n$ matrix A for which $a_{ij} = 1$ if $x_i y_j \in E(G)$ and $a_{ij} = 0$ if $x_i y_j \notin E(G)$. Given such an A we say G is the *graph for A.*

Switching the bipartition from (X, Y) to (Y, X) has the effect of changing the bipartite adjacency matrix from A to A^T, so two graphs with bipartite adjacency matrices A and A^T are actually isomorphic. The next lemma gives the full picture.

Lemma 1. *Suppose G and H are connected bipartite graphs with bipartite adjacency matrices A and B, respectively. Then $G \cong H$ if and only if there are permutation matrices K and L for which $KAL = B$ or $KAL = B^T$.*

Proof. Let A be the bipartite adjacency matrix of G relative to an ordered bipartition $(X, Y) = (\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$ of G. Let B be the bipartite adjacency matrix of H relative to an ordered bipartition $(U, V) = (\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_q\})$ of H.

Suppose there is an isomorphism $\beta : G \to H$. Because G and H are connected, they have just two bipartitions each, (X, Y) and (Y, X) for G, and (U, V) and (V, U) for H. Therefore, either $\beta(X) = U$ and $\beta(Y) = V$, or $\beta(X) = V$ and $\beta(Y) = U$.

Case 1. Suppose $\beta(X) = U$ and $\beta(Y) = V$. Then $m = p$ and $n = q$. Let κ be the permutation of the set $\{1, 2, \cdots, m\}$ for which $\beta(x_i) = u_{\kappa(i)}$, for $1 \leq i \leq m$. Let λ be the permutation of the set $\{1, 2, \cdots, n\}$ for which $\beta(y_j) = v_{\lambda(j)}$, for $1 \leq j \leq n$. Now, $a_{ij} = 1 \iff x_i y_j \in E(G) \iff \beta(x_i)\beta(y_j) \in E(H) \iff u_{\kappa(i)} v_{\lambda(j)} \in E(H) \iff b_{\kappa(i)\lambda(j)} = 1$. Therefore A and B are related by the equation $a_{ij} = b_{\kappa(i)\lambda(j)}$. Let K be the $m \times m$ permutation matrix for which left-multiplication by K permutes row i to row $\kappa(i)$, and let L be the $n \times n$ permutation matrix for which right-multiplication by L permutes column j to column $\lambda(j)$. The condition $a_{ij} = b_{\kappa(i)\lambda(j)}$ implies $KAL = B$.

Case 2. Suppose $\beta(X) = V$ and $\beta(Y) = U$. Then $m = q$ and $n = p$. Let κ be the permutation of the set $\{1, 2, \cdots, m\}$ for which $\beta(x_i) = v_{\kappa(i)}$, for $1 \leq i \leq m$. Let λ be the permutation of the set $\{1, 2, \cdots, n\}$ for which $\beta(y_j) = u_{\lambda(j)}$, for $1 \leq j \leq n$. Now, $a_{ij} = 1 \iff x_i y_j \in E(G) \iff \beta(x_i)\beta(y_j) \in E(H) \iff v_{\kappa(i)} u_{\lambda(j)} \in E(H) \iff b_{\lambda(j)\kappa(i)} = 1$. Therefore A and B are related by the equation $a_{ij} = b_{\lambda(j)\kappa(i)}$. Let K and L be as in the previous case. The condition $a_{ij} = b_{\lambda(j)\kappa(i)}$ implies $KAL = B^T$. This completes the proof that $G \cong H$ implies $KAL = B$ or $KAL = B^T$.

Conversely, suppose $KAL = B$. Let κ be the permutation of $\{1, 2, \cdots, m\}$ for which left-multiplication of A by K permutes row i to row $\kappa(i)$; let λ be the the permutation of $\{1, 2, \cdots, n\}$ for which right-multiplication of A by L permutes column j to column $\lambda(j)$. Therefore A and B are related by the equation $a_{ij} = b_{\kappa(i)\lambda(j)}$. Form the bijection $\beta : V(G) \to V(H)$, with $\beta(x_i) = u_{\kappa(i)}$ and $\beta(y_j) = v_{\lambda(j)}$. Then β is an isomorphism because $x_i y_j \in E(G) \iff a_{ij} = 1 \iff b_{\kappa(i)\lambda(j)} = 1 \iff u_{\kappa(i)} v_{\lambda(i)} \in E(H) \iff \beta(x_i)\beta(y_j) \in E(H)$.

If $KAL = B^T$, define κ and λ as above, but this time A and B are related by the equation $a_{ij} = b_{\lambda(j)\kappa(i)}$. Form the bijection $\beta : V(G) \to V(H)$, with $\beta(x_i) = v_{\kappa(i)}$ and $\beta(y_j) = u_{\lambda(j)}$. Then β is an isomorphism because $x_i y_j \in E(G) \iff a_{ij} = 1 \iff b_{\lambda(j)\kappa(i)} = 1 \iff u_{\lambda(j)} v_{\kappa(i)} \in E(H) \iff \beta(y_j)\beta(x_i) \in E(H)$. ■

The next result appeared without proof in [1], and in the form of the full adjacency matrix. Though the statement is intuitively clear and the proof is simple, the indexing is messy.

Lemma 2. *Suppose G and H are bipartite graphs with bipartitions (X, Y) and (U, V), respectively. Suppose also that, relative to an ordering of these bipartitions, their bipartite adjacency matrices are A and B, respectively. Then $G \times H$ is bipartite, with bipartitation $(\mathcal{X}, \mathcal{Y}) = ((X \times U) \cup (X \times V), (Y \times U) \cup (Y \times V))$, relative to which $G \times H$ has bipartite adjacency matrix of form*

$$M = \begin{bmatrix} 0 & A \otimes B \\ A \otimes B^T & 0 \end{bmatrix}.$$

Moreover, if G and H are connected, then $G \times H$ has exactly two components, and these components have as bipartite adjacency matrices the blocks $A \otimes B$ and $A \otimes B^T$ from M.

