# AN UPPER BOUND FOR MAXIMUM NUMBER OF EDGES IN A STRONGLY MULTIPLICATIVE GRAPH 

Chandrashekar Adiga<br>Department of Studies in Mathematics<br>University of Mysore<br>Manasagangotri, Mysore-570006, India<br>e-mail: adiga_c@yahoo.com<br>AND<br>Mahadev Smitha<br>Department of Mathematics<br>Sri Jayachamarajendra College of Engineering<br>Mysore-570006, India<br>e-mail: smithamahadev@yahoo.co.in


#### Abstract

In this note we give an upper bound for $\lambda(n)$, the maximum number of edges in a strongly multiplicative graph of order $n$, which is sharper than the upper bounds given by Beineke and Hegde [3] and Adiga, Ramaswamy and Somashekara [2], for $n \geq 28$.


Keywords: graph labelling, strongly multiplicative graphs.
2000 Mathematics Subject Classification: 05C78.

## 1. Introduction

A graph labelling is an assignment of integers to the vertices or edges, or both, subject to certain conditions. During the past forty years or so, an enormous amount of research work has been done on graph labelling and dozens of graph labelling techniques have been studied. These interesting problems have been motivated by practical problems and labelled graphs
serve as useful models for a variety of applications such as: coding theory, X-ray crystallography, circuit design etc. Recently, Beineke and Hegde [3] have studied strongly multiplicative graphs. A graph with $n$ vertices is said to be strongly multiplicative if its vertices can be labelled $1,2, \ldots, n$, so that the values on the edges, obtained as the product of labels of their end vertices, are all distinct. Beineke and Hegde [3] have shown that many graphs such as trees, wheels and grids are strongly multiplicative. They have also obtained an upper bound for the maximum number of edges $\lambda(n)$ for a given strongly multiplicative graph of order $n$. In fact they have proved that

$$
\begin{aligned}
& \lambda(4 r) \leq 6 r^{2}, \\
& \lambda(4 r+1) \leq 6 r^{2}+4 r, \\
& \lambda(4 r+2) \leq 6 r^{2}+6 r+1, \\
& \lambda(4 r+3) \leq 6 r^{2}+10 r+3 .
\end{aligned}
$$

Erdös [4] has obtained an asymptotic formula for $\lambda(n)$. Recently in [2], Adiga, Ramaswamy and Somashekara have obtained a sharper upper bound for $\lambda(n)$. In this note we obtain an upper bound for $\lambda(n)$ which is conjectured to be sharper than that upper bounds given in [2] and [3] for $n \geq 28$.

## 2. Main Result

Theorem 2.1. The maximum number of edges in a strongly multiplicative graph

$$
\lambda(n) \leq \frac{n(n-1)}{2}-\left[\sum_{m=2}^{n}\left(1+\sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^{k}}\right)-p_{2}(k)\right)\right)\right]
$$

where

$$
R\left(\frac{m}{p^{k}}\right)= \begin{cases}\frac{m}{p^{k}}-2, & \text { if } p^{2 k} \mid m \\ \frac{m}{p^{k}}-1, & \text { otherwise }\end{cases}
$$

$p$ is the smallest prime divisor of $m, \alpha$ is the highest power of $p$ such that $p^{\alpha} \mid m$, and $p_{2}(k)$ is the number of partitions of $k$ into exactly two distinct parts.

Proof. We label the vertices of the complete graph $K_{n}$ with integers $1,2, \ldots, n$. To find $\lambda(n)$ we successively delete edges whose value is duplicated with the value of another edge. Let $2 \leq m \leq n$ and $p$ be the smallest prime divisor of $m$. Suppose $\alpha$ is the highest power of $p$ such that $p^{\alpha} \mid m$. Consider the array of products

$$
\begin{array}{ccccccc}
1 . m & 1 .(m-1) & 1 .(m-3) & \cdots & 1.4 & 1.3 & 1.2, \\
2 . m & 2 .(m-1) & 2 .(m-3) & \cdots & 2.4 & 2.3, & \\
3 . m & 3 .(m-1) & 3 .(m-3) & \cdots & 3.4, & & \\
\vdots & \vdots & \vdots & & &  \tag{2.1}\\
(m-2) . m & (m-2) \cdot(m-1), & & & & \\
(m-1) . m . & & & & &
\end{array}
$$

We have

$$
\begin{array}{cccc}
1 . m=p \cdot \frac{m}{p} & 1 . m=p^{2} \cdot \frac{m}{p^{2}} & \cdots & 1 . m=p^{\alpha} \cdot \frac{m}{p^{\alpha}} \\
2 . m=2 p \cdot \frac{m}{p} & 2 . m=2 p^{2} \cdot \frac{m}{p^{2}} & \cdots & 2 . m=2 p^{\alpha} \cdot \frac{m}{p^{\alpha}} \\
\vdots & \vdots & & \vdots  \tag{2.2}\\
\left(\frac{m}{p}-1\right) m= & \left(\frac{m}{p^{2}}-1\right) m= & & \left(\frac{m}{p^{\alpha}}-1\right) m= \\
\left(\frac{m}{p}-1\right) p \frac{m}{p} & \left(\frac{m}{p^{2}}-1\right) p^{2} \frac{m}{p^{2}} & & \left(\frac{m}{p^{\alpha}}-1\right) p^{\alpha} \frac{m}{p^{\alpha}}
\end{array}
$$

Observe that in the equation $l . m=l p^{k} \cdot \frac{m}{p^{k}}$, we have $l . p^{k}=\frac{m}{p^{k}}$ if $p^{2 k} \mid m$ and $l=\frac{m}{p^{2 k}}$. Also, for every distinct integer pair $x, y$ such that $x+y=k$, the edge in the graph (and its corresponding product) is represented by two equivalent unordered pairs, for $l_{1}=l_{2}=\frac{m}{p^{k}}$.

