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A LOWER BOUND FOR THE IRREDUNDANCE NUMBER OF TREES

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Abstract

Let $\mathrm{ir}(G)$ and $\gamma(G)$ be the irredundance number and domination number of a graph G, respectively. The number of vertices and leafs of a graph G is denoted by $n(G)$ and $n_1(G)$. If T is a tree, then Lemańska [4] presented in 2004 the sharp lower bound

$$\gamma(T) \geq \frac{n(T) + 2 - n_1(T)}{3}.$$

In this paper we prove

$$\mathrm{ir}(T) \geq \frac{n(T) + 2 - n_1(T)}{3}$$

for an arbitrary tree T. Since $\gamma(T) \geq \mathrm{ir}(T)$ is always valid, this inequality is an extension and improvement of Lemańska's result.

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1. Terminology and Introduction

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* $N(v) = N(v, G)$ of the vertex v consists of the vertices adjacent to v, and the *closed neighborhood* of v is $N[v] = N[v, G] = N(v) \cup \{v\}$. For a subset

$X \subseteq V(G)$, we define $N(X) = N(X,G) = \bigcup_{v \in X} N(v)$ and $N[X] = N[X,G] = N(X) \cup X$. In addition, let $G[X]$ be the subgraph induced by X, and let $e(X)$ be the number of edges in $G[X]$. The vertex v is a *leaf* of G if $d(v,G) = 1$, and an *isolated vertex* if $d(v,G) = 0$, where $d(v) = d(v,G) = |N(v)|$ is the degree of $v \in V(G)$. Let $n_1 = n_1(G)$ be the number of leafs in a graph G. By $\delta = \delta(G)$ and $\Delta = \Delta(G)$, we denote the *minimum degree* and *maximum degree* of the graph G, respectively. If X and Y are two disjoint subsets of $V(G)$, then let $e(X,Y)$ be the number of edges with one end in X and the other in Y.

A set $D \subseteq V(G)$ is a *dominating set* of the graph G if $N[D,G] = V(G)$. The *domination number* $\gamma = \gamma(G)$ of G is the cardinality of any smallest dominating set.

Let $I \subseteq V(G)$ and $v \in I$. A vertex $u \in V(G) - I$ is an *I-external private neighbor* of v if $N(u) \cap I = \{v\}$. The set of all I-external private neighbors of v is denoted by $EPN(v,I)$ and

$$PN(v,I) = \begin{cases} EPN(v,I) \cup \{v\} & \text{if } v \text{ is isolated in } G[I] \\ EPN(v,I) & \text{otherwise.} \end{cases}$$

A subset $I \subseteq V(G)$ is *irredundant* if $PN(v,I) \neq \emptyset$ for all $v \in I$. An irredundant set I is *maximal irredundant* if for every vertex $u \in V(G) - I$, the set $I \cup \{u\}$ is not irredundant. The minimum cardinality taken over all maximal irredundant sets of G is the *irredundance number* $\mathrm{ir}(G)$ of G.

For detailed information on domination, irredundance, and related topics see the comprehensive monograph [3] by Haynes, Hedetniemi, and Slater.

Let T be a tree of maximum degree $\Delta(T) \geq 3$. If T is not isomorphic to the star $K_{1,\Delta(T)}$, then Cockayne [1] recently proved that

$$\mathrm{ir}(T) \geq \frac{2(n(T)+1)}{2\Delta(T)+3}.$$

In this note we will present the following lower bound of the irredundance number of a tree. If T is a tree, then

$$\mathrm{ir}(T) \geq \frac{n(T)+2-n_1(T)}{3}.$$

Since $\gamma(G) \geq \mathrm{ir}(G)$ is valid for an arbitrary graph G, this lower bound is an improvement of Lemańska's [4] inequality

$$\gamma(T) \geq \frac{n(T)+2-n_1(T)}{3}.$$

2. Preliminary Results

The following partition of $V(G)$ induced by the vertex subset I will be involved in the proof of the desired bound.

$$\begin{aligned} V(G) &= I \cup B \cup A \cup R \text{ (disjoint union) where} \\ B &= \{u \in V(G) - I : |N(u) \cap I| = 1\} \\ A &= \{u \in V(G) - I : |N(u) \cap I| \geq 2\} \\ R &= V(G) - N[I]. \end{aligned}$$

In addition, let $B = B_0 \cup B_1$ and $R = R_0 \cup R_1$ such that

$$\begin{aligned} B_0 &= \{u \in B : d(u) \geq 2\} \\ B_1 &= \{u \in B : d(u) = 1\} \\ R_0 &= \{u \in R : d(u) \geq 2\} \\ R_1 &= \{u \in R : d(u) = 1\}. \end{aligned}$$

In the following the cardinality of any set (except $V(G)$) denoted by any upper case letter, will be denoted by the corresponding lower case letter i.e., $|B| = b$, $|A| = a$ etc. The proof of our main result is based on a useful characterization of maximal irredundant sets by Cockayne, Grobler, Hedetniemi, and McRae [2].