Proof. Clearly $(\mathcal{X}, \mathcal{Y})$ is a bipartition, for since no edge of G has both endpoints in X or both in Y, then no edge of $G \times H$ can have both endpoints in $\mathcal{X} = (X \times U) \cup (X \times V)$ or both endpoints in $\mathcal{Y} = (Y \times U) \cup (Y \times V))$.

Say A and B come from orderings $(X, Y) = (\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$ and $(U, V) = (\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_q\})$. Order the bipartition $(\mathcal{X}, \mathcal{Y})$ of $G \times H$ as follows. The set $\mathcal{X}$ contains exactly the elements in the following list, which indexes the rows of M:
$(x_1, u_1), (x_1, u_2), \cdots, (x_1, u_p), (x_2, u_1), (x_2, u_2), \cdots, (x_2, u_p), \cdots, (x_m, u_1), (x_m, u_2), \cdots, (x_m, u_p), (x_1, v_1), (x_1, v_2), \cdots, (x_1, v_q), (x_2, v_1), (x_2, v_2), \cdots, (x_2, v_q), \cdots, (x_n, v_1), (x_n, v_2), \cdots, (x_n, v_q)$.

Set $\mathcal{Y}$ contains exactly the elements in the following list, which indexes the columns of M:
$(y_1, u_1), (y_1, u_2), \cdots, (y_1, u_p), (y_2, u_1), (y_2, u_2), \cdots, (y_2, u_p), \cdots, (y_n, u_1), (y_n, u_2), \cdots, (y_n, u_p), (y_1, v_1), (y_1, v_2), \cdots, (y_1, v_q), (y_2, v_1), (y_2, v_2), \cdots, (y_2, v_q), \cdots, (y_n, v_1), (y_n, v_2), \cdots, (y_n, v_q)$.

Now, M has block form $\begin{bmatrix} M_{UU} & M_{UV} \\ M_{VU} & M_{VV} \end{bmatrix}$ where $X \times U$ indexes the rows of the top blocks, $X \times V$ indexes the rows of the bottom blocks, $Y \times U$ indexes the columns of the left-hand blocks, and $Y \times V$ indexes the columns of the right-hand blocks. It must be shown that $M_{UU} = 0$, $M_{UV} = A \otimes B$, $M_{VU} = A \otimes B^T$ and $M_{VV} = 0$.

Notice that $M_{UU} = 0$, for its rows and columns are indexed by $X \times U$ and $Y \times U$, respectively, and no edge of $G \times H$ can have both endpoints in the union of these sets, because no edge of H joins two vertices of U. Similarly, $M_{VV} = 0$, as its rows and columns are indexed by $X \times V$ and $Y \times V$.

Next we confirm $M_{UV} = A \otimes B$. Observe that M_{UV} is a $m \times n$ array of $p \times q$ sub-blocks, where the sub-block N_{ij} in the i^{th} row and j^{th} column of this array lies in the rows $(x_i, u_1), (x_i, u_2), \cdots, (x_i, u_p)$ of M, and the columns $(y_j, v_1), (y_j, v_2), \cdots, (y_j, v_q)$ of M. Showing $M_{UV} = A \otimes B$ amounts to showing $N_{ij} = a_{ij}B$. But this is clear. The k-l entry of N_{ij} is in row (x_i, u_k) and column (y_j, v_l) of M. This entry is 1 if and only if $(x_i, u_k)(y_j, v_l) \in E(G \times H)$, if and only if $x_i y_j \in E(G)$ and $u_k v_l \in E(H)$, if and only if $a_{ij} = 1$ and $b_{kl} = 1$, if and only if $a_{ij} b_{kl} = 1$, if and only if the k-l entry of $a_{ij}B$ is 1. It follows that $N_{ij} = a_{ij}B$.

Finally we show $M_{VU} = A \otimes B^T$. Observe that M_{VU} is a $m \times n$ array of $q \times p$ sub-blocks, where the sub-block N_{ij} in the i^{th} row and j^{th} column of this array lies in the rows $(x_i, v_1), (x_i, v_2), \cdots, (x_i, u_q)$ of M, and the columns $(y_j, u_1), (y_j, u_2), \cdots, (y_j, u_p)$ of M. Showing $M_{VU} = A \otimes B^T$ amounts to showing $N_{ij} = a_{ij}B^T$. But this is clear. The k-l entry of N_{ij} is in row (x_i, v_k) and column (y_j, u_l) of M. This entry is 1 if and only if $(x_i, v_k)(y_j, u_l) \in E(G \times H)$, if and only if $x_i y_j \in E(G)$ and $u_l v_k \in E(H)$, if and only if $a_{ij} = 1$ and $b_{lk} = 1$, if and only if $a_{ij} b_{lk} = 1$, if and only if the k-l entry of $a_{ij}B^T$ is 1. It follows that $N_{ij} = a_{ij}B^T$.

Now it has been demonstrated that M is of the form stated in the lemma. To complete the proof, note that Weichsel's Theorem implies $G \times H$ has exactly two components. The structure of the matrix M implies these two components must have $A \otimes B$ and $A \otimes B^T$ as bipartite adjacency matrices. ∎

Recall that a bipartite graph has *property* π if it has a bipartition (X, Y) for which there is an automorphism α satisfying $\alpha(X) = Y$ and $\alpha(Y) = X$. The next lemma characterizes graphs with property π in terms of their bipartite adjacency matrices.

