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AN UPPER BOUND FOR MAXIMUM NUMBER OF EDGES IN A STRONGLY MULTIPLICATIVE GRAPH

Chandrashekar Adiga

Department of Studies in Mathematics
University of Mysore
Manasagangotri, Mysore–570006, India

e-mail: adiga_c@yahoo.com

and

Mahadev Smitha

Department of Mathematics
Sri Jayachamarajendra College of Engineering
Mysore–570006, India

e-mail: smithamahadev@yahoo.co.in

Abstract

In this note we give an upper bound for $\lambda(n)$, the maximum number of edges in a strongly multiplicative graph of order n, which is sharper than the upper bounds given by Beineke and Hegde [3] and Adiga, Ramaswamy and Somashekara [2], for $n \geq 28$.

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1. Introduction

A graph labelling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. During the past forty years or so, an enormous amount of research work has been done on graph labelling and dozens of graph labelling techniques have been studied. These interesting problems have been motivated by practical problems and labelled graphs serve as useful models for a variety of applications such as: coding theory, X-ray crystallography, circuit design etc. Recently, Beineke and Hegde [3] have studied strongly multiplicative graphs. A graph with n vertices is said to be strongly multiplicative if its vertices can be labelled $1, 2, \ldots, n$, so that the values on the edges, obtained as the product of labels of their end vertices, are all distinct. Beineke and Hegde [3] have shown that many graphs such as trees, wheels and grids are strongly multiplicative. They have also obtained an upper bound for the maximum number of edges $\lambda(n)$ for a given strongly multiplicative graph of order n. In fact they have proved that

$$\begin{aligned}
&\lambda(4r) \leq 6r^2, \\
&\lambda(4r+1) \leq 6r^2 + 4r, \\
&\lambda(4r+2) \leq 6r^2 + 6r + 1, \\
&\lambda(4r+3) \leq 6r^2 + 10r + 3.
\end{aligned}$$

Erdös [4] has obtained an asymptotic formula for $\lambda(n)$. Recently in [2], Adiga, Ramaswamy and Somashekara have obtained a sharper upper bound for $\lambda(n)$. In this note we obtain an upper bound for $\lambda(n)$ which is conjectured to be sharper than that upper bounds given in [2] and [3] for $n \geq 28$.

2. Main Result

Theorem 2.1. *The maximum number of edges in a strongly multiplicative graph*

$$\lambda(n) \leq \frac{n(n-1)}{2} - \left[\sum_{m=2}^{n}\left(1 + \sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^k}\right) - p_2(k)\right)\right)\right]$$

where

$$R\left(\frac{m}{p^k}\right) = \begin{cases} \frac{m}{p^k} - 2, & \text{if } p^{2k} | m, \\ \frac{m}{p^k} - 1, & \text{otherwise,} \end{cases}$$

p is the smallest prime divisor of m, α is the highest power of p such that $p^\alpha | m$, and $p_2(k)$ is the number of partitions of k into exactly two distinct parts.

Proof. We label the vertices of the complete graph K_n with integers $1, 2, \ldots, n$. To find $\lambda(n)$ we successively delete edges whose value is duplicated with the value of another edge. Let $2 \leq m \leq n$ and p be the smallest prime divisor of m. Suppose α is the highest power of p such that $p^\alpha | m$. Consider the array of products

$$\begin{array}{lllllll} 1.m & 1.(m-1) & 1.(m-3) & \cdots & 1.4 & 1.3 & 1.2, \\ 2.m & 2.(m-1) & 2.(m-3) & \cdots & 2.4 & 2.3, & \\ 3.m & 3.(m-1) & 3.(m-3) & \cdots & 3.4, & & \\ \vdots & \vdots & \vdots & & & & \\ (m-2).m & (m-2).(m-1), & & & & & \\ (m-1).m. & & & & & & \end{array} \tag{2.1}$$

