# IN-DEGREE SEQUENCE IN A GENERAL MODEL OF A RANDOM DIGRAPH 

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#### Abstract

A general model of a random digraph $D(n, \mathcal{P})$ is considered. Based on a precise estimate of the asymptotic behaviour of the distribution function of the binomial law, a problem of the distribution of extreme in-degrees of $D(n, \mathcal{P})$ is discussed.


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## 1. Introduction

We begin with a definition of a general model of a random digraph that was introduced in [6]. Let $\mathcal{P}=\left(P_{0}, \ldots, P_{n-1}\right)$ be a probability distribution, i.e., an $n$-tuple of non-negative real numbers which satisfy $\sum_{i=0}^{n-1} P_{i}=1$. Denote by $D(n, \mathcal{P})$ a random digraph on a vertex set $V=\{1,2, \ldots, n\}$ such that (here, and what follows, $N^{+}(i)$ denotes the set of images of a vertex $i$ ):
(1) each vertex "chooses" its out-degree and then its images independently of all other vertices,
(2) each vertex $i \in V$ chooses its out-degree according to the probability distribution $\mathcal{P}$, i.e.,

$$
\operatorname{Pr}\left\{\left|N^{+}(i)\right|=k\right\}=P_{k}, \quad k=0,1, \ldots, n-1
$$

(3) for every $S \subseteq V \backslash\{i\}$, with $|S|=k$, the probability that $S$ coincides with the set of images of a vertex $i$ equals

$$
\operatorname{Pr}\left\{N^{+}(i)=S\right\}=P_{k} /\binom{n-1}{k}
$$

i.e., vertex $i$ "chooses" uniformly the set of images.

In particular, if $\mathcal{P}$ is such that $P_{d}=1$ for some $d, 1 \leq d \leq n-1$, the model $D(n, \mathcal{P})$ is equivalent to a random $d$-out regular digraph $D(n, d)$. Such a digraph can also be defined as an element chosen at random from the family of all $\binom{n-1}{d}^{n}$ digraphs on $n$ labeled vertices each of out-degree $d$. (Alternatively, $D(n, d)$ can be thought as a representation of a sum of $d$ dependent random mappings as illustrated in [7].)

In a case when $\mathcal{P}$ is a binomial distribution $\mathcal{B}(n-1, p)$, i.e.,

$$
\mathcal{P}=\left(q^{n-1}, \ldots, b(k ; n-1, p), \ldots, p^{n-1}\right)
$$

where

$$
b(r ; n, p)=\binom{n}{r} p^{r} q^{n-r}
$$

the model $D(n, \mathcal{P})$ is equivalent to a random digraph $D(n, \mathcal{B})$ on $n$ labeled vertices in which each of $n(n-1)$ possible arcs appears independently with a given probability $p=1-q$.

## 2. Preliminaries

Let $X^{+}$be a discrete random variable having a probability distribution $\mathcal{P}=\left(P_{0}, P_{1}, \ldots, P_{n-1}\right):$

$$
\operatorname{Pr}\left\{X^{+}=k\right\}=P_{k}, \quad k=0,1, \ldots, n-1 .
$$

Due to the homogeneous structure of the random digraph $D(n, \mathcal{P})$, the random variable $X^{+}=X^{+}(i)$ defines the out-degree of a given vertex $i \in V=\{1,2, \ldots, n\}$ of $D(n, \mathcal{P})$. Then the probability that a given subset of vertices is contained in the set of images of vertex $i \in V$ can be expressed by appropriate factorial moment of $X^{+}$. As a matter of fact the following property is true (see [8]). Here and what follows $(n)_{k}=n(n-1) \ldots(n-k+1)$ and $E_{k}(X)$ stands for the $k$-th factorial moment of a random variable $X$.