Lemma 3. *Suppose H is a bipartite graph with bipartite adjacency matrix B. Then H has property π if and only if there are permutation matrices S and R for which $SBR = B^T$.*

Proof. Suppose $SBR = B^T$. From this, we deduce that B is square, so the partite sets of H are of equal size. Let $(U, V) = (\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_p\})$ be an ordered bipartition that gives B. Multiplication on the left by S permutes the rows of the matrix that follows it, so let σ be the permutation of $\{1, 2, \cdots, p\}$ for which S permutes row i to row $\sigma(i)$. Likewise, let ρ be the permutation for which R permutes column i to column $\rho(i)$. The condition $SBR = B^T$ implies $b_{ij} = b_{\rho(j)\sigma(i)}$ for $1 \leq i, j \leq p$. Construct a bijection $\alpha : V(H) \to V(H)$ defined as $\alpha(u_i) = v_{\sigma(i)}$ and $\alpha(v_j) = u_{\rho(j)}$ for $1 \leq i, j \leq p$. Clearly α reverses the bipartition, so we just need to show it is an isomorphism. Observe $u_i v_j \in E(H) \iff b_{ij} = 1 \iff b_{\sigma(j)\rho(i)} = 1 \iff u_{\sigma(j)} v_{\rho(i)} \in E(H) \iff \alpha(v_j)\alpha(u_i) \in E(H)$.

Conversely, suppose $H = (U, V) = (\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_p\})$ has property π, so there is an automorphism α of H satisfying $\alpha(U) = V$ and $\alpha(V) = U$. Let σ be the permutation of $\{1, 2, \cdots, p\}$ for which $\alpha(u_i) = v_{\sigma(i)}$ and let ρ be the permutation of $\{1, 2, \cdots, p\}$ for which $\alpha(v_j) = u_{\rho(j)}$. Observe $b_{ij} = 1 \iff u_i v_j \in E(H) \iff \alpha(u_i)\alpha(v_j) \in E(H) \iff v_{\sigma(i)} u_{\rho(j)} \in E(H) \iff b_{\rho(j)\sigma(i)} = 1$. Thus matrix B satisfies $b_{ij} = b_{\rho(j)\sigma(i)}$. It follows there are permutation matrices S and R with $SBR = B^T$. ■

3. Results

Now we come to our primary results. The following proposition was proved in [3]. The present proof is from the matrix point of view.

Proposition 1. *Suppose G and H are connected bipartite graphs. If one of them has property π, then the two components of $G \times H$ are isomorphic.*

Proof. Suppose one of G or H has property π, and let these graphs have bipartite adjacency matrices A and B respectively. Since $G \times H \cong H \times G$, there is no loss of generality in assuming that it is the the second factor H that has property π. According to Lemma 3, there are permutation matrices S and R for which $SBR = B^T$. Let I_m and I_n be identity matrices whose orders are equal to the number of rows and columns of A, respectively. Then $I_m \otimes S$ and $I_n \otimes R$ are permutation matrices for which $(I_m \otimes S)(A \otimes B)(I_n \otimes R) = A \otimes (SBR) = A \otimes B^T$. By Lemma 1, graphs with bipartite adjacency matrices $A \otimes B$ and $A \otimes B^T$ are isomorphic. By Lemma 2, the components of $G \times H$ are isomorphic. ■

The next proposition, which will help prove our main result, shows the converse of Proposition 1 holds if one of the factors is a complete bipartite graph. Without loss of generality, we may assume it is the first factor that is complete.

Proposition 2. *Suppose G is a complete bipartite graph and H is a connected bipartite graph. If the two components of $G \times H$ are isomorphic, then one of G or H has property π.*

Proof. Let $G = K_{mn}$, so its bipartite adjacency matrix A is an $m \times n$ matrix, every entry of which is 1. Let H have bipartite adjacency matrix B. Lemmas 1 and 2 imply that either $K(A \otimes B)L = A \otimes B^T$ or $K(A \otimes B)L = (A \otimes B^T)^T = A^T \otimes B$, for permutation matrices K and L.

Case 1. $K(A \otimes B)L = A \otimes B^T$. Writing the tensor products in block form, this equation is

$$K \begin{bmatrix} B & \cdots & B \\ \vdots & & \vdots \\ B & \cdots & B \end{bmatrix} L = \begin{bmatrix} K \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} & \cdots & K \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix} \end{bmatrix} L = \begin{bmatrix} B^T & \cdots & B^T \\ \vdots & & \vdots \\ B^T & \cdots & B^T \end{bmatrix}.$$

Note that, in particular, B must be square. Suppose $\mathbf{v}$ is a vector which is equal to exactly p columns of $K \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$. Then $\mathbf{v}$ is equal to exactly np columns of $K \begin{bmatrix} B & \cdots & B \\ \vdots & & \vdots \\ B & \cdots & B \end{bmatrix}$. Since multiplying this on the right by L just permutes the columns, it follows that $\mathbf{v}$ equals exactly np columns of $\begin{bmatrix} B^T & \cdots & B^T \\ \vdots & & \vdots \\ B^T & \cdots & B^T \end{bmatrix}$, from which we infer that each of the n column-blocks $\begin{bmatrix} B^T \\ \vdots \\ B^T \end{bmatrix}$ of this matrix has exactly p columns equal to $\mathbf{v}$. Consequently, if a column-vector appears with multiplicity p in $K \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$, then it appears with multiplicity p in $\begin{bmatrix} B^T \\ \vdots \\ B^T \end{bmatrix}$, so, as they have the same number of columns, these matrices differ only by a permutation of their columns. Thus there is a permutation matrix R for which $K \begin{bmatrix} BR \\ \vdots \\ BR \end{bmatrix} = \begin{bmatrix} B^T \\ \vdots \\ B^T \end{bmatrix}$.

Now let $\mathbf{w}$ be a row-vector that is equal to exactly q rows of BR, which means it is equal to exactly mq rows of $\begin{bmatrix} BR \\ \vdots \\ BR \end{bmatrix}$. Since multiplication on the left by K just permutes the rows, exactly mq rows of $K\begin{bmatrix} BR \\ \vdots \\ BR \end{bmatrix} = \begin{bmatrix} B^T \\ \vdots \\ B^T \end{bmatrix}$ are equal to $\mathbf{w}$, whence exactly q rows of B^T are equal to $\mathbf{w}$. It follows that the multiset of rows of BR equals the multiset of rows B^T, so these matrices differ only by a permutation of their rows, so there is a permutation matrix S with $SBR = B^T$. By Lemma 3, H has property π.

