# A LOWER BOUND FOR THE IRREDUNDANCE NUMBER OF TREES 

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#### Abstract

Let $\operatorname{ir}(G)$ and $\gamma(G)$ be the irredundance number and domination number of a graph $G$, respectively. The number of vertices and leafs of a graph $G$ is denoted by $n(G)$ and $n_{1}(G)$. If $T$ is a tree, then Lemańska [4] presented in 2004 the sharp lower bound $$
\gamma(T) \geq \frac{n(T)+2-n_{1}(T)}{3}
$$

In this paper we prove $$
\operatorname{ir}(T) \geq \frac{n(T)+2-n_{1}(T)}{3}
$$ for an arbitrary tree $T$. Since $\gamma(T) \geq \operatorname{ir}(T)$ is always valid, this inequality is an extension and improvement of Lemańska's result. Keywords: irredundance, tree, domination. 2000 Mathematics Subject Classification: 05C69.


## 1. Terminology and Introduction

We consider finite, undirected, and simple graphs $G$ with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph $G$ is called the order of $G$ and is denoted by $n=n(G)$. The open neighborhood $N(v)=N(v, G)$ of the vertex $v$ consists of the vertices adjacent to $v$, and the closed neighborhood of $v$ is $N[v]=N[v, G]=N(v) \cup\{v\}$. For a subset
$X \subseteq V(G)$, we define $N(X)=N(X, G)=\bigcup_{v \in X} N(v)$ and $N[X]=N[X, G]$ $=N(X) \cup X$. In addition, let $G[X]$ be the subgraph induced by $X$, and let $e(X)$ be the number of edges in $G[X]$. The vertex $v$ is a leaf of $G$ if $d(v, G)=1$, and an isolated vertex if $d(v, G)=0$, where $d(v)=d(v, G)=$ $|N(v)|$ is the degree of $v \in V(G)$. Let $n_{1}=n_{1}(G)$ be the number of leafs in a graph $G$. By $\delta=\delta(G)$ and $\Delta=\Delta(G)$, we denote the minimum degree and maximum degree of the graph $G$, respectively. If $X$ and $Y$ are two disjoint subsets of $V(G)$, then let $e(X, Y)$ be the number of edges with one end in $X$ and the other in $Y$.

A set $D \subseteq V(G)$ is a dominating set of the graph $G$ if $N[D, G]=V(G)$. The domination number $\gamma=\gamma(G)$ of $G$ is the cardinality of any smallest dominating set.

Let $I \subseteq V(G)$ and $v \in I$. A vertex $u \in V(G)-I$ is an $I$-external private neighbor of $v$ if $N(u) \cap I=\{v\}$. The set of all $I$-external private neighbors of $v$ is denoted by $E P N(v, I)$ and

$$
P N(v, I)= \begin{cases}E P N(v, I) \cup\{v\} & \text { if } v \text { is isolated in } G[I] \\ E P N(v, I) & \text { otherwise. }\end{cases}
$$

A subset $I \subseteq V(G)$ is irredundant if $P N(v, I) \neq \emptyset$ for all $v \in I$. An irredundant set $I$ is maximal irredundant if for every vertex $u \in V(G)-I$, the set $I \cup\{u\}$ is not irredundant. The minimum cardinality taken over all maximal irredundant sets of $G$ is the irredundance number $\operatorname{ir}(G)$ of $G$.

For detailed information on domination, irredundance, and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater.

Let $T$ be a tree of maximum degree $\Delta(T) \geq 3$. If $T$ is not isomorphic to the star $K_{1, \Delta(T)}$, then Cockayne [1] recently proved that

$$
\operatorname{ir}(T) \geq \frac{2(n(T)+1)}{2 \Delta(T)+3}
$$

In this note we will present the following lower bound of the irredundance number of a tree. If $T$ is a tree, then

$$
\operatorname{ir}(T) \geq \frac{n(T)+2-n_{1}(T)}{3}
$$

Since $\gamma(G) \geq \operatorname{ir}(G)$ is valid for an arbitrary graph $G$, this lower bound is an improvement of Lemańska's [4] inequality

$$
\gamma(T) \geq \frac{n(T)+2-n_{1}(T)}{3}
$$

## 2. Preliminary Results

The following partition of $V(G)$ induced by the vertex subset $I$ will be involved in the proof of the desired bound.

