

IN-DEGREE SEQUENCE IN A GENERAL MODEL OF A RANDOM DIGRAPH

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Abstract

A general model of a random digraph $D(n, \mathcal{P})$ is considered. Based on a precise estimate of the asymptotic behaviour of the distribution function of the binomial law, a problem of the distribution of extreme in-degrees of $D(n, \mathcal{P})$ is discussed.

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1. Introduction

We begin with a definition of a general model of a random digraph that was introduced in [6]. Let $\mathcal{P} = (P_0, \dots, P_{n-1})$ be a probability distribution, i.e., an n -tuple of non-negative real numbers which satisfy $\sum_{i=0}^{n-1} P_i = 1$. Denote by $D(n, \mathcal{P})$ a random digraph on a vertex set $V = \{1, 2, \dots, n\}$ such that (here, and what follows, $N^+(i)$ denotes the set of images of a vertex i):

- (1) each vertex "chooses" its out-degree and then its images independently of all other vertices,
- (2) each vertex $i \in V$ chooses its out-degree according to the probability distribution \mathcal{P} , i.e.,

$$Pr\{|N^+(i)| = k\} = P_k, \quad k = 0, 1, \dots, n-1,$$

- (3) for every $S \subseteq V \setminus \{i\}$, with $|S| = k$, the probability that S coincides with the set of images of a vertex i equals

$$Pr\{N^+(i) = S\} = P_k / \binom{n-1}{k}$$

i.e., vertex i "chooses" uniformly the set of images.

In particular, if \mathcal{P} is such that $P_d = 1$ for some $d, 1 \leq d \leq n-1$, the model $D(n, \mathcal{P})$ is equivalent to a random d -out regular digraph $D(n, d)$. Such a digraph can also be defined as an element chosen at random from the family of all $\binom{n-1}{d}^n$ digraphs on n labeled vertices each of out-degree d . (Alternatively, $D(n, d)$ can be thought as a representation of a sum of d dependent random mappings as illustrated in [7].)

In a case when \mathcal{P} is a binomial distribution $\mathcal{B}(n-1, p)$, i.e.,

$$\mathcal{P} = (q^{n-1}, \dots, b(k; n-1, p), \dots, p^{n-1})$$

where

$$b(r; n, p) = \binom{n}{r} p^r q^{n-r}$$

the model $D(n, \mathcal{P})$ is equivalent to a random digraph $D(n, \mathcal{B})$ on n labeled vertices in which each of $n(n-1)$ possible arcs appears independently with a given probability $p = 1 - q$.

2. Preliminaries

Let X^+ be a discrete random variable having a probability distribution $\mathcal{P} = (P_0, P_1, \dots, P_{n-1})$:

$$Pr\{X^+ = k\} = P_k, \quad k = 0, 1, \dots, n-1.$$

Due to the homogeneous structure of the random digraph $D(n, \mathcal{P})$, the random variable $X^+ = X^+(i)$ defines the out-degree of a given vertex $i \in V = \{1, 2, \dots, n\}$ of $D(n, \mathcal{P})$. Then the probability that a given subset of vertices is contained in the set of images of vertex $i \in V$ can be expressed by appropriate factorial moment of X^+ . As a matter of fact the following property is true (see [8]). Here and what follows $(n)_k = n(n-1) \dots (n-k+1)$ and $E_k(X)$ stands for the k -th factorial moment of a random variable X .

Property 1. *For a given $i, 1 \leq i \leq n$, let $U \subseteq V \setminus \{i\}$ and $|U| = t \geq 1$. Then*

$$\Pr\{U \subseteq N^+(i)\} = \frac{1}{(n-1)_t} E_t(X^+).$$

In particular, if $t = 1$ the above property defines an arc occurrence probability in digraph $D(n, \mathcal{P})$. Let

$$E^+ = E^+(\mathcal{P}) = \sum_{k=0}^{n-1} k P_k.$$

Then the probability of an arc in $D(n, \mathcal{P})$ is given by

$$(1) \quad p^* = \frac{E^+(\mathcal{P})}{n-1}.$$

Now let $X^- = X^-(i)$ be the in-degree of a given vertex $i \in \{1, 2, \dots, n\}$ of $D(n, \mathcal{P})$. Clearly, the probability distribution of X^- depends on \mathcal{P} . We have the following result (see [8]).