Property 1. For a given $i, 1 \leq i \leq n$, let $U \subseteq V \backslash\{i\}$ and $|U|=t \geq 1$. Then

$$
\operatorname{Pr}\left\{U \subseteq N^{+}(i)\right\}=\frac{1}{(n-1)_{t}} E_{t}\left(X^{+}\right)
$$

In particular, if $t=1$ the above property defines an arc occurrence probability in digraph $D(n, \mathcal{P})$. Let

$$
E^{+}=E^{+}(\mathcal{P})=\sum_{k=0}^{n-1} k P_{k}
$$

Then the probability of an arc in $D(n, \mathcal{P})$ is given by

$$
\begin{equation*}
p^{*}=\frac{E^{+}(\mathcal{P})}{n-1} \tag{1}
\end{equation*}
$$

Now let $X^{-}=X^{-}(i)$ be the in-degree of a given vertex $i \in\{1,2, \ldots, n\}$ of $D(n, \mathcal{P})$. Clearly, the probability distribution of $X^{-}$depends on $\mathcal{P}$. We have the following result (see [8]).

Property 2. For $i=1,2, \ldots, n$ the random variable $X^{-}(i)$ has binomial distribution $\mathcal{B}\left(n-1, p^{*}\right)$.

In contrast with out-degrees of vertices of $D(n, \mathcal{P})$, the random variables $X^{-}(i), i=1,2, \ldots, n$, are not, in general, independent. The only case when these variables are independent is when $X^{+}$is binomially distributed (see [8]).

The main aim of our paper is to study the probabilistic properties of extreme in-degrees of the random digraph $D(n, \mathcal{P})$. We show that the indegree sequence of $D(n, \mathcal{P})$ behaves similarly to the degree sequence of the classical model of a random graph (see [11]). Our results generalize those presented in [10].

Let $G_{n}$ be an arbitrary random graph model defined on $n$ vertices. If $\pi$ is a graph property then the assertion " $G_{n}$ has property $\pi$ asymptotically almost surely (a.a.s.)" means

$$
\lim _{n \rightarrow \infty} P\left(G_{n} \text { has property } \pi\right)=1
$$

The symbols $o, O$ and $\sim$ are used with respect to $n \rightarrow \infty$.

Consider "degree" sequence $d_{(1)} \leq d_{(2)} \leq \cdots \leq d_{(n)}$ of $G_{n}$. If $G_{n}$ is a simple (directed) graph then by the "degree" sequence we mean sequence of degrees (in-degrees or out-degrees) written in non-decreasing order. Denote by $X_{r}, Y_{s}$ and $Z_{t}$ the number of vertices of "degree" $=r, \leq s$ and $\geq t$ in $G_{n}$, respectively.

Let $B(s ; n, p)$ denote probability of at most $s$ successes in the binomial distribution. Similarly, let $F(t ; n, p)$ denote probability of at least $t$ successes in such distribution. In the proofs of our main results we will need a very precise etimate of the asymptotic behaviour of the distribution function of the binomial law with parameters $n$ and $p$, where $p=p(n)=o(1)$ and $n p / \log n \rightarrow \infty$ as $n \rightarrow \infty$ (see [5] and [12]).

Consider the equation

$$
(1+z) \log (1+z)+\frac{1}{a}(1-a z) \log (1-a z)=u
$$

where $0 \leq u<\infty$ and $a \geq 0$. It is known (see e.g. [5]) that this equation has a negative solution $z(u, a)$ and a positive solution $y(u, a)$, which in some neighbourhood of zero are given by the power series

$$
\begin{equation*}
z(u, a)=-\left(\frac{2 u}{1+a}\right)^{\frac{1}{2}}+\sum_{i=2}^{\infty}(-1)^{i} f_{i}(a)\left(\frac{2 u}{1+a}\right)^{i / 2} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
y(u, a)=-\left(\frac{2 u}{1+a}\right)^{\frac{1}{2}}+\sum_{i=2}^{\infty} f_{i}(a)\left(\frac{2 u}{1+a}\right)^{i / 2} \tag{3}
\end{equation*}
$$

in which

$$
\begin{array}{r}
f_{i+1}(a)=\frac{(-1)^{i}}{i+1} \sum \frac{(-1)^{k}(i+1)(i+3) \ldots(i+2 k-1)}{k_{1}!\ldots k_{i}!(2 \cdot 3)^{k_{1}} \ldots[(i+1)(i+2)]^{k_{i}}} \\
\times \frac{\left(1-a^{2}\right)^{k_{1}}\left(1+a^{3}\right)^{k_{2}} \ldots\left[1+(-1)^{i} a^{i+1}\right]^{k_{i}}}{(1+a)^{k}}
\end{array}
$$