$$
\begin{aligned}
V(G) & =I \cup B \cup A \cup R \text { (disjoint union) where } \\
B & =\{u \in V(G)-I:|N(u) \cap I|=1\} \\
A & =\{u \in V(G)-I:|N(u) \cap I| \geq 2\} \\
R & =V(G)-N[I]
\end{aligned}
$$

In addition, let $B=B_{0} \cup B_{1}$ and $R=R_{0} \cup R_{1}$ such that

$$
\begin{aligned}
& B_{0}=\{u \in B: d(u) \geq 2\} \\
& B_{1}=\{u \in B: d(u)=1\} \\
& R_{0}=\{u \in R: d(u) \geq 2\} \\
& R_{1}=\{u \in R: d(u)=1\}
\end{aligned}
$$

In the following the cardinality of any set (except $V(G)$ ) denoted by any upper case letter, will be denoted by the corresponding lower case letter i.e., $|B|=b,|A|=a$ etc. The proof of our main result is based on a useful characterization of maximal irredundant sets by Cockayne, Grobler, Hedetniemi, and McRae [2].

Theorem 2.1 ([2] 1997). Let I be an irredundant set in a graph G. The set $I$ is maximal irredundant if and only if for each $w \in N[R]$, there exists a vertex $v \in I$ such that

$$
\begin{equation*}
P N(v, I) \subseteq N[w] \tag{1}
\end{equation*}
$$

If (1) is satisfied we say that $w$ annihilates $v$.

Suppose that $\mathcal{F}\left(i, n_{1}\right)$ is the set of forests of maximum order which have $n_{1}$ leafs and a maximal irredundant set of size $i$.

Lemma 2.2. Let $I$ be a maximal irredundant set of size $i$ of the forest $G \in \mathcal{F}\left(i, n_{1}\right)$. For each $w \in R$, there exists exactly one $v \in I$ such that $w$ annihilates $v$.

Proof. In view of Theorem 2.1, there exists a vertex $v \in I$ such that $w$ annihilates $v$, that means $P N(v, I) \subseteq N[w]$. Suppose that there exist two different vertices $v_{1}, v_{2} \in I$ such that $w$ annihilates $v_{1}$ as well as $v_{2}$. Let $\left\{u_{1}\right\}=N(w) \cap N\left(v_{1}\right)$ and $\left\{u_{2}\right\}=N(w) \cap N\left(v_{2}\right)$. Form the graph $G_{1}$ by deleting the edge $w u_{2}$ and adding a vertex $w_{2}$ to the set $R$ and the new edges $w w_{2}$ and $u_{2} w_{2}$. Since $w_{2}$ annihilates $v_{2}$ in $G_{1}$, the set $I$ is, by Theorem 2.1, furthermore a maximal irredundant set of the tree $G_{1}$ with $n_{1}$ leafs. This is a contradiction to the hypothesis that $G \in \mathcal{F}\left(i, n_{1}\right)$, and the proof of Lemma 2.2 is complete.

## 3. Main Result

Theorem 3.1. If $T$ is a tree of order $n$ with $n_{1}$ leafs, then

$$
\begin{equation*}
\operatorname{ir}(T) \geq \frac{n+2-n_{1}}{3} \tag{2}
\end{equation*}
$$

Proof. Since the result is immediate for $n \leq 3$, we assume in the following that $n \geq 4$. It is evident that it is enough to prove inequality (3) for $T \in \mathcal{F}\left(i, n_{1}\right)$. Thus let now $T \in \mathcal{F}\left(i, n_{1}\right)$, and let $I$ be a maximal irredundant set of size $i$. It is well-known that $|V(T)|-1=|E(T)|$, and thus we deduce that

$$
\begin{aligned}
|V(T)|-1 & =i+b_{0}+a+r_{0}+b_{1}+r_{1}-1 \\
& =e\left(B_{0}\right)+e\left(B_{0}, A\right)+e(A)+e(I)+e\left(R_{0}\right)+e\left(B_{0}, R_{0}\right)+e\left(A, R_{0}\right) \\
& +e\left(I, B_{0}\right)+e(I, A)+b_{1}+r_{1}
\end{aligned}
$$