Case 2. $K(A \otimes B)L = A^T \otimes B$. It immediately follows that A is square, so G is a complete bipartite graph with partite sets of equal size. Obviously, then, G has property π. ∎

A *square* in G is a subgraph isomorphic to a cycle on four vertices. Such a square $x_i y_j x_k y_l x_i$ in G produces the following configuration in the bipartite adjacency matrix A, where rows i and k and columns j and l are indicated.

$$\begin{bmatrix} & 1 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 1 & \end{bmatrix}$$

We call such a configuration a *square* in A. A graph is called *square-free* if it has no squares, and this is equivalent to its bipartite adjacency matrix having no squares. Here is a proof that the converse of Proposition 1 holds for square-free graphs.

Proposition 3. *Suppose G and H are square-free connected bipartite graphs. If the components of $G \times H$ are isomorphic, then one of G or H has property π.*

Proof.[†] By Proposition 2, we may assume neither G nor H is a complete bipartite graph, so, as neither is a star, both have paths of length 3. Choose ordered bipartitions $(X, Y) = (\{x_1, x_2, \cdots, x_m\}, \{y_1, y_2, \cdots, y_n\})$ for G, and $(U, V) = (\{u_1, u_2, \cdots, u_p\}, \{v_1, v_2, \cdots, v_q\})$ for H, with the properties that $x_2 y_1 x_1 y_2$ and $u_2 v_1 u_1 v_2$ are paths of length 3 in G and H, respectively. Let A and B be the corresponding bipartite adjacency matrices for G and H, respectively. Then both matrices have the following configuration in the

[†]For this proof, we often denote the entry a_{ij} of a matrix A as A_{ij}, especially when the matrix is denoted by more than one symbol. Thus, for example, the i-j entry of A^T is $(A^T)_{ij}$ and the (i,k)-(j,l) entry of $A \otimes B$ is $(A \otimes B)_{(i,k)(j,l)}$.

upper-left corner: $\begin{bmatrix} 1 & 1 & \cdots \\ 1 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$. Note that the entry in the second row and column must be 0, for otherwise the four 1's would correspond to a square in G or H.

By Lemma 2, the two components of $G \times H$ have matrices $A \otimes B$ and $A \otimes B^T$. Since the components are isomorphic, Lemma 1 implies that there are permutation matrices K and L for which either

$$K(A \otimes B)L = A \otimes B^T \qquad \text{or} \qquad K(A \otimes B)L = (A \otimes B^T)^T = A^T \otimes B.$$

We want to show that if one of these equations holds, then one of G or H has property π. But observe that it is only necessary to prove this for the first equation. For suppose $K(A \otimes B)L = A \otimes B^T$ implies G or H has property π. Suppose also $K(A \otimes B)L = A^T \otimes B$. Choose permutation matrices M and N for which $M(A \otimes B)N = B \otimes A$ and $M(A^T \otimes B)N = B \otimes A^T$. The equation $K(A \otimes B)L = A^T \otimes B$ becomes $(MKM^{-1})(M(A \otimes B)N)(N^{-1}LN) = M(A^T \otimes B)N$, which reduces to $(MKM^{-1})(B \otimes A)(N^{-1}LN) = B \otimes A^T$. This has form $K(A \otimes B)L = A \otimes B^T$, with the roles of A and B reversed. It follows one of G or H has property π.

Thus assume $K(A \otimes B)L = A \otimes B^T$. To complete the proof, we must show one of G or H has property π. If we are lucky, and both K and L are identity matrices, then $A \otimes B = A \otimes B^T$, whence $B = B^T$, and H has property π by Lemma 3. In general, much more care is required, but it will be proved that H still has property π.

From $K(A \otimes B)L = A \otimes B^T$ it follows that B is square, so $p = q$. Using the indexing from Lemma 2, label the rows of $A \otimes B$, consecutively, with the list of ordered pairs

$$(1,1), (1,2), \cdots, (1,p), (2,1), (2,2), \cdots, (2,p), \cdots, (m,1), (m,2), \cdots, (m,p).$$

Likewise, label its columns with the list of ordered pairs

$$(1,1), (1,2), \cdots, (1,p), (2,1), (2,2), \cdots, (2,p), \cdots, (n,1), (n,2), \cdots, (n,p).$$

Regarding $A \otimes B$ as the matrix A with each entry a_{ij} replaced by $a_{ij}B$, for a fixed i and j the block of $A \otimes B$ in rows (i,k) and columns (j,l) (for $1 \leq k, l \leq p$) is the matrix $a_{ij}B$ in the a_{ij} position. We call this the *i-j block* of $A \otimes B$. Likewise, the i-j block of $A \otimes B^T$ is $a_{ij}B^T$.

The permutation matrix K permutes the rows of $A \otimes B$. The corresponding permutation κ on the set of row labels can be written componentwise as $\kappa(i,k) = (\kappa_A(i,k), \kappa_B(i,k))$. (That is, K moves row (i,k) to row $(\kappa_A(i,k), \kappa_B(i,k))$, so row (i,k) of $A \otimes B$ equals row $\kappa(i,k)$ of $K(A \otimes B)$.) Likewise, L permutes the columns of $A \otimes B$ by the permutation $\lambda(j,l) = (\lambda_A(j,l), \lambda_B(j,l))$. The equation $K(A \otimes B)L = A \otimes B^T$ means

$$
\begin{aligned}
(A \otimes B)_{(i,k)(j,l)} &= (A \otimes B^T)_{\kappa(i,k)\lambda(j,l)} \\
&= (A \otimes B^T)_{(\kappa_A(i,k),\kappa_B(i,k))(\lambda_A(j,l),\lambda_B(j,l))}.
\end{aligned} \tag{1}
$$

Permutations κ and λ respect the blocks of $A \otimes B$ in the manner described below.