where $k=k_{1}+k_{2}+\ldots k_{i}$ and the summation is over all non-negative integers $k_{1}, \ldots, k_{i}$ such that $k_{1}+2 k_{2}+\cdots+i k_{i}=i$. In particular,
(4) $z(u, a)=-\left(\frac{2 u}{1+a}\right)^{\frac{1}{2}}+\frac{1-a}{3(1+a)} u+\frac{\sqrt{2}}{36} \frac{1+4 a+a^{2}}{(1+a)^{3 / 2}} u^{3 / 2}+\ldots$
and
(5) $y(u, a)=\left(\frac{2 u}{1+a}\right)^{\frac{1}{2}}+\frac{1-a}{3(1+a)} u-\frac{\sqrt{2}}{36} \frac{1+4 a+a^{2}}{(1+a)^{3 / 2}} u^{3 / 2}+\ldots$

Now put

$$
\begin{equation*}
u=u(n, p)=\frac{1}{n p}\left(\log n-\frac{1}{2} \log \log n\right) . \tag{6}
\end{equation*}
$$

In proofs of our main results we will need the following lemma giving a very precise asymptotic behaviour of binomial distribution (see [12]).

Lemma 1. Let $m=n p=\omega(n) \log n$ where $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$ in such a way that $p=p(n)=o(1)$. Assume that $x=x(n)$ satisfies $x^{2}=$ $o(\min \{\omega(n), \log n\})$, and put

$$
\begin{aligned}
& s=m+m z\left(u, \frac{p}{q}\right)-\left(\frac{m}{2 \log n}\right)^{1 / 2}(x-\log \sqrt{4 \pi}+o(1)) \\
& t=m+m y\left(u, \frac{p}{q}\right)+\left(\frac{m}{2 \log n}\right)^{1 / 2}(x-\log \sqrt{4 \pi}+o(1))
\end{aligned}
$$

where $u$ is given by (6). Then

$$
\begin{equation*}
n B(s ; n, p) \sim n F(t ; n, p) \sim e^{-x} \tag{7}
\end{equation*}
$$

and

$$
\begin{equation*}
n b(s ; n, p) \sim n b(t ; n, p) \sim\left(\frac{2 \log n}{n p q}\right)^{1 / 2} e^{-x} \tag{8}
\end{equation*}
$$

## 3. Main Results

Let $X_{r}^{-}=X_{r}^{-}(\mathcal{P})$ denote the number of vertices of in-degree $r$ in a general model of a random digraph $D(n, \mathcal{P})$. Then by Property 2 we have

Property 3. The expected value of $X_{r}^{-}$equals

$$
E\left(X_{r}^{-}\right)=n b\left(r ; n-1, p^{*}\right)
$$

where $p^{*}$ is given by (1).

Now let us put $Y_{s}^{-}=Y_{s}^{-}(\mathcal{P})$ and $Z_{t}^{-}=Z_{t}^{-}(\mathcal{P})$ for the number of vertices of in-degree of at most $s$ and at least $t$ in $D(n, \mathcal{P})$, respectively. The following two lemmas, which proofs will be shown in the next section, are the basic tool in proving our main results.

## Lemma 2.

$$
\begin{equation*}
E\left(Y_{s}^{-}\right)=n B\left(s ; n-1, p^{*}\right) \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
E\left(Z_{t}^{-}\right)=n F\left(t ; n-1, p^{*}\right) . \tag{10}
\end{equation*}
$$

Lemma 3. (i) If $r=o(n)$ then

$$
E_{2}\left(X_{r}^{-}\right) \leq n^{2} b^{2}\left(r ; n-1, p^{*}\right)(1+o(1))
$$

(ii) If $E^{+}=E^{+}(\mathcal{P})=o(n), s<n p^{*}, t>n p^{*}$ and $t=o(n)$ then

$$
E_{2}\left(Y_{s}^{-}\right) \leq n^{2} B^{2}\left(s ; n-1, p^{*}\right)(1+o(1))
$$

and

$$
E_{2}\left(Z_{t}^{-}\right) \leq n^{2} F^{2}\left(t ; n-1, p^{*}\right)(1+o(1))
$$