Property 2. *For $i = 1, 2, \dots, n$ the random variable $X^-(i)$ has binomial distribution $\mathcal{B}(n-1, p^*)$.*

In contrast with out-degrees of vertices of $D(n, \mathcal{P})$, the random variables $X^-(i)$, $i = 1, 2, \dots, n$, are not, in general, independent. The only case when these variables are independent is when X^+ is binomially distributed (see [8]).

The main aim of our paper is to study the probabilistic properties of extreme in-degrees of the random digraph $D(n, \mathcal{P})$. We show that the in-degree sequence of $D(n, \mathcal{P})$ behaves similarly to the degree sequence of the classical model of a random graph (see [11]). Our results generalize those presented in [10].

Let G_n be an arbitrary random graph model defined on n vertices. If π is a graph property then the assertion " G_n has property π asymptotically almost surely (a.a.s.)" means

$$\lim_{n \rightarrow \infty} P(G_n \text{ has property } \pi) = 1.$$

The symbols o , O and \sim are used with respect to $n \rightarrow \infty$.

Consider "degree" sequence $d_{(1)} \leq d_{(2)} \leq \dots \leq d_{(n)}$ of G_n . If G_n is a simple (directed) graph then by the "degree" sequence we mean sequence of degrees (in-degrees or out-degrees) written in non-decreasing order. Denote by X_r , Y_s and Z_t the number of vertices of "degree" $= r, \leq s$ and $\geq t$ in G_n , respectively.

Let $B(s; n, p)$ denote probability of at most s successes in the binomial distribution. Similarly, let $F(t; n, p)$ denote probability of at least t successes in such distribution. In the proofs of our main results we will need a very precise estimate of the asymptotic behaviour of the distribution function of the binomial law with parameters n and p , where $p = p(n) = o(1)$ and $np/\log n \rightarrow \infty$ as $n \rightarrow \infty$ (see [5] and [12]).

Consider the equation

$$(1+z)\log(1+z) + \frac{1}{a}(1-az)\log(1-az) = u$$

where $0 \leq u < \infty$ and $a \geq 0$. It is known (see e.g. [5]) that this equation has a negative solution $z(u, a)$ and a positive solution $y(u, a)$, which in some neighbourhood of zero are given by the power series

$$(2) \quad z(u, a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \sum_{i=2}^{\infty} (-1)^i f_i(a) \left(\frac{2u}{1+a}\right)^{i/2}$$

and

$$(3) \quad y(u, a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \sum_{i=2}^{\infty} f_i(a) \left(\frac{2u}{1+a}\right)^{i/2}$$

in which

$$f_{i+1}(a) = \frac{(-1)^i}{i+1} \sum \frac{(-1)^k (i+1)(i+3) \dots (i+2k-1)}{k_1! \dots k_i! (2 \cdot 3)^{k_1} \dots [(i+1)(i+2)]^{k_i}} \\ \times \frac{(1-a^2)^{k_1} (1+a^3)^{k_2} \dots [1+(-1)^i a^{i+1}]^{k_i}}{(1+a)^k}$$

where $k = k_1 + k_2 + \dots + k_i$ and the summation is over all non-negative integers k_1, \dots, k_i such that $k_1 + 2k_2 + \dots + ik_i = i$. In particular,

$$(4) \quad z(u, a) = -\left(\frac{2u}{1+a}\right)^{\frac{1}{2}} + \frac{1-a}{3(1+a)}u + \frac{\sqrt{2}}{36} \frac{1+4a+a^2}{(1+a)^{3/2}} u^{3/2} + \dots$$

and

$$(5) \quad y(u, a) = \left(\frac{2u}{1+a} \right)^{\frac{1}{2}} + \frac{1-a}{3(1+a)}u - \frac{\sqrt{2}}{36} \frac{1+4a+a^2}{(1+a)^{3/2}} u^{3/2} + \dots$$

Now put

$$(6) \quad u = u(n, p) = \frac{1}{np} \left(\log n - \frac{1}{2} \log \log n \right).$$

In proofs of our main results we will need the following lemma giving a very precise asymptotic behaviour of binomial distribution (see [12]).