Next let $B_{0}=X \cup Y$ and $R_{0}=R_{0}^{\prime} \cup R_{0}^{\prime \prime}$ with

$$
\begin{aligned}
X & =\left\{u \in B_{0}: N(u) \cap R \neq \emptyset\right\} \\
Y & =\left\{u \in B_{0}: N(u) \cap R=\emptyset\right\} \\
R_{0}^{\prime} & =\left\{w \in R_{0}:|N(w) \cap B|=1\right\} \\
R_{0}^{\prime \prime} & =\left\{w \in R_{0}:|N(w) \cap B| \geq 2\right\}
\end{aligned}
$$

Furthermore, we define the set $X_{0} \subseteq X$ as follows: If $u \in X$ is adjacent to the vertex $w \in R$, then $u$ is also adjacent to the vertex $v$ with the property that $w$ annihilates $v$. Finally, let $X_{1}=X-X_{0}$.

If $u \in Y$, then $d(u) \geq 2$. Because of $|N(u) \cap I|=1$ and $|N(u) \cap R|=0$, the vertex $u$ is adjacent to a vertex of $A \cup B_{0}$. This easily leads to

$$
\begin{equation*}
\frac{y}{2} \leq e\left(B_{0}\right)+e\left(B_{0}, A\right) \tag{4}
\end{equation*}
$$

Let $u \in X_{0}$ and $v \in I$ the unique neighbor of $u$ in $I$. By the definition of $X_{0}$, there exists a vertex $w \in R \cap N(u)$ such that $P N(v, I) \subseteq N[w]$. This implies that $v$ is no isolated vertex in the subgraph $G[I]$, because otherwise we would arrive at the contradiction $\{v\} \subseteq P N(v, I) \nsubseteq N[w]$. Let $u_{1} \neq u$ be a further vertex in $X_{0}$. Suppose that $v$ is also the unique neighbor of $u_{1}$ in $I$. Since $T$ is a tree, we observe that $u_{1}$ and $w$ are not adjacent. This leads to the contradiction $\left\{u_{1}\right\} \subseteq P N(v, I) \nsubseteq N[w]$. Altogether, we conclude that

$$
\begin{equation*}
\frac{x_{0}}{2} \leq e(I) . \tag{5}
\end{equation*}
$$

According to Theorem 2.1, each vertex $w \in R_{0}^{\prime}$ annihilates a vertex $v$ in $I$. Hence each vertex $w \in R_{0}^{\prime}$ is adjacent to a vertex $u \in B_{0}$. Moreover, in view of Lemma 2.2, the vertex $u$ is unique and thus $\left|R_{0}^{\prime}\right| \leq e\left(R_{0}^{\prime}, B_{0}\right)$. The condition $d(w) \geq 2$ shows that $w$ is adjacent to a further vertex in $A \cup R$. Since, by Theorem 2.1, $R_{1}$ is not possible, $w$ is adjacent to a vertex in $A \cup R_{0}$. We obtain the minimum number of edges if each $w \in R_{0}^{\prime}$ is adjacent to exactly one vertex of $R_{0}^{\prime}$ and $w$ has no neighbor in $A \cup R_{0}^{\prime \prime}$. This yields

$$
\frac{\left|R_{0}^{\prime}\right|}{2} \leq e\left(R_{0}^{\prime}\right)+e\left(R_{0}^{\prime}, R_{0}^{\prime \prime}\right)+e\left(R_{0}^{\prime}, A\right)
$$

and thus

$$
\begin{align*}
\frac{3\left|R_{0}^{\prime}\right|}{2} & \leq e\left(R_{0}^{\prime}, B_{0}\right)+e\left(R_{0}^{\prime}\right)+e\left(R_{0}^{\prime}, R_{0}^{\prime \prime}\right)+e\left(R_{0}^{\prime}, A\right) \\
& \leq e\left(R_{0}^{\prime}, B_{0}\right)+e\left(R_{0}\right)+e\left(R_{0}^{\prime}, A\right) \tag{6}
\end{align*}
$$