Let

$$
d_{(1)}^{-} \leq d_{(2)}^{-} \leq \cdots \leq d_{(n)}^{-}
$$

be the in-degree sequence of vertices in a random digraph $D(n, \mathcal{P})$. The first result shows that for any fixed $i \geq 2$ the first $i$-th and the last $i$-th terms of the in-degree sequence of $D(n, \mathcal{P})$ are asymptotically almost surely strictly increasing. For the sake of simplicity let us denote

$$
\begin{align*}
& s=s(n, \mathcal{P})=(1+z(u, a)) E^{+}  \tag{11}\\
& t=t(n, \mathcal{P})=(1+y(u, a)) E^{+} \tag{12}
\end{align*}
$$

and

$$
\begin{equation*}
\varphi=\varphi(n, \mathcal{P})=\left(\frac{E^{+}}{2 \log n}\right)^{1 / 2} x(n) \tag{13}
\end{equation*}
$$

where power series $z(u, a)$ and $y(u, a)$ are given by (2) and (3), respectively and $x(n)$ is a sequence tending to infinity arbitrary slowly as $n \rightarrow \infty$.

Theorem 1. Let $\mathcal{P}=\left(P_{0}, P_{1}, \ldots, P_{n-1}\right)$ be such that

$$
E^{+}=\omega(n) \log (n)=o(n)
$$

where $\omega(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then for any fixed $i \geq 2$

$$
\begin{equation*}
s-\varphi<d_{(1)}^{-}<\cdots<d_{(i)}^{-}<s+\varphi \quad \text { a.a.s. } \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
t-\varphi<d_{(n-i+1)}^{-}<\cdots<d_{(n)}^{-}<t+\varphi \quad \text { a.a.s. } \tag{15}
\end{equation*}
$$

where $s$ and $t$ are given by (11) and (12) with

$$
\begin{equation*}
u=u(n, \mathcal{P})=\frac{1}{E^{+}}\left(\log n-\frac{1}{2} \log \log n\right) \tag{16}
\end{equation*}
$$

$$
\begin{equation*}
a=a(n, \mathcal{P})=\frac{E^{+}}{n-1-E^{+}} \tag{17}
\end{equation*}
$$

and $\varphi$ is given by (13).
Proof. Put $r=s-\varphi$. Then by Lemma 2 we have

$$
E\left(Y_{r}^{-}\right)=n B\left(s-\varphi ; n-1, p^{*}\right)
$$

Since

$$
p^{*}=\frac{E^{+}}{n-1}=\frac{\omega(n) \log n}{n-1}
$$

and

$$
s-\varphi=\frac{\omega(n) \log n}{n-1}(1+z(u, a))-\left(\frac{\omega(n)}{2(n-1)}\right)^{1 / 2} x(n)
$$

so by Lemma 1

$$
\begin{align*}
E\left(Y_{r}^{-}\right) & \sim e^{-x(n)} \\
& =o(1) \tag{18}
\end{align*}
$$

Consequently

$$
\begin{aligned}
\operatorname{Pr}\left(d_{(1)} \leq s-\varphi\right) & =\operatorname{Pr}\left(Y_{r}^{-} \geq 1\right) \\
& \leq E\left(Y_{r}^{-}\right) \\
& =o(1) .
\end{aligned}
$$

Now let us put $r=s+\varphi$. Then

$$
\begin{equation*}
E\left(Y_{r}^{-}\right)=\sim e^{x(n)} \rightarrow \infty \text { as } n \rightarrow \infty \tag{19}
\end{equation*}
$$

Moreover, routine calculations show that by (4), (11), (13), (16) and (17) we have

$$
r=s+\varphi<n p^{*}(1+o(1)) .
$$

So by Lemma 2 and 3

$$
E_{2}\left(Y_{r}^{-}\right) \leq E^{2}\left(Y_{r}^{-}\right)(1+o(1))
$$

which implies that

$$
\operatorname{Var}\left(Y_{r}^{-}\right) \leq E\left(Y_{r}^{-}\right)+o\left(E^{2}\left(Y_{r}^{-}\right)\right) .
$$