Lemma 1. *Let $m = np = \omega(n) \log n$ where $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$ in such a way that $p = p(n) = o(1)$. Assume that $x = x(n)$ satisfies $x^2 = o(\min\{\omega(n), \log n\})$, and put*

$$s = m + mz \left(u, \frac{p}{q} \right) - \left(\frac{m}{2 \log n} \right)^{1/2} (x - \log \sqrt{4\pi} + o(1))$$

$$t = m + my \left(u, \frac{p}{q} \right) + \left(\frac{m}{2 \log n} \right)^{1/2} (x - \log \sqrt{4\pi} + o(1))$$

where u is given by (6). Then

$$(7) \quad nB(s; n, p) \sim nF(t; n, p) \sim e^{-x}$$

and

$$(8) \quad nb(s; n, p) \sim nb(t; n, p) \sim \left(\frac{2 \log n}{npq} \right)^{1/2} e^{-x}. \quad \blacksquare$$

3. Main Results

Let $X_r^- = X_r^-(\mathcal{P})$ denote the number of vertices of in-degree r in a general model of a random digraph $D(n, \mathcal{P})$. Then by Property 2 we have

Property 3. *The expected value of X_r^- equals*

$$E(X_r^-) = nb(r; n-1, p^*)$$

where p^* is given by (1).

Now let us put $Y_s^- = Y_s^-(\mathcal{P})$ and $Z_t^- = Z_t^-(\mathcal{P})$ for the number of vertices of in-degree of at most s and at least t in $D(n, \mathcal{P})$, respectively. The following two lemmas, which proofs will be shown in the next section, are the basic tool in proving our main results.

Lemma 2.

$$(9) \quad E(Y_s^-) = nB(s; n-1, p^*)$$

and

$$(10) \quad E(Z_t^-) = nF(t; n-1, p^*).$$

Lemma 3. (i) If $r = o(n)$ then

$$E_2(X_r^-) \leq n^2 b^2(r; n-1, p^*)(1 + o(1)).$$

(ii) If $E^+ = E^+(\mathcal{P}) = o(n)$, $s < np^*$, $t > np^*$ and $t = o(n)$ then

$$E_2(Y_s^-) \leq n^2 B^2(s; n-1, p^*)(1 + o(1))$$

and

$$E_2(Z_t^-) \leq n^2 F^2(t; n-1, p^*)(1 + o(1)).$$

Let

$$d_{(1)}^- \leq d_{(2)}^- \leq \cdots \leq d_{(n)}^-$$

be the in-degree sequence of vertices in a random digraph $D(n, \mathcal{P})$. The first result shows that for any fixed $i \geq 2$ the first i -th and the last i -th terms of the in-degree sequence of $D(n, \mathcal{P})$ are asymptotically almost surely strictly increasing. For the sake of simplicity let us denote

$$(11) \quad s = s(n, \mathcal{P}) = (1 + z(u, a))E^+$$

$$(12) \quad t = t(n, \mathcal{P}) = (1 + y(u, a))E^+$$

and

$$(13) \quad \varphi = \varphi(n, \mathcal{P}) = \left(\frac{E^+}{2 \log n} \right)^{1/2} x(n)$$

where power series $z(u, a)$ and $y(u, a)$ are given by (2) and (3), respectively and $x(n)$ is a sequence tending to infinity arbitrary slowly as $n \rightarrow \infty$.

Theorem 1. *Let $\mathcal{P} = (P_0, P_1, \dots, P_{n-1})$ be such that*

$$E^+ = \omega(n) \log(n) = o(n),$$

where $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then for any fixed $i \geq 2$

$$(14) \quad s - \varphi < d_{(1)}^- < \dots < d_{(i)}^- < s + \varphi \quad a.a.s.$$

and

$$(15) \quad t - \varphi < d_{(n-i+1)}^- < \dots < d_{(n)}^- < t + \varphi \quad a.a.s.$$

where s and t are given by (11) and (12) with

$$(16) \quad u = u(n, \mathcal{P}) = \frac{1}{E^+} \left(\log n - \frac{1}{2} \log \log n \right)$$

$$(17) \quad a = a(n, \mathcal{P}) = \frac{E^+}{n - 1 - E^+}$$

and φ is given by (13).

Proof. Put $r = s - \varphi$. Then by Lemma 2 we have

$$E(Y_r^-) = nB(s - \varphi; n - 1, p^*).$$

Since

$$p^* = \frac{E^+}{n - 1} = \frac{\omega(n) \log n}{n - 1}$$

and

$$s - \varphi = \frac{\omega(n) \log n}{n - 1} (1 + z(u, a)) - \left(\frac{\omega(n)}{2(n - 1)} \right)^{1/2} x(n)$$

so by Lemma 1

$$(18) \quad \begin{aligned} E(Y_r^-) &\sim e^{-x(n)} \\ &= o(1). \end{aligned}$$

Consequently

$$\begin{aligned} \Pr(d_{(1)} \leq s - \varphi) &= \Pr(Y_r^- \geq 1) \\ &\leq E(Y_r^-) \\ &= o(1). \end{aligned}$$