Consider a path $u_i v_k u_j v_l$ of length 3 in H. In B, this is indicated by $b_{ik} = b_{jk} = b_{jl} = 1$, making the upper-left four blocks of $A \otimes B$ look as follows (up to the order of i and j, and k and l). Relevant rows and columns are labeled.

$$
\begin{array}{c|ccccc|ccccc|}
 & & (1,k) & & (1,l) & & & (2,k) & & (2,l) & \\
\hline
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
(1,j) & \cdots & 1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 & \cdots \\
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
(1,i) & \cdots & 1 & \cdots & 0 & \cdots & \cdots & 1 & \cdots & 0 & \cdots \\
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
\hline
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
(2,j) & \cdots & 1 & \cdots & 1 & \cdots & \cdots & 0 & \cdots & 0 & \cdots \\
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
(2,i) & \cdots & 1 & \cdots & 0 & \cdots & \cdots & 0 & \cdots & 0 & \cdots \\
 & & \vdots & & \vdots & & & \vdots & & \vdots & \\
\hline
\end{array}
\quad A \otimes B\,. \tag{2}
$$

We make two claims about the indices of such a path $u_i v_k u_j v_l$.

Claim A. $\kappa_A(1,i) = \kappa_A(1,j) \iff \lambda_A(1,k) = \lambda_A(1,l)$.

Claim B. $\kappa_B(1,i) = \kappa_B(1,j) \iff \lambda_B(1,k) = \lambda_B(1,l)$.

First we prove Claim A. Suppose $\kappa_A(1,i) = \kappa_A(1,j) = r$. Let $s = \lambda_A(1,k)$ and $t = \lambda_A(1,l)$. If $s \neq t$ then, from Diagram (2) above, the r-s and the r-t

blocks of $K(A \otimes B)L = A \otimes B^T$ would look as follows (up to the order of the rows and the order of the blocks).

$$
\begin{array}{c|ccc|c|ccc|}
 & & \lambda(1,k) & & & & \lambda(1,l) & \\ \cline{2-4}\cline{6-8}
\kappa(1,j) & \cdots & 1 & \cdots & \cdots & \cdots & 1 & \cdots \\
 & & \vdots & & & & \vdots & \\
\kappa(1,i) & \cdots & 1 & \cdots & \cdots & \cdots & 0 & \cdots \\
 & & \vdots & & & & \vdots & \\
\kappa(2,j) & \cdots & 1 & \cdots & \cdots & \cdots & 1 & \cdots \\ \cline{2-4}\cline{6-8}
\end{array}
$$

Notice $\kappa_A(2,j) = r$, for otherwise the 1's in rows $\kappa(1,j)$ and $\kappa(2,j)$ would produce a square in A.

Now, the columns $\lambda(1,k)$ and $\lambda(1,l)$ of the blocks are columns in copies of B^T. By the presence of the 0, these are different columns of B^T. By inspection, then, B^T has two columns which have two 1's in the same positions, a contradiction, since H is square-free. Hence $s = t$, so $\lambda_A(1,k) = \lambda_A(1,l)$. Conversely, if $\lambda_A(1,k) = \lambda_A(1,l)$, then the same argument — with the roles of columns and rows reversed — shows $\kappa_A(1,i) = \kappa_A(1,j)$. This proves Claim A.

To prove Claim B, suppose $\kappa_B(1,i) = \kappa_B(1,j)$. Then a portion of $K(A \otimes B)L$ would be as follows, for the reasons outlined below. Each block containing a 1 is a B^T.

$$
\begin{array}{c|ccccc|c|ccccc|}
 & & \lambda(1,k) & & \lambda(2,k) & & & & & \lambda(1,l) & & \\ \cline{2-6}\cline{8-12}
\kappa(1,j) & \cdots & 1 & \cdots & 1 & \cdots & & \cdots & \cdots & 1 & \cdots & \cdots \\
 & & \vdots & & \vdots & & \cdots & & & \vdots & & \\
\kappa(2,j) & \cdots & 1 & \cdots & 0 & \cdots & & \cdots & \cdots & 1 & \cdots & \cdots \\ \cline{2-6}\cline{8-12}
 & & & \vdots & & & & & & \vdots & & \\ \cline{2-6}\cline{8-12}
 & & \vdots & & \vdots & & & & & \vdots & & \\
\kappa(1,i) & \cdots & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\
 & & \vdots & & \vdots & & & & & \vdots & & \\ \cline{2-6}\cline{8-12}
\end{array}
$$

From $\kappa_B(1,i) = \kappa_B(1,j)$, it follows immediately from injectivity of κ that $\kappa_A(1,i) \neq \kappa_A(1,j)$, as indicated in the diagram. Now, $\lambda_A(1,k) \neq \lambda_A(1,l)$, as indicated, for otherwise row $\kappa_B(1,i)$ of the lower-left block would have a 0 in column $\lambda_B(1,l)$, while the same row $\kappa_B(1,j)$ of the upper-left block would have a 1 in column $\lambda_B(1,l)$ — a contradiction since both blocks equal B^T. Also, $\kappa_A(1,j) = \kappa_A(2,j)$ for otherwise rows $\kappa(1,j)$ and $\kappa(2,j)$ and columns $\lambda(1,k)$ and $\lambda(1,l)$ would yield a square in A. Finally, column $\lambda_B(1,k)$ of the upper-left B^T and column $\lambda_B(1,l)$ of the upper-right B^T both have 1's in their rows $\kappa_B(1,j)$ and $\kappa_B(2,j)$. Therefore these are the same columns of B^T — meaning $\lambda_B(1,k) = \lambda_B(1,l)$ — since B^T has no squares. Conversely, if $\lambda_B(1,k) = \lambda_B(1,l)$, interchanging rows and columns of the above argument shows $\kappa_B(1,i) = \kappa_B(1,j)$. The proof of Claim B is complete.

