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SOME RESULTS ON TOTAL DOMINATION IN DIRECT PRODUCTS OF GRAPHS *

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Abstract

Upper and lower bounds on the total domination number of the direct product of graphs are given. The bounds involve the $\{2\}$-total domination number, the total 2-tuple domination number, and the open packing number of the factors. Using these relationships one exact total domination number is obtained. An infinite family of graphs is constructed showing that the bounds are best possible. The domination number of direct products of graphs is also bounded from below.

Keywords: direct product, total domination, k-tuple domination, open packing, domination.

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1. Introduction

Recently, the total domination number γ_t of the direct product of graphs attracted quite some attention [1, 2, 11]. The primary goal is to exactly determine this graph invariant on direct products. A very nice result of this kind is the main result of Rall from [11] asserting that for any tree T with at least one edge and any graph H without isolated vertices, $\gamma_t(T \times H) = \gamma_t(T)\gamma_t(H)$. Similar result also holds for graphs with equal total domination number and open packing number. In [2] the authors computed the total domination number when one factor is complete and the other factor a cycle, or when both factors are cycles.

Since the exact problem is quite difficult in general, it is also of interest to have good lower and upper bounds on the total domination number of the product in terms of invariants of its factors. Two such lower bounds were proved in [2, 11] and will be restated in the next section. On the other hand, the total domination number of factors can be used to bound the domination number of the product, cf. [1, 11].

In the next section we present definitions and concepts needed in this paper. For more information and details concerning graph domination parameters we refer to [4] and for these invariants studied on graph products see [10]. Then, in Section 3, we propose the following relationship between the total domination number of the direct product and the total $\{2\}$-domination numbers of the factors:

$$\gamma_t(G \times H) \geq \max\{\gamma_t^{\{2\}}(G), \gamma_t^{\{2\}}(H)\}\,.$$

In a special case we also find an upper bound for the total domination number of the product involving the total 2-tuple domination number of the factors. This in particular enables us to alternatively obtain the total domination number of the product of a cycle and a complete graph first computed in [2]. Graphs that attain these bounds are also constructed. Finally, in Section 4, we show how one can adopt our approach to bound the domination number of direct products of graphs in term of the $\{2\}$-domination numbers of the factors.

2. Preliminaries

For a graph $G = (V, E)$ with vertex set V and edge set E, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. By $\delta(G)$ we denote the smallest degree of G, that is, $\delta(G) = \min_v |N(v)|$.

A set $S \subseteq V$ is a *dominating set* if each vertex in $V - S$ is adjacent to at least one vertex of S. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set. Similarly, $S \subseteq V$ is a *total dominating set* if each vertex in V is adjacent to at least one vertex of S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set.

Let $G = (V, E)$ be a graph. For a real-valued function $f\colon V \to \mathbb{R}$ the *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Let $k \geq 1$, then a function $f\colon V \to \{0, 1, \ldots, k\}$ is called a $\{k\}$-*dominating function* if for every $v \in V$, $f(N[v]) \geq k$. The $\{k\}$-*domination number* $\gamma^{\{k\}}(G)$ of G is the minimum weight of a $\{k\}$-dominating function. Similarly, $f\colon V \to \{0, 1, \ldots, k\}$ is called a *total* $\{k\}$-*dominating function* if for every $v \in V$, $f(N(v)) \geq k$. The *total* $\{k\}$-*domination number* $\gamma_t^{\{k\}}(G)$ of G is the minimum weight of a total $\{k\}$-dominating function.

Yet another related concept introduced in [3] (see also [8, 9]) is the following. $S \subseteq V$ is a k-*tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq k$. In other words, either v is in S and has at least $k - 1$ neighbors in S or v is in $V \setminus S$ and has at least k neighbors in S. The k-*tuple domination number* $\gamma^{(\times k)}(G)$ is the minimum cardinality of a k-tuple dominating set of G. Note that $\gamma_t(G) \leq \gamma^{(\times 2)}(G)$. Finally, $S \subseteq V$ is a *total* k-*tuple dominating set* of G if for every vertex $v \in V$, $|N(v) \cap S| \geq k$, that is, v is dominated by at least k neighbors in S. The *total* k-*tuple domination number* $\gamma_t^{(\times k)}$ is the minimum cardinality of a total k-tuple dominating set of G.

The 2-*packing* number $\rho(G)$ of a graph G is the maximum cardinality of a vertex subset X of G such that $N[u] \cap N[v] = \emptyset$ for any different vertices $u, v \in X$. An *open packing* of a graph G is a set S of vertices such that the sets $N(x)$, $x \in S$, are pairwise disjoint. The *open packing number* $\rho^\circ(G)$ is the maximal cardinality of an open packing on G.

