Discussiones Mathematicae
Graph Theory 26 (2006) 141–147

DECOMPOSING COMPLETE GRAPHS INTO CUBES

Saad I. El-Zanati and C. Vanden Eynden

4520 Mathematics Department
Illinois State University
Normal, Illinois 61790–4520, USA

Abstract

This paper concerns when the complete graph on n vertices can be decomposed into d-dimensional cubes, where d is odd and n is even. (All other cases have been settled.) Necessary conditions are that n be congruent to 1 modulo d and 0 modulo 2^d. These are known to be sufficient for d equal to 3 or 5. For larger values of d, the necessary conditions are asymptotically sufficient by Wilson's results. We prove that for each odd d there is an infinite arithmetic progression of even integers n for which a decomposition exists. This lends further weight to a long-standing conjecture of Kotzig.

Keywords: graph decomposition, graph factorization, d-cube.

2000 Mathematics Subject Classification: 05C70, 05B30.

1. Introduction

A sequence $H_1, H_2, \ldots, H_n$ of graphs with union G is called a *decomposition* of G if each edge of G is in H_i for exactly one i, and in this case we write $G = H_1 + H_2 + \cdots + H_n$. If in addition the subgraphs H_i are all isomorphic to H, then we write $G = nH$, and say that H *divides* G. We call such a decomposition an *H-decomposition* of G. If G_1 is a subgraph of G that includes all the vertices of G and each component of G_1 is isomorphic to H, then we call G_1 an *H-factor* of G. We denote the complete graph on n vertices by K_n, and the complete bipartite graph with j vertices on one side and k on the other by $K_{j,k}$. If $m \leq n$ by $K_n \backslash K_m$ we mean the complete graph on a set of n vertices with all edges internal to some subset of m vertices (called the *hole*) removed. By a *k-set* we mean a set with k elements.

The *d-cube*, denoted Q_d, is the graph whose vertices can be labelled with all the binary d-tuples, such that two vertices are adjacent if and only if they differ in a single coordinate. It is easy to see that Q_d is d-regular, bipartite, and has 2^d vertices and $d2^{d-1}$ edges.

The decomposition of graphs is the focus of a great deal of research (see [2] for a thorough discussion of the subject). In particular, decompositions of K_n into smaller complete graphs and decompositions of K_n into cycles have received much attention. In 1979, Anton Kotzig initiated interest in d-cube decompositions of complete graphs by asking for which values of d and n there exists a Q_d-decomposition of K_n (Problem 15 of [12]). In 1981 he established necessary conditions on d and n for the existence of Q_d-decompositions of K_n for all d and proved the sufficiency of these conditions for some cases [13].

Since Q_d is d-regular with 2^d vertices and $d2^{d-1}$ edges, it is easy to see that the following are necessary conditions for the existence of a d-cube decomposition of K_n:

(1) if $n > 1$ then $n \geq 2^d$,

(2) $d \mid n-1$, and

(3) $d2^d \mid n(n-1)$.

For a fixed d, these necessary conditions require that n lies in certain congruence classes modulo d. In 1981, Kotzig [13] proved the following results.

Theorem 1. *If there exists a Q_d-decomposition of K_n, then*

(a) *if d is even, then $n \equiv 1 \pmod{d2^d}$;*

(b) *if d is odd, then either*

  (i) *$n \equiv 1 \pmod{d2^d}$, or*

  (ii) *$n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$.*

Theorem 2. *There is a Q_d-decomposition of K_n if $n \equiv 1 \pmod{d2^d}$.*

These two theorems established the sufficiency of conditions (1) through (3) for the cases when d is even and when d is odd and n is odd. Sufficiency of these conditions in the case $d = 3$ was shown by Maheo [14] in 1980. Recently, the case $d = 5$ was settled by Bryant *et al.* [4]. This however still leaves the following unsolved problem.

Problem 1. Let $d > 5$ be odd and let n be such that $n \equiv 0 \pmod{2^d}$ and $n \equiv 1 \pmod{d}$. Show that $Q_d | K_n$.

Although this problem has been cited often in the literature (see for example [2, 10, 11, 12]), little progress was made on the case d odd and n even until recently. Of course the well-known 1975 theorem of Wilson [15] implies that for each d we have $Q_d \mid K_n$ for all sufficiently large n satisfying conditions (1) through (3). A new technique for Q_d-decompositions using partitions of vector spaces into linearly independent sets was introduced in [6] in 1998. This technique was used in [8] to give, for each odd d, an explicit infinite sequence of even values of n such that $Q_d \mid K_n$.