Now we are ready to show that H has property π. The proof involves two cases, depending on whether or not $\kappa_A(1,2) = \kappa_A(1,1)$.

Case 1. Suppose $\kappa_A(1,2) = \kappa_A(1,1)$. Set $r = \kappa_A(1,1)$, and $s = \lambda_A(1,1)$. Consider a path of form $u_2v_1u_1v_{i_1}u_{i_2}v_{i_3}u_{i_4}\cdots$, which begins with edges u_2v_1 and v_1u_1, but is otherwise arbitrary. Then Claim A applied to path $u_2v_1u_1v_{i_1}$ gives $\lambda_A(1,1) = \lambda_A(1,i_1)$. Combining this with Claim A applied to the path $u_{i_2}v_{i_1}u_1v_1$ gives $\kappa_A(1,i_2) = \kappa_A(1,1)$. *This* combined with Claim A applied to the path $u_1v_{i_1}u_{i_2}v_{i_3}$ gives $\lambda_A(1,i_1) = \lambda_A(1,i_3)$, which applied to $u_{i_4}v_{i_3}u_{i_2}v_{i_1}$ produces $\kappa_A(1,i_4) = \kappa_A(1,i_2)$. Continuing in this fashion, $r = \kappa_A(1,1) = \kappa_A(1,i_2) = \kappa_A(1,i_4) = \cdots$ and $s = \lambda_A(1,1) = \lambda_A(1,i_1) = \lambda_A(1,i_3) = \cdots$. Since every vertex of H is on such a path (or on a path $v_2u_1v_1u_{i_1}v_{i_2}u_{i_3}v_{i_4}\cdots$, to which a similar argument applies, beginning with the fact $\lambda_A(1,1) = \lambda_A(1,2)$ which follows from Claim A applied to $u_2v_1u_1v_2$) we have

$$\kappa_A(1,k) = \kappa_A(1,1) = r \quad \text{for} \quad 1 \le k \le p, \tag{3}$$

$$\lambda_A(1,l) = \lambda_A(1,1) = s \quad \text{for} \quad 1 \le l \le p. \tag{4}$$

Now, the map $k \mapsto \kappa_B(1,k)$ is a permutation of $\{1,2,\cdots,p\}$, for it is injective, as $\kappa_B(1,k) = \kappa_B(1,l)$ means $\kappa(1,k) = (\kappa_A(1,k),\kappa_B(1,k)) = (\kappa_A(1,l),\kappa_B(1,l)) = \kappa(1,l)$ so $k = l$ by injectivity of κ. Similarly, $l \mapsto \lambda_B(1,l)$ is a permutation of $\{1,2,\cdots,p\}$. Equations (1), (3) and (4) yield

$$\begin{aligned} b_{kl} &= (A \otimes B)_{(1,k)(1,l)} = (A \otimes B^T)_{(\kappa_A(1,k),\kappa_B(1,k))(\lambda_A(1,l),\lambda_B(1,l))} \\ &= (A \otimes B^T)_{(r,\kappa_B(1,k))(s,\lambda_B(1,l))} = (B^T)_{\kappa_B(1,k),\lambda_B(1,l)}. \end{aligned}$$

Hence if S is the permutation matrix that permutes row k of B to row $\kappa_B(1,k)$, and R is the permutation matrix that permutes column l of B to column $\lambda_B(1,l)$, then $SBR = B^T$, so H has property π by Lemma 3. This completes Case 1.

Here is the intuitive picture of what happened in Case 1, and of what will happen in Case 2. In $G \times H$, the fiber over an edge of G consists of two disjoint copies of H, one in each component. In Case 1, permutations κ and λ correspond to an isomorphism between the components, and Equations (3) and (4) imply that the fiber over an edge of G is mapped into the fiber over a (possibly different) edge of G. In Case 2, the fiber over an edge of G will get mapped into the fiber over an edge of H, forcing $G \cong H$. The greater symmetry makes the situation slightly more complex.

Case 2. Suppose $\kappa_A(1,2) \neq \kappa_A(1,1)$. This is illustrated schematically below. The upper-left corner of $A \otimes B$ is on the left. Each block containing a 1 is a B. On the right are the $\kappa_A(1,1)$-$\lambda_A(1,1)$ and $\kappa_A(1,2)$-$\lambda_A(1,1)$ blocks of $K(A \otimes B)L$, both of which are equal to B^T.

$$\begin{array}{c|ccc|ccc|c}
 & (1,1) & (1,2) & & (2,1) & (2,2) & & \\ \hline
(1,1) & 1 & 1 & \cdots & 1 & 1 & \cdots & \\
(1,2) & 1 & 0 & \cdots & 1 & 0 & \cdots & \cdots \\
 & \vdots & \vdots & & \vdots & \vdots & & \\ \hline
(2,1) & 1 & 1 & \cdots & 0 & 0 & \cdots & \\
(2,2) & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots \\
 & \vdots & \vdots & & \vdots & \vdots & & \\ \hline
 & & & \vdots & & \vdots & &
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|ccc|}
 & \lambda(1,1) & & \lambda(2,1) \\ \hline
 & \vdots & & \vdots \\
\kappa(1,1) & 1 & \cdots & 1 \\
 & \vdots & & \vdots \\ \hline
 & \vdots & & \vdots \\ \hline
 & \vdots & & \vdots \\
\kappa(1,2) & 1 & \cdots & 1 \\
 & \vdots & & \vdots \\ \hline
\end{array}$$