Finally, recall that the direct product $G \times H$ is the graph defined by $V(G \times H) = V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent if and only if g_1g_2 and h_1h_2 are edges of G and H, respectively. Let g be a vertex of G, then the subgraph of $G \times H$ induced by $\{g\} \times V(H)$ is called a *fiber* and denoted ${}^gH$. Similarly one defines the fiber G^h for a vertex h of H. Note that if the factors graphs are without loops, then the fibers of their direct product are discrete. Note also that the direct product is commutative and associative; for more information on the direct product see [5, 6].

3. Bounding Total Domination Numbers

Let G and H be graphs with no isolated vertices. Then Rall [11] proved the following lower bound:

$$\gamma_t(G \times H) \geq \max\{\rho^\circ(G)\gamma_t(H), \rho^\circ(H)\gamma_t(G)\}\,, \tag{1}$$

while El-Zahar, Gravier, and Klobučar [2] followed with:

$$\gamma_t(G \times H) \geq \max\left\{\frac{|G|}{\Delta(G)}\gamma_t(H), \frac{|H|}{\Delta(H)}\gamma_t(G)\right\}. \tag{2}$$

None of the bounds (1) and (2) follows from the other. For this sake note that for $n \geq 3$, $\rho^\circ(K_n) = 1$, $\gamma_t(K_n) = 2$, so (1) gives $\gamma_t(K_n \times K_n) \geq 2$ while (2) implies $\gamma_t(K_n \times K_n) \geq 3$. (In fact, $\gamma_t(K_n \times K_n) = 3$ for $n \geq 3$, cf. [1].) On the other hand, for any $n \geq 2$, $\rho^\circ(K_{1,n}) = 2$, $\gamma_t(K_{1,n}) = 2$, so (1) gives $\gamma_t(K_{1,n} \times K_{1,n}) \geq 4$ while (2) only gives $\gamma_t(K_{1,n} \times K_{1,n}) \geq 3$.

We now give another lower bound on the total domination number of direct products.

Theorem 3.1. *For any nontrivial connected graphs G and H we have*

$$\gamma_t(G \times H) \geq \max\left\{\gamma_t^{\{2\}}(G), \gamma_t^{\{2\}}(H)\right\}. \tag{3}$$

Proof. Let S be a minimum total dominating set of $G \times H$. Define an integer function f on $V(G)$ with

$$f(u) = \min\{2,\ |S \cap {}^uH|\ \}\,.$$

We claim that f is a total $\{2\}$-dominating function of G.

Let u be an arbitrary vertex of G and let $V(H) = \{v_1, \ldots, v_n\}$. Recall that H is nontrivial, hence $n \geq 2$. Since S is a total dominating set, there exists a vertex (x, v_i) that dominates (u, v_1). Note that $x \neq u$ and $i \neq 1$. Consider the vertex (u, v_i). It is dominated by some vertex (y, v_j), where $y \neq u$ and $j \neq i$. If $x = y$, then since $i \neq j$ we have $f(x) = 2$, and hence $f(N(u)) \geq 2$. And if $x \neq y$, then $f(x) \geq 1$, $f(y) \geq 1$, and therefore $f(N(u)) \geq 2$ again. Thus f is a total $\{2\}$-dominating function of G with $w(f) \leq |S|$, hence $\gamma_t(G \times H) \geq \gamma_t^{\{2\}}(G)$. By the commutativity of the direct product the inequality follows. ■

To see that the lower bound (3) can be simultaneously better than (1) and (2) consider the following example. For $n \geq 3$, let M_n be the graph obtained from n copies of K_3 such that in each copy one vertex is selected and these vertices are then identified. Then we have $\rho^\circ(M_n) = 1$, $\gamma_t(M_n) = 2$, and $\gamma_t^{\{2\}}(M_n) = 4$. Then (3) gives $\gamma_t(M_n \times M_n) \geq 4$, while (2) implies $\gamma_t(M_n \times M_n) \geq 3$ and (1) $\gamma_t(M_n \times M_n) \geq 2$.

On the other hand, suppose that $\rho^\circ(G) \geq 2$. Then

$$\gamma_t(G \times H) \geq \rho^\circ(G)\gamma_t(H) \geq 2\gamma_t(H) \geq \gamma_t^{\{2\}}(H)\,,$$

hence (3) follows from (1) as soon as $\rho^\circ(G) \geq 2$. It would be nice to have a lower bound that would cover the three above bounds. However the provided examples show that this task might be difficult.