Theorem 2.1 ([2] 1997)**.** *Let I be an irredundant set in a graph G. The set I is maximal irredundant if and only if for each $w \in N[R]$, there exists a vertex $v \in I$ such that*

$$PN(v, I) \subseteq N[w]. \tag{1}$$

If (1) is satisfied we say that w annihilates v.

Suppose that $\mathcal{F}(i, n_1)$ is the set of forests of maximum order which have n_1 leafs and a maximal irredundant set of size i.

Lemma 2.2. *Let I be a maximal irredundant set of size i of the forest $G \in \mathcal{F}(i, n_1)$. For each $w \in R$, there exists exactly one $v \in I$ such that w annihilates v.*

Proof. In view of Theorem 2.1, there exists a vertex $v \in I$ such that w annihilates v, that means $PN(v, I) \subseteq N[w]$. Suppose that there exist two different vertices $v_1, v_2 \in I$ such that w annihilates v_1 as well as v_2. Let $\{u_1\} = N(w) \cap N(v_1)$ and $\{u_2\} = N(w) \cap N(v_2)$. Form the graph G_1 by deleting the edge wu_2 and adding a vertex w_2 to the set R and the new edges ww_2 and u_2w_2. Since w_2 annihilates v_2 in G_1, the set I is, by Theorem 2.1, furthermore a maximal irredundant set of the tree G_1 with n_1 leafs. This is a contradiction to the hypothesis that $G \in \mathcal{F}(i, n_1)$, and the proof of Lemma 2.2 is complete. ■

3. Main Result

Theorem 3.1. *If T is a tree of order n with n_1 leafs, then*

$$\mathrm{ir}(T) \geq \frac{n + 2 - n_1}{3}. \tag{2}$$

Proof. Since the result is immediate for $n \leq 3$, we assume in the following that $n \geq 4$. It is evident that it is enough to prove inequality (3) for $T \in \mathcal{F}(i, n_1)$. Thus let now $T \in \mathcal{F}(i, n_1)$, and let I be a maximal irredundant set of size i. It is well-known that $|V(T)| - 1 = |E(T)|$, and thus we deduce that

$$\begin{aligned} |V(T)| - 1 &= i + b_0 + a + r_0 + b_1 + r_1 - 1 \\ &= e(B_0) + e(B_0, A) + e(A) + e(I) + e(R_0) + e(B_0, R_0) + e(A, R_0) \\ &\quad + e(I, B_0) + e(I, A) + b_1 + r_1. \end{aligned} \tag{3}$$

Next let $B_0 = X \cup Y$ and $R_0 = R_0' \cup R_0''$ with

$$\begin{aligned} X &= \{u \in B_0 : N(u) \cap R \neq \emptyset\} \\ Y &= \{u \in B_0 : N(u) \cap R = \emptyset\} \\ R_0' &= \{w \in R_0 : |N(w) \cap B| = 1\} \\ R_0'' &= \{w \in R_0 : |N(w) \cap B| \geq 2\}. \end{aligned}$$

Furthermore, we define the set $X_0 \subseteq X$ as follows: If $u \in X$ is adjacent to the vertex $w \in R$, then u is also adjacent to the vertex v with the property that w annihilates v. Finally, let $X_1 = X - X_0$.

If $u \in Y$, then $d(u) \geq 2$. Because of $|N(u) \cap I| = 1$ and $|N(u) \cap R| = 0$, the vertex u is adjacent to a vertex of $A \cup B_0$. This easily leads to

$$\frac{y}{2} \leq e(B_0) + e(B_0, A). \tag{4}$$

Let $u \in X_0$ and $v \in I$ the unique neighbor of u in I. By the definition of X_0, there exists a vertex $w \in R \cap N(u)$ such that $PN(v, I) \subseteq N[w]$. This implies that v is no isolated vertex in the subgraph $G[I]$, because otherwise we would arrive at the contradiction $\{v\} \subseteq PN(v, I) \not\subseteq N[w]$. Let $u_1 \neq u$ be a further vertex in X_0. Suppose that v is also the unique neighbor of u_1 in I. Since T is a tree, we observe that u_1 and w are not adjacent. This leads to the contradiction $\{u_1\} \subseteq PN(v, I) \not\subseteq N[w]$. Altogether, we conclude that