Theorem 3 [8]**.** *Let d be odd and let s be the order of* $2 \pmod d$*. If r is any integer with $r \geq d/s$, then $Q_d \mid K_{2^{rs}}$.*

Other articles dealing with various d-cube decompositions include [1, 3] and [9].

In this paper we prove that for each odd d there is an infinite arithmetic progression of even integers n for which a Q_d-decomposition of K_n exists.

2. Preliminaries

Let Z_2 be the field of order 2. We denote Z_2^m, regarded as a vector space over Z_2, by V_m. Note that we can think of V_m as the vertex set of Q_m. We denote by $\langle S \rangle$ the subspace of V_m generated by $S \subseteq V_m$. For $a \in V_m$ and $A, B \subseteq V_m$ we define $a+B = \{a+b : b \in B\}$, we define $A+B = \cup_{a \in A}(a+B)$. If A and B are subsets of V_m with $0 \notin B$, then let $G(A,B)$ be the graph with vertex set $A \bigcup (A+B)$ and edge set $\{\{a, a+b\} : a \in A, b \in B\}$.

The following is the $k = 2$ case of Lemma 1 of [6].

Theorem 4. *Suppose B is a linearly independent subset of V_m with d elements. Then $G(V_m, B)$ is a Q_d-factor of the complete graph on V_m.*

The following somewhat more general result appears in [4], but we repeat the short proof here.

Lemma 5. *Suppose $A, B \subseteq V_m$, with $A \supseteq A+B$, $|B| = d$, and B linearly independent. Then $G(A,B)$ is a Q_d-factor of the complete graph on A.*

Proof. Note that $G(\langle B \rangle, B) \cong Q_d$ by Theorem 4.

Now $A \supseteq A+B \supseteq (A+B)+B \supseteq \ldots$, and so $A \supseteq A + \langle B \rangle$, implying $A = A + \langle B \rangle$. Also if $a \in A$, then $G(a + \langle B \rangle, B) = a + G(\langle B \rangle, B) \cong Q_d$ by the above. Furthermore the sets $a + < B >$ for $a \in A$ are cosets of $< B >$, and so either identical or disjoint. Thus $G(A, B) = G(A + < B >, B) = \bigcup_{a \in A} G(a + < B >, B)$, which is the vertex disjoint union of copies of Q_d. ■

In [8] we prove a lemma (Lemma 3), which becomes the following when applied to V_m.

Theorem 6. *Let W be a subspace of V_m, and let $d_1, d_2, \ldots, d_t$ be integers with $1 \leq d_i \leq m$ for $1 \leq i \leq t$ and $\sum_i d_i = |V_m \backslash W|$. Then $V_m \backslash W$ can be partitioned into linearly independent sets $X_1, X_2, \ldots, X_t$ such that $|X_i| = d_i$ for $1 \leq i \leq t$.*

Likewise Theorem 5 of [8] becomes the following when we take $k = 2$ and $j = n = m$.

Theorem 7. *Let $d_1, d_2, \ldots, d_t$ be integers such that $1 \leq d_i \leq m$ for $1 \leq i \leq t$ and $\sum_{i=1}^{t} d_i = 2^m - 1$. Then K_{2^m} can be decomposed into a Q_{d_1}-factor, a Q_{d_2}-factor, …, and a Q_{d_t}-factor.*

3. Main Results

Theorem 8. *Let d, a and b be integers with $0 < d \leq a < b$ such that $2^a - 1 \equiv 2^b - 1 \equiv r \pmod{d}$, where $0 \leq r < d$. Then $K_{2^b} \backslash K_{2^a}$ can be written as a Q_r-factor on the non-hole vertices plus a graph divisible by Q_d.*

Proof. Let W be the subspace of V_b consisting of all vectors $(x_1, x_2, \ldots, x_b)$ such that $x_1 = x_2 = \ldots = x_{b-a} = 0$. Clearly W has 2^a vectors and is isomorphic to V_a. We will take the vertex set of $K_{2^b} \backslash K_{2^a}$ to be V_b, with hole W.

Let $2^a - 1 = qd + r$. By Theorem 6 we can partition $W \backslash \{0\}$ into linearly independent sets $B_1, B_2, \ldots, B_q$, R, with $|B_i| = d$ for all i and $|R| = r$, and partition $V_b \backslash W$ into linearly independent d-sets $C_1, C_2, \ldots, C_s$, where $s = (2^b - 2^a)/d$.

Note that the hypotheses of Lemma 5 on A and B apply to each graph $G(V_b \backslash W, R)$, $G(V_b \backslash W, B_i)$, and $G(V_b, C_i)$. Thus the graph $G(V_b \backslash W, R)$ is a Q_r-factor of the complete graph on $V_b \backslash W$, and the graphs $G(V_b \backslash W, B_i)$, and $G(V_b, C_i)$ are Q_d-factors of the complete graphs on $V_b \backslash W$ and V_b, respectively, for all appropriate i.