Theorem 3.2. *Let G be a graph with $\delta(G) \geq 2$ and let $n \geq \gamma_t^{(\times 2)}(G)$. Then*

$$\gamma_t(G \times K_n) \leq \gamma_t^{(\times 2)}(G)\,. \tag{4}$$

Proof. Let $S = \{s_1, \ldots, s_k\}$ be a minimum total 2-tuple dominating set of G and let $\{v_1, \ldots, v_n\}$ be the vertex set of K_n. We claim that $T = \{(s_i, v_i) \mid i = 1, \ldots, k\}$ is a minimum total dominating set of $G \times K_n$. Note first that T is well defined since $n \geq \gamma_t^{(\times 2)}(G) = k$. Let (x, v_t) be an arbitrary vertex of $G \times K_n$ and assume that x is dominated by vertices s_i and s_j. Then s_i, s_j and x are pairwise different vertices. Suppose without

loss of generality that $t \neq i$. Then (x, v_t) is dominated by (s_i, v_i), and so T is a total dominating set of $G \times K_n$. We conclude that $\gamma_t(G \times K_n) \leq \gamma_t^{(\times 2)}(G)$. ■

Corollary 3.3 [2]. *For any $m \geq n \geq 3$, $\gamma_t(C_n \times K_m) = n$.*

Proof. Clearly, $\gamma_t^{(\times 2)}(C_n) = n$, hence $\gamma_t(C_n \times K_m) \leq n$ by Theorem 3.2. On the other hand, the lower bound easily follows from (2). ■

Using inequality (4) we next construct examples where the lower bound (2) is optimal. Let G_n be the graph obtained from the complete graph K_n by adding a vertex x_e for each edge $e = uv$ of K_n, and joining x_e with u and v. (See Figure 1 where G_4 is drawn.)

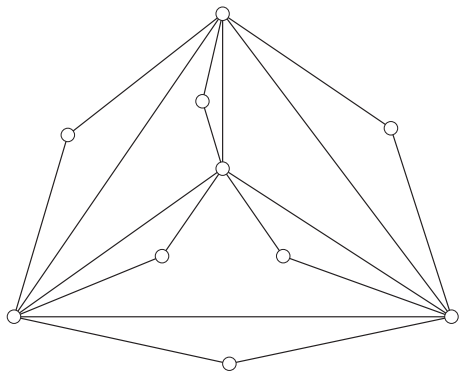

Figure 1. Graph G_4

We claim that for $n \geq 3$, $\gamma_t(G_n \times K_n) = n$. It is easy to check that $\gamma_t^{(\times 2)}(G_n) = n$, hence by (4), $\gamma_t(G_n \times K_n) \leq n$. On the other hand, (2) implies that for any $n \geq 3$,

$$\gamma_t(G_n \times K_n) \geq \frac{|K_n|}{\Delta(K_n)} \gamma_t(G_n) = n\,.$$

We conclude this section with one more lower bound. We don't know whether (5) eventually follows from (1). However, for a given graph G it might be easier to evaluate $\gamma_t^{(\times 2)}(G)$ than $\rho^\circ(G)$ and $\gamma_t(G)$. Moreover, the below proof technique is somehow nonstandard and might be useful in other situations.

Theorem 3.4. *Let G and H be graphs. If $\delta(G) \geq 2$ and $\Delta(G) < \gamma_t(H)$, then*

$$\gamma_t(G \times H) \geq \gamma_t^{(\times 2)}(G)\,. \tag{5}$$

Proof. Let S be a minimum total dominating set of $G \times H$. We are going to construct a total 2-tuple dominating set X of G as follows.

Let $V(H) = \{v_1, \ldots, v_n\}$. For an arbitrary vertex u of G we proceed as follows. Let $(x, v_i) \in S$ be a vertex that dominates (u, v_1), where $x \neq u$ and $i \neq 1$. Consider the vertex (u, v_i). It is dominated by some vertex (y, v_j), where $y \neq u$ and $j \neq i$. If $x \neq y$, then set $x, y \in X$. Suppose $x = y$. If some vertex (u, v_k) with $k \neq i, j$ is dominated by a vertex (z, v_ℓ), where $z \neq x$, then we put $x, z \in X$. So assume that all vertices of ${}^uH$ are dominated by vertices of ${}^xH$. Then select an arbitrary neighbor $w \neq x$ of u and set $x, w \in X$. (Note that w exists since $\delta(G) \geq 2$.)