Thus by Chebyshev's inequality

$$
\operatorname{Pr}\left(Y_{r}^{-} \leq \frac{1}{2} E\left(Y_{r}^{-}\right)\right) \leq \frac{4 \operatorname{Var}\left(Y_{r}^{-}\right)}{E^{2}\left(Y_{r}^{-}\right)}=o(1) .
$$

Consequently, for any fixed $i \geq 1$

$$
\begin{aligned}
\operatorname{Pr}\left(d_{(i)}^{-} \leq s+\varphi\right) & =\operatorname{Pr}\left(Y_{r}^{-} \geq i\right) \\
& \geq 1-o(1) .
\end{aligned}
$$

To show that the sequence is stricly increasing we have to show that probability that there are at least two vertices of equal in-degree $\leq s+\varphi$ tends to zero as $n \rightarrow \infty$. We have

$$
\sum_{k=0}^{s+\varphi} \operatorname{Pr}\left(X_{k}^{-} \geq 2\right) \leq \sum_{k=0}^{s+\varphi} E_{2}\left(X_{k}^{-}\right) .
$$

Since, by Lemma 3,

$$
E_{2}\left(X_{r}^{-}\right) \leq E^{2}\left(X_{r}^{-}\right)(1+o(1))
$$

so applying Lemma 1 we obtain

$$
\begin{aligned}
\sum_{k=0}^{s+\varphi} E_{2}\left(X_{k}^{-}\right) & \leq \sum_{k=0}^{s+\varphi} n^{2} b^{2}\left(k, n-1, p^{*}\right)(1+o(1)) \\
& \leq n b\left(s+\varphi ; n-1, p^{*}\right) n B\left(s+\varphi ; n-1, p^{*}\right)(1+o(1)) \\
& \sim\left(\frac{2 \log n}{n p^{*} q^{*}}\right)^{1 / 2} e^{-2 x(n)} \\
& =o(1)
\end{aligned}
$$

which completes the proof of (14). The proof of (15) follows analogously.
The above theorem gives a very precise estimate of the in-degree distribution of $D(n, \mathcal{P})$ in a case when the out-degree distribution $\mathcal{P}=\left(P_{0}, P_{1}, \ldots, P_{n-1}\right)$ satisfies the condition

$$
E^{+}(\mathcal{P})=\sum_{k=0}^{n-1} k P_{k}=\omega(n) \log n
$$

The disadvantage of this result is the complicated form for given bounds which are expressed by appropriate power series. It appears that if $E^{+}(\mathcal{P})$ tends to infinity a bit faster than $\omega(n) \log n$ much more pleasant estimates for in-degree sequence can be given. Now let

$$
\begin{align*}
& s=E^{+}-\left(2 n p^{*} q^{*} \log n\right)^{1 / 2}+\left(\frac{n p^{*} q^{*}}{8 \log n}\right)^{1 / 2} \log \log n  \tag{20}\\
& t=E^{+}+\left(2 n p^{*} q^{*} \log n\right)^{1 / 2}-\left(\frac{n p^{*} q^{*}}{8 \log n}\right)^{1 / 2} \log \log n
\end{align*}
$$

and

$$
\begin{equation*}
\varphi(n)=\left(\frac{n p^{*} q^{*}}{2 \log n}\right)^{1 / 2} x(n) \tag{22}
\end{equation*}
$$

where $x(n) \rightarrow \infty$ as $n \rightarrow \infty$ but $x(n)=o(\log \log n)$.