Assume that $w \in R_{0}^{\prime \prime}$. Again Theorem 2.1 implies that $w$ annihilates a vertex $v$ in $I$. Hence $w$ is adjacent to a vertex $u \in X_{0}$. In view of Lemma 2.2, the vertex $u$ is unique and thus

$$
\begin{equation*}
\left|R_{0}^{\prime \prime}\right|=e\left(R_{0}^{\prime \prime}, X_{0}\right) . \tag{7}
\end{equation*}
$$

In addition, the definition of $R_{0}^{\prime \prime}$ shows that $w$ is adjacent to a further vertex $u^{\prime} \in X_{1}$, and each vertex $u^{\prime \prime} \in X_{1}$ is adjacent to a vertex in $R_{0}^{\prime \prime}$. Hence it
follows that

$$
\begin{equation*}
e\left(R_{0}^{\prime \prime}, X_{1}\right) \geq \max \left\{\left|R_{0}^{\prime \prime}\right|,\left|X_{1}\right|\right\} \geq \frac{\left|R_{0}^{\prime \prime}\right|}{2}+\frac{\left|X_{1}\right|}{2} . \tag{8}
\end{equation*}
$$

Combining (5) - (9) with the inequality

$$
e\left(R_{0}^{\prime \prime}, X_{0}\right)+e\left(R_{0}^{\prime \prime}, X_{1}\right)+e\left(R_{0}^{\prime}, B_{0}\right) \leq e\left(R_{0}, B_{0}\right)
$$

we arrive at

$$
\begin{aligned}
\frac{b_{0}}{2}+\frac{3 r_{0}}{2} & =\frac{x_{0}}{2}+\frac{x_{1}}{2}+\frac{y}{2}+\frac{3\left|R_{0}^{\prime}\right|}{2}+\frac{3\left|R_{0}^{\prime \prime}\right|}{2} \\
& =\frac{x_{0}}{2}+\frac{y}{2}+\frac{3\left|R_{0}^{\prime}\right|}{2}+\left|R_{0}^{\prime \prime}\right|+\frac{x_{1}}{2}+\frac{\left|R_{0}^{\prime \prime}\right|}{2} \\
& \leq e(I)+e\left(B_{0}\right)+e\left(B_{0}, A\right)+e\left(R_{0}^{\prime}, B_{0}\right)+e\left(R_{0}\right) \\
& +e\left(R_{0}^{\prime}, A\right)+e\left(R_{0}^{\prime \prime}, X_{0}\right)+e\left(R_{0}^{\prime \prime}, X_{1}\right) \\
& \leq e(I)+e\left(B_{0}\right)+e\left(B_{0}, A\right)+e\left(R_{0}, B_{0}\right)+e\left(R_{0}\right)+e\left(R_{0}, A\right)+e(A) .
\end{aligned}
$$

Now we deduce from (4)

$$
\begin{aligned}
i+b_{0}+a+r_{0}-1 & =e\left(B_{0}\right)+e\left(B_{0}, A\right)+e(A)+e(I)+e\left(R_{0}\right)+e\left(B_{0}, R_{0}\right) \\
& +e\left(A, R_{0}\right)+e\left(I, B_{0}\right)+e(I, A) \\
& \geq \frac{b_{0}}{2}+\frac{3 r_{0}}{2}+b_{0}+2 a \\
& \geq \frac{3 b_{0}}{2}+\frac{3 r_{0}}{2}+\frac{3 a}{2} .
\end{aligned}
$$

This implies $2 i-2 \geq b_{0}+a+r_{0}$ and thus

$$
2 i-2+n_{1} \geq b_{0}+a+r_{0}+n_{1} .
$$

Since by definition $n_{1} \geq b_{1}+r_{1}$, we obtain

$$
3 i-2+n_{1} \geq i+b_{0}+a+r_{0}+b_{1}+r_{1}=n
$$

and this leads to

$$
i \geq \frac{n+2-n_{1}}{3}
$$

Since the last bound is valid for all maximal irredundant sets $I$ with $|I|=i$, the desired inequality (3) is proved.

Remark 3.2. If $T$ is a tree, then Lemańska [4] has proved that

$$
\gamma(T)=\frac{n(T)+2-n_{1}(T)}{3}
$$

if and only if the distance between each pair of distinct leafs in $T$ is congruent 2 modulo 3. An analyses of the proof of Theorem 3.1 shows that we obtain equality in (3) for exactly the same family of trees.

## References

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