$$\frac{x_0}{2} \leq e(I). \tag{5}$$

According to Theorem 2.1, each vertex $w \in R_0'$ annihilates a vertex v in I. Hence each vertex $w \in R_0'$ is adjacent to a vertex $u \in B_0$. Moreover, in view of Lemma 2.2, the vertex u is unique and thus $|R_0'| \leq e(R_0', B_0)$. The condition $d(w) \geq 2$ shows that w is adjacent to a further vertex in $A \cup R$. Since, by Theorem 2.1, R_1 is not possible, w is adjacent to a vertex in $A \cup R_0$. We obtain the minimum number of edges if each $w \in R_0'$ is adjacent to exactly one vertex of R_0' and w has no neighbor in $A \cup R_0''$. This yields

$$\frac{|R_0'|}{2} \leq e(R_0') + e(R_0', R_0'') + e(R_0', A)$$

and thus

$$\begin{aligned} \frac{3|R_0'|}{2} &\leq e(R_0', B_0) + e(R_0') + e(R_0', R_0'') + e(R_0', A) \\ &\leq e(R_0', B_0) + e(R_0) + e(R_0', A). \end{aligned} \tag{6}$$

Assume that $w \in R_0''$. Again Theorem 2.1 implies that w annihilates a vertex v in I. Hence w is adjacent to a vertex $u \in X_0$. In view of Lemma 2.2, the vertex u is unique and thus

$$|R_0''| = e(R_0'', X_0). \tag{7}$$

In addition, the definition of R_0'' shows that w is adjacent to a further vertex $u' \in X_1$, and each vertex $u'' \in X_1$ is adjacent to a vertex in R_0''. Hence it follows that

$$e(R_0'', X_1) \geq \max\{|R_0''|, |X_1|\} \geq \frac{|R_0''|}{2} + \frac{|X_1|}{2}. \tag{8}$$

Combining (5) – (9) with the inequality

$$e(R_0'', X_0) + e(R_0'', X_1) + e(R_0', B_0) \leq e(R_0, B_0)$$

we arrive at

$$\begin{aligned}
\frac{b_0}{2} + \frac{3r_0}{2} &= \frac{x_0}{2} + \frac{x_1}{2} + \frac{y}{2} + \frac{3|R_0'|}{2} + \frac{3|R_0''|}{2} \\
&= \frac{x_0}{2} + \frac{y}{2} + \frac{3|R_0'|}{2} + |R_0''| + \frac{x_1}{2} + \frac{|R_0''|}{2} \\
&\leq e(I) + e(B_0) + e(B_0, A) + e(R_0', B_0) + e(R_0) \\
&+ e(R_0', A) + e(R_0'', X_0) + e(R_0'', X_1) \\
&\leq e(I) + e(B_0) + e(B_0, A) + e(R_0, B_0) + e(R_0) + e(R_0, A) + e(A).
\end{aligned}$$

Now we deduce from (4)

$$\begin{aligned}
i + b_0 + a + r_0 - 1 &= e(B_0) + e(B_0, A) + e(A) + e(I) + e(R_0) + e(B_0, R_0) \\
&+ e(A, R_0) + e(I, B_0) + e(I, A) \\
&\geq \frac{b_0}{2} + \frac{3r_0}{2} + b_0 + 2a \\
&\geq \frac{3b_0}{2} + \frac{3r_0}{2} + \frac{3a}{2}.
\end{aligned}$$

This implies $2i - 2 \geq b_0 + a + r_0$ and thus

$$2i - 2 + n_1 \geq b_0 + a + r_0 + n_1.$$

Since by definition $n_1 \geq b_1 + r_1$, we obtain

$$3i - 2 + n_1 \geq i + b_0 + a + r_0 + b_1 + r_1 = n,$$

and this leads to

$$i \geq \frac{n + 2 - n_1}{3}.$$

Since the last bound is valid for all maximal irredundant sets I with $|I| = i$, the desired inequality (3) is proved. ■

Remark 3.2. If T is a tree, then Lemańska [4] has proved that

$$\gamma(T) = \frac{n(T) + 2 - n_1(T)}{3}$$

if and only if the distance between each pair of distinct leafs in T is congruent 2 modulo 3. An analyses of the proof of Theorem 3.1 shows that we obtain equality in (3) for exactly the same family of trees.

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