Now we claim that the graph $K_{2^b}\backslash K_{2^a}$, interpreted as the complete graph on V_b with all edges internal to W removed, consists of the r-factor $G(V_b\backslash W, R)$ of $V_b\backslash W$ along with $(\bigcup_{i=1}^q G(V_b\backslash W, B_i))\bigcup(\bigcup_{i=1}^s G(V_b, C_i))$.

If A and B satisfy the hypotheses of Lemma 5, then the graph $G(A,B)$ contains $|A||B|/2$ edges. Thus $G(V_b\backslash W, R)$, $G(V_b\backslash W, B_i)$, and $G(V_b, C_i)$ contain $(2^b-2^a)r/2$, $(2^b-2^a)d/2$, and $2^b d/2$ edges, respectively. Then

$$G(V_b\backslash W, R)\bigcup\left(\bigcup_{i=1}^{q} G(V_b\backslash W, B_i)\right)\bigcup\left(\bigcup_{i=1}^{s} G(V_b, C_i)\right)$$

contains

$$\begin{aligned}\frac{(2^b-2^a)r}{2}+q\frac{(2^b-2^a)d}{2}+s\frac{2^b d}{2} &= \frac{(2^b-2^a)(2^a-1)}{2}+\frac{(2^b-2^a)2^b}{2}\\ &= \frac{2^b(2^b-1)}{2}-\frac{2^a(2^a-1)}{2}\end{aligned}$$

edges, which is the correct number of edges in $K_{2^b}\backslash K_{2^a}$. Thus it suffices to show that if x and y are distinct elements of V_b, but not both in W, then the edge $\{x,y\}$ is included in the above union. We can assume that $x\notin W$.

First assume that $y-x\in W$. Then $y-x$ is in R or B_i for some i, and $\{x,y\}$ is an edge of $G(V_b\backslash W, R)$ or $G(V_b\backslash W, B_i)$, respectively.

Now assume that $y-x\notin W$. Then $y-x\in C_i$ for some i, and $\{x,y\}$ is an edge of $G(V_b, C_i)$. ■

The following is Theorem 4 of [7]

Theorem 9. *There exists a d-cube decomposition of $K_{xd2^{d-1},yd2^{d-1}}$ for all positive integers x, y, and d.*

Theorem 10. *Let d and a be integers with d odd and $0<d\le a$ such that $2^a-1\equiv r \pmod{d}$, where $0\le r<d$. Let s be the order of 2 modulo d and set $b=a+s$. Then for any nonnegative integer k, $K_{2^a+k(2^b-2^a)}$ can be decomposed into a Q_r-factor and a graph divisible by Q_d.*

Proof. Let $2^a-1=dq+r$. Then by Theorem 7 the graph K_{2^a} can be decomposed into a Q_r-factor and q Q_d-factors. Likewise by Theorem 8 the graph $K_{2^b}\backslash K_{2^a}$ can be written as a Q_r-factor on its nonhole vertices plus a graph divisible by Q_d. Let $2^s-1=dt$. Then by Theorem 9 with $x=y=2^{a-d+1}t$ the graph $K_{2^b-2^a,2^b-2^a}$ is divisible by Q_d.

Now consider the vertex set of $K_{2^a+k(2^b-2^a)}$ to be partitioned into a 2^a-set X and k (2^b-2^a)-sets $Y_1, Y_2, \ldots, Y_k$. We can consider $K_{2^a+k(2^b-2^a)}$ as the union of the complete graph K_{2^a} on X, k complete graphs with holes $K_{2^b}\backslash K_{2^a}$ on the sets $X \bigcup Y_i$ with hole X, and $\binom{k}{2}$ complete bipartite graphs $K_{2^b-2^a,2^b-2^a}$ with bipartite sets Y_i and Y_j, $i \neq j$. By the previous paragraph these graphs taken together decompose into a Q_r-factor and a graph divisible by Q_d. ∎

Now we can show that if d is odd there exists an infinite arithmetic progression of integers n such that Q_d divides K_n.

Theorem 11. *Let d be any odd positive integer, let s be the order of 2 modulo d and let t be the least integer not less than d/s. Then Q_d divides K_n where $n = 2^{st} + k(2^{st+s} - 2^{st})$.*

Proof. We take $a = st$ in Theorem 10. Then $r = 0$ and so only d-cubes are involved in the decomposition. ∎

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Received 5 May 2005
Revised 19 September 2005