We have

$$\begin{array}{cccc} 1.m = p.\frac{m}{p} & 1.m = p^2.\frac{m}{p^2} & \cdots & 1.m = p^\alpha.\frac{m}{p^\alpha}, \\ 2.m = 2p.\frac{m}{p} & 2.m = 2p^2.\frac{m}{p^2} & \cdots & 2.m = 2p^\alpha.\frac{m}{p^\alpha}, \\ \vdots & \vdots & & \vdots \\ \left(\frac{m}{p}-1\right)m = & \left(\frac{m}{p^2}-1\right)m = & \cdots & \left(\frac{m}{p^\alpha}-1\right)m = \\ \left(\frac{m}{p}-1\right)p\frac{m}{p} & \left(\frac{m}{p^2}-1\right)p^2\frac{m}{p^2} & & \left(\frac{m}{p^\alpha}-1\right)p^\alpha\frac{m}{p^\alpha}. \end{array} \tag{2.2}$$

Observe that in the equation $l.m = lp^k.\frac{m}{p^k}$, we have $l.p^k = \frac{m}{p^k}$ if $p^{2k}|m$ and $l = \frac{m}{p^{2k}}$. Also, for every distinct integer pair x, y such that $x + y = k$, the edge in the graph (and its corresponding product) is represented by two equivalent unordered pairs, for $l_1 = l_2 = \frac{m}{p^k}$.

From (2.2) and the above two observations it follows that the total number of repetitions in the other columns of products in the first column of (2.1) is greater than or equal to $1 + \sum_{k=1}^{\alpha}(R(\frac{m}{p^k}) - p_2(k))$. Since K_n has $\frac{n(n-1)}{2}$ edges, it follows that

$$\lambda(n) \leq \frac{n(n-1)}{2} - \left[\sum_{m=2}^{n}\left(1 + \sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^k}\right) - p_2(k)\right)\right)\right]. \qquad \blacksquare$$

Remarks. (a) We have

$$p_2(k) = \left[\frac{k-1}{2}\right].$$

(b) Although our upper bound

$$B_n = \frac{n(n-1)}{2} - \left[\sum_{m=2}^{n} \left(1 + \sum_{k=1}^{\alpha} \left(R\left(\frac{m}{p^k}\right) - p_2(k) \right) \right) \right]$$

for $\lambda(n)$ looks complicated, we have a recurrence formula to calculate B_n:

$$B_{n+1} = B_n + (n+1) + \sum_{k=1}^{\alpha_{n+1}} \left(R\left(\frac{n+1}{p^k}\right) - p_2(k) \right)$$

where p is the smallest prime divisor of $n+1$ and α_{n+1} is the highest power of p such that $p^{\alpha_{n+1}} | n+1$.

n	$\lambda(n)$	upper bound for $\lambda(n)$ using our theorem	upper bound for $\lambda(n)$ given by Adiga *et al.*	upper bound for $\lambda(n)$ given by Beineke and Hegde
28	251	280	283	294
29	279	309	312	322
30	291	325	327	337
31	321	356	358	367
32	338	368	375	384
33	357	391	398	416
34	374	409	415	433
35	393	438	444	467
36	406	450	463	486
37	442	487	500	522
38	461	507	519	541
39	481	534	546	579
40	496	544	567	600
41	536	585	608	640
42	554	607	629	661
43	596	650	672	703
44	618	664	695	726
45	639	692	726	770
46	662	716	749	793
47	708	763	796	839
48	726	774	821	864
49	763	818	864	912
50	786	844	889	937
51	817	895	924	987
52	850	911	951	1014
53	903	964	1004	1066
54	928	992	1051	1093

(c) Our upper bound for $\lambda(n)$ can be further improved by considering other prime divisors (we have considered only smallest prime divisor).

(d) The following table suggests that the upper bound for $\lambda(n)$ given in the above theorem is sharper than the upper bounds given by Beineke and Hegde [3] and Adiga *et al.* [2].

n	$\lambda(n)$	upper bound for $\lambda(n)$ using our theorem	upper bound for $\lambda(n)$ given by Adiga *et al.*	upper bound for $\lambda(n)$ given by Beineke and Hegde
55	959	1037	1106	1147
56	981	1054	1135	1176
57	1013	1093	1174	1232
58	1042	1123	1203	1261
59	1099	1182	1262	1319
60	1117	1200	1293	1350

The values of column 2 of the above table were obtained on using the formula for $\lambda(n)$ [1].

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