Now let us put $r = s + \varphi$. Then

$$(19) \quad E(Y_r^-) \sim e^{x(n)} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Moreover, routine calculations show that by (4), (11), (13), (16) and (17) we have

$$r = s + \varphi < np^*(1 + o(1)).$$

So by Lemma 2 and 3

$$E_2(Y_r^-) \leq E^2(Y_r^-)(1 + o(1))$$

which implies that

$$\text{Var}(Y_r^-) \leq E(Y_r^-) + o(E^2(Y_r^-)).$$

Thus by Chebyshev's inequality

$$\Pr\left(Y_r^- \leq \frac{1}{2}E(Y_r^-)\right) \leq \frac{4\text{Var}(Y_r^-)}{E^2(Y_r^-)} = o(1).$$

Consequently, for any fixed $i \geq 1$

$$\begin{aligned} \Pr(d_{(i)}^- \leq s + \varphi) &= \Pr(Y_r^- \geq i) \\ &\geq 1 - o(1). \end{aligned}$$

To show that the sequence is strictly increasing we have to show that probability that there are at least two vertices of equal in-degree $\leq s + \varphi$ tends to zero as $n \rightarrow \infty$. We have

$$\sum_{k=0}^{s+\varphi} \Pr(X_k^- \geq 2) \leq \sum_{k=0}^{s+\varphi} E_2(X_k^-).$$

Since, by Lemma 3,

$$E_2(X_r^-) \leq E^2(X_r^-)(1 + o(1))$$

so applying Lemma 1 we obtain

$$\begin{aligned} \sum_{k=0}^{s+\varphi} E_2(X_k^-) &\leq \sum_{k=0}^{s+\varphi} n^2 b^2(k, n-1, p^*)(1 + o(1)) \\ &\leq nb(s + \varphi; n-1, p^*)nB(s + \varphi; n-1, p^*)(1 + o(1)) \\ &\sim \left(\frac{2 \log n}{np^*q^*} \right)^{1/2} e^{-2x(n)} \\ &= o(1) \end{aligned}$$

which completes the proof of (14). The proof of (15) follows analogously. ■

The above theorem gives a very precise estimate of the in-degree distribution of $D(n, \mathcal{P})$ in a case when the out-degree distribution $\mathcal{P} = (P_0, P_1, \dots, P_{n-1})$ satisfies the condition

$$E^+(\mathcal{P}) = \sum_{k=0}^{n-1} kP_k = \omega(n) \log n.$$

The disadvantage of this result is the complicated form for given bounds which are expressed by appropriate power series. It appears that if $E^+(\mathcal{P})$ tends to infinity a bit faster than $\omega(n) \log n$ much more pleasant estimates for in-degree sequence can be given. Now let

$$(20) \quad s = E^+ - (2np^*q^* \log n)^{1/2} + \left(\frac{np^*q^*}{8 \log n} \right)^{1/2} \log \log n$$

$$(21) \quad t = E^+ + (2np^*q^* \log n)^{1/2} - \left(\frac{np^*q^*}{8 \log n} \right)^{1/2} \log \log n$$

and

$$(22) \quad \varphi(n) = \left(\frac{np^*q^*}{2 \log n} \right)^{1/2} x(n)$$

where $x(n) \rightarrow \infty$ as $n \rightarrow \infty$ but $x(n) = o(\log \log n)$.

Theorem 2. *Let $E^+ \geq [\gamma(n)(\log n)^3]$, $\gamma(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then for any fixed $i \geq 1$*

$$(23) \quad s - \varphi \leq d_{(1)} < d_{(2)} < \cdots < d_{(i)} \leq s + \varphi \quad a.a.s.$$

and

$$(24) \quad t - \varphi \leq d_{(n-i+1)} < \cdots < d_{(n-1)} < d_{(n)} \leq t + \varphi \quad a.a.s.$$

where s, t and φ are given by (20), (21) and (22), respectively.