$A \otimes B$ $\qquad\qquad$ $A \otimes B^T$

Notice $\lambda_A(1,1) = \lambda_A(2,1)$, for otherwise there would be a square in A. Given this, $\kappa_B(1,1) = \kappa_B(1,2)$, for the diagram indicates rows $\kappa_A(1,1)$ and $\kappa_A(1,2)$ of B^T each have two 1's in identical positions. Therefore

$$\kappa_B(1,2) = \kappa_B(1,1), \tag{5}$$

$$\lambda_A(1,1) = \lambda_A(2,1). \tag{6}$$

Consider a path of form $u_2 v_1 u_1 v_{i_1} u_{i_2} v_{i_3} u_{i_4} \cdots$. Claim B and equation (5) applied to the first four vertices produces $\lambda_B(1,1) = \lambda_B(1,i_1)$. Combining this with Claim B applied to the path $u_{i_2} v_{i_1} u_1 v_1$ gives $\kappa_B(1,i_2) = \kappa_B(1,1)$. *This* combined with Claim B applied to the path $u_1 v_{i_1} u_{i_2} v_{i_3}$ gives $\lambda_B(1,i_1) =$

$\lambda_B(1, i_3)$, which applied to $u_{i_4} v_{i_3} u_{i_2} v_{i_1}$ produces $\kappa_B(1, i_4) = \kappa_B(1, i_2)$. Continuing in this fashion, $\kappa_B(1, 1) = \kappa_B(1, i_2) = \kappa_B(1, i_4) = \cdots$ and $\lambda_B(1, 1) = \lambda_B(1, i_1) = \lambda_B(1, i_3) = \cdots$. Since every vertex of H is on such a path (or on a path $v_2 u_1 v_1 u_{i_1} v_{i_2} u_{i_3} v_{i_4} \cdots$, to which a similar argument applies, beginning with the fact $\lambda_B(1, 2) = \lambda_B(1, 1)$ which follows from Claim B applied to Equation (6) and path $u_2 v_1 u_1 v_2$) it follows

$$\kappa_B(1, k) = \kappa_B(1, 1) \quad \text{for } 1 \le k \le p, \tag{7}$$

$$\lambda_B(1, l) = \lambda_B(1, 1) \quad \text{for } 1 \le l \le p. \tag{8}$$

Two more equations of this type are needed. They will depend on the following claim.

Claim C. If $x_i y_k x_j y_l$ is a path in G, then $\kappa_A(i, 1) = \kappa_A(j, 1) \iff \lambda_A(k, 1) = \lambda_A(l, 1)$.

To prove this, suppose $\kappa_A(i, 1) = \kappa_A(j, 1)$. The situation is illustrated in the following diagram. The i-k, j-k, i-l and j-l blocks of $A \otimes B$ are indicated on the left, and a portion of $A \otimes B^T$ appears on the right. The upper-left block of this is the $\kappa_A(j, 1)$-$\lambda_A(k, 1)$ block of $A \otimes B^T$, and since $\kappa_A(i, 1) = \kappa_A(j, 1)$, both rows $\kappa(i, 1)$ and $\kappa(j, 1)$ are aligned with this block. The other rows and columns are in their indicated positions for the following reasons.

$$
\begin{array}{c|cc|c|cc|}
 & (k,1) & (k,2) & & (l,1) & (l,2) \\
\cline{2-3}\cline{5-6}
(j,1) & 1 & 1 & & 1 & 1 \\
(j,2) & 1 & 0 & \cdots & 1 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots \\
\cline{2-3}\cline{5-6}
\multicolumn{1}{c}{} & \multicolumn{2}{c}{\vdots} & \multicolumn{1}{c}{} & \multicolumn{2}{c}{\vdots} \\
\cline{2-3}\cline{5-6}
(i,1) & 1 & 1 & & 0 & 0 \\
(i,2) & 1 & 0 & \cdots & 0 & 0 \\
 & \vdots & \vdots & & \vdots & \vdots \\
\cline{2-3}\cline{5-6}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c|ccc|c|c|}
 & \lambda(k,1) & & \lambda(l,1) & & \lambda(k,2) \\
\cline{2-4}\cline{6-6}
\kappa(j,1) & 1 & \cdots & 1 & & 1 \\
 & \vdots & & \vdots & \cdots & \vdots \\
\kappa(i,1) & 1 & \cdots & 0 & & 1 \\
\cline{2-4}\cline{6-6}
\multicolumn{1}{c}{} & \multicolumn{3}{c}{\vdots} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\vdots} \\
\cline{2-4}\cline{6-6}
 & \vdots & & \vdots & & \vdots \\
\kappa(j,2) & 1 & \cdots & 1 & \cdots & 0 \\
 & \vdots & & \vdots & & \vdots \\
\cline{2-4}\cline{6-6}
\end{array}
$$

$A \otimes B$ $\qquad\qquad$ $A \otimes B^T$

First, $\lambda_A(k, 2) \neq \lambda_A(k, 1)$, as indicated, for otherwise there would be a square in the upper-right block. Next, $\kappa_A(j, 2) \neq \kappa_A(j, 1)$, as indicated, for otherwise column $\lambda_B(k,1)$ of the B^T on the upper-left would differ from column $\lambda_B(k,2)$ of the B^T on the upper-right, for they would differ in the $\kappa_B(j,2)$ position — a contradiction since these two columns both have 1's in their $\kappa_B(j,1)$ and $\kappa_B(i,1)$ positions. Finally, $\lambda_A(k,1) = \lambda_A(l,1)$, for otherwise there would be a square in A. Conversely, if $\lambda_A(k,1) = \lambda_A(l,1)$, the same argument with rows and columns reversed gives $\kappa_A(i,1) = \kappa_A(j,1)$. This finishes the proof of Claim C.