Theorem 2. Let $E^{+} \geq\left[\gamma(n)(\log n)^{3}\right], \gamma(n) \rightarrow \infty$ as $n \rightarrow \infty$. Then for any fixed $i \geq 1$

$$
\begin{equation*}
s-\varphi \leq d_{(1)}<d_{(2)}<\cdots<d_{(i)} \leq s+\varphi \quad \text { a.a.s. } \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
t-\varphi \leq d_{(n-i+1)}<\cdots<d_{(n-1)}<d_{(n)} \leq t+\varphi \quad \text { a.a.s. } \tag{24}
\end{equation*}
$$

where $s, t$ and $\varphi$ are given by (20), (21) and (22), respectively.
Proof. Put

$$
r=E^{+}-v \sqrt{n p^{*} q^{*}}
$$

where

$$
v=v(n)=(2 \log n)^{1 / 2}-\left(\frac{1}{2 \log n}\right)^{1 / 2}\left(\frac{1}{2} \log \log n-x(n)\right) .
$$

Then the assumption $n p^{*} \geq \gamma(n)(\log n)^{3}$ implies

$$
\frac{v^{3}}{\sqrt{n p^{*} q^{*}}} \leq\left(\frac{8}{\gamma(n)}\right)^{1 / 2}=o(1)
$$

so applying the classical DeMoivre-Laplace formula (see Feller [4] Chapter 7) we obtain

$$
\begin{aligned}
E\left(Y_{r}^{-}\right) & \sim \frac{n}{\sqrt{2 \pi}} \frac{1}{v} e^{-\frac{v^{2}}{2}} \\
& \sim \frac{1}{\sqrt{2 \pi}} e^{-x(n)} \\
& =o(1) .
\end{aligned}
$$

Now putting

$$
z=E^{+}-w \sqrt{n p^{*} q^{*}}
$$

where

$$
w=w(n)=(2 \log n)^{1 / 2}-\left(\frac{1}{2 \log n}\right)^{1 / 2}\left(\frac{1}{2} \log \log n+x(n)\right)
$$

we have

$$
E\left(Y_{z}^{-}\right) \rightarrow \infty \text { as } n \rightarrow \infty
$$

and

$$
\sum_{k=0}^{z} \operatorname{Pr}\left(X_{k}^{-} \geq 2\right)=o(1)
$$

Therefore the same argument as in the proof of Theorem 1 implies the first part of our result. The second part follows analogously.

## 4. Proofs of Lemmas

## Proof of Lemma 2.

$$
\begin{aligned}
E\left(Y_{t}^{-}\right) & =n \sum_{k=0}^{t}\binom{n-1}{k}\left\{\sum_{\substack{a_{0}, \ldots, a_{n-1} \geq 0 \\
a_{0}+\ldots+a_{n-1}=k}}\binom{n-1}{a_{0}, \ldots, a_{n-1}} \prod_{j=0}^{n-1}\left(P_{j} \frac{j}{n-1}\right)^{a_{j}}\right. \\
& \left.\times \sum_{\substack{b_{0}, \ldots, b_{n-1} \geq 0 \\
b_{0}+\cdots+b_{n-1}=n-k-1}}\binom{n-1}{b_{0}, \ldots, b_{n-1}} \prod_{j=0}^{n-1}\left[\left(1-\frac{j}{n-1}\right) P_{j}\right]^{b_{j}}\right\} \\
= & n \sum_{k=0}^{t}\binom{n-1}{k}\left[\left(\sum_{j=0}^{n-1} P_{j} \frac{j}{n-1}\right)^{k}\left(\sum_{j=0}^{n-1}\left(1-P_{j} \frac{j}{n-1}\right)\right)^{n-k-1}\right] .
\end{aligned}
$$

Hence

$$
p^{*}=\sum_{j=0}^{n-1} P_{j} \frac{j}{n-1}
$$

we have

$$
\begin{aligned}
E\left(Y_{t}^{-}\right) & =n \sum_{k=0}^{t}\binom{n-1}{k}\left(p^{*}\right)^{k}\left(q^{*}\right)^{n-k-1} \\
& =n B\left(t ; n-1, p^{*}\right) .
\end{aligned}
$$

Proof of (10) is analogous.