Proof. Put

$$r = E^+ - v\sqrt{np^*q^*}$$

where

$$v = v(n) = (2\log n)^{1/2} - \left(\frac{1}{2\log n}\right)^{1/2} \left(\frac{1}{2}\log\log n - x(n)\right).$$

Then the assumption $np^* \geq \gamma(n)(\log n)^3$ implies

$$\frac{v^3}{\sqrt{np^*q^*}} \leq \left(\frac{8}{\gamma(n)}\right)^{1/2} = o(1)$$

so applying the classical DeMoivre-Laplace formula (see Feller [4] Chapter 7) we obtain

$$\begin{aligned} E(Y_r^-) &\sim \frac{n}{\sqrt{2\pi}} \frac{1}{v} e^{-\frac{v^2}{2}} \\ &\sim \frac{1}{\sqrt{2\pi}} e^{-x(n)} \\ &= o(1). \end{aligned}$$

Now putting

$$z = E^+ - w\sqrt{np^*q^*}$$

where

$$w = w(n) = (2\log n)^{1/2} - \left(\frac{1}{2\log n}\right)^{1/2} \left(\frac{1}{2}\log\log n + x(n)\right)$$

we have

$$E(Y_z^-) \rightarrow \infty \text{ as } n \rightarrow \infty$$

and

$$\sum_{k=0}^z Pr(X_k^- \geq 2) = o(1).$$

Therefore the same argument as in the proof of Theorem 1 implies the first part of our result. The second part follows analogously. ■

4. Proofs of Lemmas

Proof of Lemma 2.

$$\begin{aligned} E(Y_t^-) &= n \sum_{k=0}^t \binom{n-1}{k} \left\{ \sum_{\substack{a_0, \dots, a_{n-1} \geq 0 \\ a_0 + \dots + a_{n-1} = k}} \binom{n-1}{a_0, \dots, a_{n-1}} \prod_{j=0}^{n-1} \left(P_j \frac{j}{n-1} \right)^{a_j} \right. \\ &\quad \times \left. \sum_{\substack{b_0, \dots, b_{n-1} \geq 0 \\ b_0 + \dots + b_{n-1} = n-k-1}} \binom{n-1}{b_0, \dots, b_{n-1}} \prod_{j=0}^{n-1} \left[\left(1 - \frac{j}{n-1} \right) P_j \right]^{b_j} \right\} \\ &= n \sum_{k=0}^t \binom{n-1}{k} \left[\left(\sum_{j=0}^{n-1} P_j \frac{j}{n-1} \right)^k \left(\sum_{j=0}^{n-1} \left(1 - P_j \frac{j}{n-1} \right) \right)^{n-k-1} \right]. \end{aligned}$$

Hence

$$p^* = \sum_{j=0}^{n-1} P_j \frac{j}{n-1}$$

we have

$$\begin{aligned} E(Y_t^-) &= n \sum_{k=0}^t \binom{n-1}{k} (p^*)^k (q^*)^{n-k-1} \\ &= nB(t; n-1, p^*). \end{aligned}$$

Proof of (10) is analogous. ■

Proof of Lemma 3. We show part (i). Let \mathcal{L} denotes the set of all arcs in $D(n, \mathcal{P})$. Let A be the event that two given vertices from V , say v_1 and v_2 , have the in-degree equal to r in $D(n, \mathcal{P})$. Then

$$(25) \quad E_2(X_r^-) = (n)_2 Pr(A).$$

Let

$$B(v_1) = \{v \in V \setminus \{v_1, v_2\} : (v, v_1) \in \mathcal{L}\}$$

and

$$B(v_2) = \{v \in V \setminus \{v_1, v_2\} : (v, v_2) \in \mathcal{L}\}.$$

Then considering the event A_1 that $(v_1, v_2) \notin \mathcal{L}$ and $(v_2, v_1) \notin \mathcal{L}$, we have clearly that

$$|B(v_1)| = |B(v_2)| = r \text{ and } |B(v_1) \cap B(v_2)| = k$$

for $k = f, \dots, r$, where $f = \max\{0, 2r - (n - 2)\}$ and

$$\begin{aligned} Pr(A_1) &= \left[P_1 \frac{\binom{n-2}{1}}{\binom{n-1}{1}} + \dots + P_{n-1} \frac{\binom{n-2}{n-1}}{\binom{n-1}{n-1}} \right]^2 \\ &= \left[P_1 \left(1 - \frac{1}{n-1}\right) + \dots + P_{n-1} \left(1 - \frac{n-1}{n-1}\right) \right]^2 \\ &= \left[1 - \frac{1}{n-1} \sum_{i=1}^{n-1} i P_i \right]^2 \\ &= (q^*)^2. \end{aligned}$$