From (2.2) and the above two observations it follows that the total number of repetitions in the other columns of products in the first column of (2.1) is greater than or equal to $1+\sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^{k}}\right)-p_{2}(k)\right)$. Since $K_{n}$ has $\frac{n(n-1)}{2}$ edges, it follows that

$$
\lambda(n) \leq \frac{n(n-1)}{2}-\left[\sum_{m=2}^{n}\left(1+\sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^{k}}\right)-p_{2}(k)\right)\right)\right]
$$

Remarks. (a) We have

$$
p_{2}(k)=\left[\frac{k-1}{2}\right] .
$$

(b) Although our upper bound

$$
B_{n}=\frac{n(n-1)}{2}-\left[\sum_{m=2}^{n}\left(1+\sum_{k=1}^{\alpha}\left(R\left(\frac{m}{p^{k}}\right)-p_{2}(k)\right)\right)\right]
$$

for $\lambda(n)$ looks complicated, we have a recurrence formula to calculate $B_{n}$ :

$$
B_{n+1}=B_{n}+(n+1)+\sum_{k=1}^{\alpha_{n+1}}\left(R\left(\frac{n+1}{p^{k}}\right)-p_{2}(k)\right)
$$

where $p$ is the smallest prime divisor of $n+1$ and $\alpha_{n+1}$ is the highest power of $p$ such that $p^{\alpha_{n+1}} \mid n+1$.

| $n$ | $\lambda(n)$ | upper bound for <br> $\lambda(n)$ using <br> our theorem | upper bound for <br> $\lambda(n)$ given by <br> Adiga et al. | upper bound for <br> $\lambda(n)$ given by <br> Beineke and Hegde |
| :---: | :---: | :---: | :---: | :---: |
| 28 | 251 | 280 | 283 | 294 |
| 29 | 279 | 309 | 312 | 322 |
| 30 | 291 | 325 | 327 | 337 |
| 31 | 321 | 356 | 358 | 367 |
| 32 | 338 | 368 | 375 | 384 |
| 33 | 357 | 391 | 398 | 416 |
| 34 | 374 | 409 | 415 | 433 |
| 35 | 393 | 438 | 444 | 467 |
| 36 | 406 | 450 | 463 | 486 |
| 37 | 442 | 487 | 500 | 522 |
| 38 | 461 | 507 | 519 | 541 |
| 39 | 481 | 534 | 546 | 579 |
| 40 | 496 | 544 | 567 | 600 |
| 41 | 536 | 585 | 608 | 640 |
| 42 | 554 | 607 | 629 | 661 |
| 43 | 596 | 650 | 672 | 703 |
| 44 | 618 | 664 | 695 | 726 |
| 45 | 639 | 692 | 726 | 770 |
| 46 | 662 | 716 | 749 | 793 |
| 47 | 708 | 763 | 796 | 839 |
| 48 | 726 | 774 | 821 | 864 |
| 49 | 763 | 818 | 864 | 912 |
| 50 | 786 | 844 | 889 | 937 |
| 51 | 817 | 895 | 924 | 987 |
| 52 | 850 | 911 | 951 | 1014 |
| 53 | 903 | 964 | 1004 | 1066 |
| 54 | 928 | 992 | 1051 | 1093 |

(c) Our upper bound for $\lambda(n)$ can be further improved by considering other prime divisors (we have considered only smallest prime divisor).
(d) The following table suggests that the upper bound for $\lambda(n)$ given in the above theorem is sharper than the upper bounds given by Beineke and Hegde [3] and Adiga et al. [2].

| $n$ | $\lambda(n)$ | upper bound for <br> $\lambda(n)$ using <br> our theorem | upper bound for <br> $\lambda(n)$ given by <br> Adiga et al. | upper bound for <br> $\lambda(n)$ given by <br> Beineke and Hegde |
| :---: | :---: | :---: | :---: | :---: |
| 55 | 959 | 1037 | 1106 | 1147 |
| 56 | 981 | 1054 | 1135 | 1176 |
| 57 | 1013 | 1093 | 1174 | 1232 |
| 58 | 1042 | 1123 | 1203 | 1261 |
| 59 | 1099 | 1182 | 1262 | 1319 |
| 60 | 1117 | 1200 | 1293 | 1350 |

The values of column 2 of the above table were obtained on using the formula for $\lambda(n)[1]$.

## Acknowledgement

The authors thank the referees for their useful suggestions which improved the quality of the paper.

## References

[1] C. Adiga, H.N. Ramaswamy and D.D. Somashekara, On strongly multiplicative graphs, South East Asian J. Math. \& Math. Sc. 2 (2003) 45-47.
[2] C. Adiga, H.N. Ramaswamy and D.D. Somashekara, A note on strongly multiplicative graphs, Discuss. Math. Graph Theory 24 (2004) 81-83.
[3] L.W. Beineke and S.M. Hegde, Strongly multiplicative graphs, Discuss. Math. Graph Theory 21 (2001) 63-76.
[4] P. Erdös, An asymptotic inequality in the theory of numbers, Vestnik Leningrad. Univ. 15 (1960) 41-49.