Claim C applied to Equation (6) and the path $x_2y_1x_1y_2$ of G gives $\kappa_A(2,1) = \kappa_A(1,1)$. Now consider an arbitrary path of form $x_2y_1x_1y_{i_1}x_{i_2}y_{i_3}x_{i_4}\cdots$ in G. Claim C applied to equation $\kappa_A(2,1) = \kappa_A(1,1)$ and path $x_2y_1x_1y_{i_1}$ gives $\lambda_A(1,1) = \lambda_A(i_1,1)$. Combining this with Claim C applied to the path $x_{i_2}y_{i_1}x_1y_1$ gives $\kappa_A(i_2,1) = \kappa_A(1,1)$. *This* combined with Claim C applied to the path $x_1y_{i_1}x_{i_2}y_{i_3}$ gives $\lambda_A(i_1,1) = \lambda_A(i_3,1)$, which applied to $x_{i_4}y_{i_3}x_{i_2}y_{i_1}$ produces $\kappa_A(i_4,1) = \kappa_A(i_2,1)$. Continuing in this fashion, $\kappa_A(1,1) = \kappa_A(i_2,1) = \kappa_A(i_4,1) = \cdots$ and $\lambda_A(1,1) = \lambda_A(i_1,1) = \lambda_A(i_3,1) = \cdots$. Since every vertex of G is on such a path (or on a path $y_2x_1y_1x_{i_1}y_{i_2}x_{i_3}y_{i_4}\cdots$, to which a similar argument applies, beginning with Equation (6)), it follows

$$\kappa_A(i,1) = \kappa_A(1,1) \quad \text{for } 1 \le i \le m, \tag{9}$$

$$\lambda_A(j,1) = \lambda_A(1,1) \quad \text{for } 1 \le j \le n. \tag{10}$$

Now, the map $k \mapsto \kappa_A(1,k)$ is an injection of $\{1,2,\cdots,p\}$ into $\{1,2,\cdots,m\}$, for, using Equation (7), $\kappa_A(1,k) = \kappa_A(1,l)$ means $\kappa(1,k) = (\kappa_A(1,k), \kappa_B(1,k)) = (\kappa_A(1,l), \kappa_B(1,l)) = \kappa(1,l)$, so $k = l$ by injectivity of κ. Similarly, map $i \mapsto \kappa_B(i,1)$ is an injection of $\{1,2,\cdots,m\}$ into $\{1,2,\cdots,p\}$, for, using Equation (9), $\kappa_B(i,1) = \kappa_B(j,1)$ means $\kappa(i,1) = (\kappa_A(i,1), \kappa_B(i,1)) = (\kappa_A(j,1), \kappa_B(j,1)) = \kappa(j,1)$, so $i = j$ by injectivity of κ. It follows that $m = p$, and maps $k \mapsto \kappa_A(1,k)$ and $i \mapsto \kappa_B(i,1)$ are permutations of $\{1,2,\cdots,p\}$. By the same argument, using λ and Equations (8) and (10), maps $j \mapsto \lambda_B(j,1)$ and $l \mapsto \lambda_A(1,l)$ are permutations of $\{1,2,\cdots,p\}$. In particular, this means both A and B are $p \times p$ matrices.

Let C be the $p\times p$ sub-matrix of $A\otimes B^T$ whose i-j entry is the $(i, \kappa_B(1,1))$-$(j, \lambda_B(1,1))$ entry of $A \otimes B^T$. In other words, the i-j entry of C is the $\kappa_B(1,1)$-$\lambda_B(1,1)$ entry of the i-j block of $A \otimes B^T$. As the $(1,1)$-$(1,1)$ entry of $A \otimes B$ is 1, the $(\kappa_A(1,1), \kappa_B(1,1))$-$(\lambda_A(1,1), \lambda_B(1,1))$ entry of $K(A \otimes B)L = A \otimes B^T$ is 1. Hence the $\kappa_B(1,1)$-$\lambda_B(1,1)$ entry of B^T is 1, whence it follows $C = A$. Likewise, let D be the $p \times p$ sub-matrix of $A \otimes B$ whose i-j entry equals the $(i,1)$-$(j,1)$ entry of $A \otimes B$, so $D = A$. Equations (1), (7) and (8) yield

$$\begin{aligned} b_{kl} = (A \otimes B)_{(1,k)(1,l)} &= (A \otimes B^T)_{(\kappa_A(1,k),\kappa_B(1,k))(\lambda_A(1,l),\lambda_B(1,l))} \\ &= (A \otimes B^T)_{(\kappa_A(1,k),\kappa_B(1,1))(\lambda_A(1,l),\lambda_B(1,1))} \\ &= c_{\kappa_A(1,k)\lambda_A(1,l)} = a_{\kappa_A(1,k)\lambda_A(1,l)}. \end{aligned}$$

This implies there are permutation matrices S' and R' for which

$$S'BR' = A. \tag{11}$$

Equations (1), (9) and (10) yield

$$\begin{aligned} a_{ij} &= d_{ij} = (A \otimes B)_{(i,1)(j,1)} = (A \otimes B^T)_{(\kappa_A(i,1),\kappa_B(i,1))(\lambda_A(j,1),\lambda_B(j,1))} \\ &= (A \otimes B^T)_{(\kappa_A(1,1),\kappa_B(i,1))(\lambda_A(1,1),\lambda_B(j,1))} = (B^T)_{\kappa_B(i,1)\lambda_B(j,1)}. \end{aligned}$$

This implies there are permutation matrices S'' and R'' for which

$$S''AR'' = B^T. \tag{12}$$

From Equations (11) and (12), $(S''S')B(R'R'') = B^T$, so H has property π by Lemma 3. ■

The proof of Proposition 3 hinges on using the equation $K(A \otimes B)L = A \otimes B^T$ to derive $SBR = B^T$. The square-free hypothesis makes the permutations in the first equation tame enough to easily imply the second. But, as Proposition 2 suggests, the square-free hypothesis is by no means necessary. The following conjecture, if proved, would prove the converse of Proposition 1 in complete generality.

Conjecture. Suppose G and H are connected bipartite graphs with bipartite adjacency matrices A and B. If there are permutation matrices K and L for which $K(A \otimes B)L = A \otimes B^T$, then there are permutation matrices S and R for which $SBR = B^T$.

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