Analogously denoting by A_2, A_3 and A_4 the events corresponding to the case

- $(v_1, v_2) \notin \mathcal{L}$ and $(v_2, v_1) \in \mathcal{L}$
- $(v_1, v_2) \in \mathcal{L}$ and $(v_2, v_1) \notin \mathcal{L}$
- $(v_1, v_2) \in \mathcal{L}$ and $(v_2, v_1) \in \mathcal{L}$,

respectively we have

$$Pr(A_2) = Pr(A_3) = p^* q^*$$

and

$$Pr(A_4) = (p^*)^2.$$

Furthermore, let B_j stand for the event that a given vertex from the set $V \setminus \{v_1, v_2\}$ emanates j ($j = 0, 1, 2$) arcs to vertices $\{v_1, v_2\}$. Assume that for $j = 1$ it is known to which vertex, v_1 or v_2 , an arc is coming to. Then for $j = 0, 1, 2$ we have

$$Pr(B_j) = \sum_{i=j}^{n-j} P_i \frac{\binom{n-3}{i-j}}{\binom{n-1}{i}} \quad j = 0, 1, 2.$$

In particular

$$\begin{aligned} Pr(B_1) &= \sum_{i=1}^{n-2} P_i \frac{\binom{n-3}{i-1}}{\binom{n-1}{i}} \\ &= \sum_{i=1}^{n-2} P_i \frac{i(n-i-1)}{(n-1)(n-2)} \\ &= \sum_{i=1}^{n-2} P_i \frac{i(n-1)}{(n-1)(n-2)} - \sum_{i=1}^{n-2} P_i \frac{i^2}{(n-1)(n-2)} \\ &\leq \sum_{i=0}^{n-1} P_i \frac{i}{(n-1)} - \sum_{i=1}^{n-2} P_i \frac{i^2}{(n-1)(n-2)} \\ &\leq p^* - (p^*)^2 = p^* q^*. \end{aligned}$$

Similarly we get that $Pr(B_0) \leq (q^*)^2$ and $Pr(B_2) \leq (p^*)^2$. Consequently, with

$$H(a, b, c, e)$$

$$= \binom{n-2}{a} \sum_{k=b}^c \binom{a}{k} \binom{n-2-a}{c-k} Pr(B_2)^k Pr(B_1)^{2(r-k)-e} Pr(B_0)^{n-2-2r+k+e},$$

$f = \max\{0, 2r+2-n\}$, $g = \max\{0, 2r+1-n\}$ and $h = \max\{0, 2r-n\}$ we have

$$Pr(A|A_1) = H(r, f, r, 0)$$

$$Pr(A|A_2) = Pr(A|A_3) = H(r, g, r-1, 1)$$

$$Pr(A|A_4) = H(r-1, h, r-1, 2).$$

Applying the well-known relation

$$\sum_{k=0}^c \binom{a}{c} \binom{n-2-a}{c-k} = \binom{n-2}{c}$$

we obtain the following estimate

$$\begin{aligned} & Pr(A|A_1)Pr(A_1) \\ &= \binom{n-2}{r} \sum_{k=f}^r \binom{r}{k} \binom{n-2-r}{r-k} Pr(B_2)^k Pr(B_1)^{2(r-k)} Pr(B_0)^{n-2r-2+k} (q^*)^2 \\ &\leq \binom{n-2}{r} \sum_{k=f}^r \binom{r}{k} \binom{n-2-r}{r-k} (p^*)^{2r} (q^*)^{2(n-r-1)} \\ &\leq \binom{n-2}{r} \binom{n-2}{r} (p^*)^{2r} (q^*)^{2(n-r-1)} \\ &= \left[\binom{n-1}{r} (p^*)^r (q^*)^{n-r-1} \right]^2 \left(1 - \frac{r}{n-1} \right)^2 \\ &= b^2(n-1; r, p^*) \left(1 + O^2\left(\frac{r}{n}\right) \right). \end{aligned}$$

Analogously

$$\begin{aligned} Pr(A|A_2)Pr(A_2) &= Pr(A|A_3)Pr(A_3) \\ &= b^2(n-1; r, p^*) \frac{r}{n-1} \left(1 - \frac{r}{n-1} \right) \end{aligned}$$

and

$$Pr(A|A_4)Pr(A_4) = b^2(n-1; r, p^*) \frac{r^2}{(n-1)^2}.$$

Thus by the assumption that $r = o(n)$ we get

$$\begin{aligned} Pr(A) &= \sum_{i=1}^4 Pr(A|A_i)Pr(A_i) \\ &\leq b^2(n-1; r, p^*)(1 + o(1)) \end{aligned}$$

and consequently by (25)

$$E_2(X_r^-) \leq n^2 b^2 (n-1; r, p^*) (1 + o(1)).$$

Proof of part (ii) is analogous. ■

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