Proof of Lemma 3. We show part (i). Let $\mathcal{L}$ denotes the set of all arcs in $D(n, \mathcal{P})$. Let $A$ be the event that two given vertices from $V$, say $v_{1}$ and $v_{2}$, have the in-degree equal to $r$ in $D(n, \mathcal{P})$. Then

$$
\begin{equation*}
E_{2}\left(X_{r}^{-}\right)=(n)_{2} \operatorname{Pr}(A) . \tag{25}
\end{equation*}
$$

Let

$$
B\left(v_{1}\right)=\left\{v \in V \backslash\left\{v_{1}, v_{2}\right\}:\left(v, v_{1}\right) \in \mathcal{L}\right\}
$$

and

$$
B\left(v_{2}\right)=\left\{v \in V \backslash\left\{v_{1}, v_{2}\right\}:\left(v, v_{2}\right) \in \mathcal{L}\right\}
$$

Then considering the event $A_{1}$ that $\left(v_{1}, v_{2}\right) \notin \mathcal{L}$ and $\left(v_{2}, v_{1}\right) \notin \mathcal{L}$, we have clearly that

$$
\left|B\left(v_{1}\right)\right|=\left|B\left(v_{2}\right)\right|=r \text { and }\left|B\left(v_{1}\right) \cap B\left(v_{2}\right)\right|=k
$$

for $k=f, \ldots, r$, where $f=\max \{0,2 r-(n-2)\}$ and

$$
\begin{aligned}
\operatorname{Pr}\left(A_{1}\right) & =\left[P_{1} \frac{\binom{n-2}{1}}{\binom{n-1}{1}}+\cdots+P_{n-1} \frac{\binom{n-2}{n-1}}{\binom{n-1}{n-1}}\right]^{2} \\
& =\left[P_{1}\left(1-\frac{1}{n-1}\right)+\cdots+P_{n-1}\left(1-\frac{n-1}{n-1}\right)\right]^{2} \\
& =\left[1-\frac{1}{n-1} \sum_{i=1}^{n-1} i P_{i}\right]^{2} \\
& =\left(q^{*}\right)^{2} .
\end{aligned}
$$

Analogously denoting by $A_{2}, A_{3}$ and $A_{4}$ the events corresponding to the case

- $\left(v_{1}, v_{2}\right) \notin \mathcal{L}$ and $\left(v_{2}, v_{1}\right) \in \mathcal{L}$
- $\left(v_{1}, v_{2}\right) \in \mathcal{L}$ and $\left(v_{2}, v_{1}\right) \notin \mathcal{L}$
- $\left(v_{1}, v_{2}\right) \in \mathcal{L}$ and $\left(v_{2}, v_{1}\right) \in \mathcal{L}$,
respectively we have

$$
\operatorname{Pr}\left(A_{2}\right)=\operatorname{Pr}\left(A_{3}\right)=p^{*} q^{*}
$$

and

$$
\operatorname{Pr}\left(A_{4}\right)=\left(p^{*}\right)^{2} .
$$

Furthermore, let $B_{j}$ stand for the event that a given vertex from the set $V \backslash\left\{v_{1}, v_{2}\right\}$ emanates $j(j=0,1,2)$ arcs to vertices $\left\{v_{1}, v_{2}\right\}$. Assume that for $j=1$ it is known to which vertex, $v_{1}$ or $v_{2}$, an arc is coming to. Then for $j=0,1,2$ we have

$$
\operatorname{Pr}\left(B_{j}\right)=\sum_{i=j}^{n-j} P_{i} \frac{\binom{n-3}{i-j}}{\binom{n-1}{i}} \quad j=0,1,2 .
$$

In particular

$$
\begin{aligned}
\operatorname{Pr}\left(B_{1}\right) & =\sum_{i=1}^{n-2} P_{i} \frac{\binom{n-3}{i-1}}{\binom{n-1}{i}} \\
& =\sum_{i=1}^{n-2} P_{i} \frac{i(n-i-1)}{(n-1)(n-2)} \\
& =\sum_{i=1}^{n-2} P_{i} \frac{i(n-1)}{(n-1)(n-2)}-\sum_{i=1}^{n-2} P_{i} \frac{i^{2}}{(n-1)(n-2)} \\
& \leq \sum_{i=0}^{n-1} P_{i} \frac{i}{(n-1)}-\sum_{i=1}^{n-2} P_{i} \frac{i^{2}}{(n-1)(n-2)} \\
& \leq p^{*}-\left(p^{*}\right)^{2}=p^{*} q^{*} .
\end{aligned}
$$

Similarly we get that $\operatorname{Pr}\left(B_{0}\right) \leq\left(q^{*}\right)^{2}$ and $\operatorname{Pr}\left(B_{2}\right) \leq\left(p^{*}\right)^{2}$. Consequently, with
$H(a, b, c, e)$
$=\binom{n-2}{a} \sum_{k=b}^{c}\binom{a}{k}\binom{n-2-a}{c-k} \operatorname{Pr}\left(B_{2}\right)^{k} \operatorname{Pr}\left(B_{1}\right)^{2(r-k)-e} \operatorname{Pr}\left(B_{0}\right)^{n-2-2 r+k+e}$,
$f=\max \{0,2 r+2-n\}, g=\max \{0,2 r+1-n\}$ and $h=\max \{0,2 r-n\}$ we have

$$
\begin{aligned}
& \operatorname{Pr}\left(A \mid A_{1}\right)=H(r, f, r, 0) \\
& \operatorname{Pr}\left(A \mid A_{2}\right)=\operatorname{Pr}\left(A \mid A_{3}\right)=H(r, g, r-1,1) \\
& \operatorname{Pr}\left(A \mid A_{4}\right)=H(r-1, h, r-1,2) .
\end{aligned}
$$

Applying the well-known relation

$$
\sum_{k=0}^{c}\binom{a}{c}\binom{n-2-a}{c-k}=\binom{n-2}{c}
$$

we obtain the following estimate
$\operatorname{Pr}\left(A \mid A_{1}\right) \operatorname{Pr}\left(A_{1}\right)$
$=\binom{n-2}{r} \sum_{k=f}^{r}\binom{r}{k}\binom{n-2-r}{r-k} \operatorname{Pr}\left(B_{2}\right)^{k} \operatorname{Pr}\left(B_{1}\right)^{2(r-k)} \operatorname{Pr}\left(B_{0}\right)^{n-2 r-2+k}\left(q^{*}\right)^{2}$
$\leq\binom{ n-2}{r} \sum_{k=f}^{r}\binom{r}{k}\binom{n-2-r}{r-k}\left(p^{*}\right)^{2 r}\left(q^{*}\right)^{2(n-r-1)}$
$\leq\binom{ n-2}{r}\binom{n-2}{r}\left(p^{*}\right)^{2 r}\left(q^{*}\right)^{2(n-r-1)}$
$=\left[\binom{n-1}{r}\left(p^{*}\right)^{r}\left(q^{*}\right)^{n-r-1}\right]^{2}\left(1-\frac{r}{n-1}\right)^{2}$
$=b^{2}\left(n-1 ; r, p^{*}\right)\left(1+O^{2}\left(\frac{r}{n}\right)\right)$.
Analogously

$$
\begin{aligned}
\operatorname{Pr}\left(A \mid A_{2}\right) \operatorname{Pr}\left(A_{2}\right) & =\operatorname{Pr}\left(A \mid A_{3}\right) \operatorname{Pr}\left(A_{3}\right) \\
& =b^{2}\left(n-1 ; r, p^{*}\right) \frac{r}{n-1}\left(1-\frac{r}{n-1}\right)
\end{aligned}
$$

and

$$
\operatorname{Pr}\left(A \mid A_{4}\right) \operatorname{Pr}\left(A_{4}\right)=b^{2}\left(n-1 ; r, p^{*}\right) \frac{r^{2}}{(n-1)^{2}} .
$$

Thus by the assumption that $r=o(n)$ we get

$$
\begin{aligned}
\operatorname{Pr}(A) & =\sum_{i=1}^{4} \operatorname{Pr}\left(A \mid A_{i}\right) \operatorname{Pr}\left(A_{i}\right) \\
& \leq b^{2}\left(n-1 ; r, p^{*}\right)(1+o(1))
\end{aligned}
$$

and consequently by (25)

$$
E_{2}\left(X_{r}^{-}\right) \leq n^{2} b^{2}\left(n-1 ; r, p^{*}\right)(1+o(1))
$$

Proof of part (ii) is analogous.

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