Clearly, X is a total 2-tuple dominating set. We claim that $|X| \leq |S|$. For this sake we will construct an injection $f : X \to S$ in the following way. Let $x \in X$ be a vertex for which we have $|{}^xH \cap S| \geq 1$ and let i be the smallest index for which $(x, v_i) \in S$. Then set $f(x) = (x, v_i)$. Let next $x \in X$ but ${}^xH \cap S = \emptyset$. Then there exists a neighbor y_x of x such that y_x is adjacent to y'_x such that $|{}^{y'_x}H \cap S| \geq \gamma_t(H)$. Let i be the smallest integer such that $(y'_x, v_i) \in S$ and (y'_x, v_i) is an image of no vertex of X under the map f. Then set $f(x) = (y'_x, v_i)$. We need to show that f is well-defined since then f will be injective. Suppose on the contrary that for some vertex x such an assignment is not possible. In such a case x is adjacent to y_x such that y_x is adjacent to y'_x. Moreover, there exist vertices $x_2, \ldots, x_{\gamma_t(H)}$ adjacent to y_{x_i}, respectively, and any y_{x_i}, $2 \leq i \leq \gamma_t(H)$, is adjacent to y'_x. It follows that the degree of y'_x is at least $\gamma_t(H)$, which is not possible by the theorem assumption. ■

4. A Remark on Domination in Direct Products

In this concluding section we give a lower bound for the usual domination number of direct products of graphs. The bound is given in the same spirit as our previous bounds for the total domination number.

Theorem 4.1. *For any nontrivial connected graphs G and H,*

$$\gamma(G \times H) \geq \max\{\gamma^{\{2\}}(G), \gamma^{\{2\}}(H)\}\,. \tag{6}$$

Proof. Let S be a minimum dominating set of $G \times H$. Define an integer function f on $V(G)$ with

$$f(u) = \min\{2,\ |S \cap {}^uH|\ \}\,.$$

We claim that f is a $\{2\}$-dominating function of G. Let u be an arbitrary vertex of G and let $V(H) = \{v_1, \ldots, v_n\}$. Recall that H is nontrivial, hence $n \geq 2$.

Case 1. For $f(u) = 2$ there is nothing to be shown.

Case 2. Suppose $f(u) = 1$. We may assume without loss of generality that $(u, v_1) \in S$. Then $(u, v_i) \notin S$, for $i \geq 2$. Since S is a dominating set, (u, v_2) is dominated by some vertex $(x, v_j) \in S$, where $x \neq u$. Hence $f(x) \geq 1$ and x is adjacent to u. It follows that $f(N[u]) \geq 2$.

Case 3. Suppose $f(u) = 0$. Then $(u, v_1) \notin S$ and $(u, v_2) \notin S$, therefore there exist vertices x_1, x_2 of G and v_i, v_j of H such that (x_1, v_i) dominates (u, v_1) and (x_2, v_j) dominates (u, v_2). If $x_1 \neq x_2$, then $f(x_1) \geq 1$ and $f(x_2) \geq 1$, hence $f(N[u]) \geq 2$. Suppose $x_1 = x_2$. In the case that $i \neq j$ we infer that $f(x_1) = 2$ and consequently $f(N[u]) \geq 2$. The final case to consider is when $(x_1, v_i) = (x_2, v_j)$, that is, when (u, v_1) and (u, v_2) are both dominated by (x_1, v_i). Observe now that $i \neq 1, 2$. Then the vertex (u, v_i) must be dominated by some vertex (y, v_s) where $s \neq i$. If $y = x_1$, then $f(x_1) = 2$ and so $f(N[u]) \geq 2$. And if $y \neq x_1$, then $f(x_1) \geq 1$ and $f(y) \geq 1$, hence we conclude again that $f(N[u]) \geq 2$.

Thus we have proved that f is a $\{2\}$-dominating function of G. Since $w(f) \leq |S|$, we conclude that $\gamma(G \times H) \geq \gamma^{\{2\}}(G)$. By the commutativity of the direct product the result follows. ■

In [11] Rall proved that for any graphs G and H with no isolated vertices,

$$\gamma(G \times H) \geq \max\{\rho(G)\gamma_t(H), \rho(H)\gamma_t(G)\}\,. \tag{7}$$

If $\rho(G) \geq 2$, then we have $\gamma(G \times H) \geq 2\gamma_t(H) \geq \gamma^{\{2\}}(H)$, hence (6) follows from (7). On the other hand, (6) can give a better estimation for some "small" graphs. Consider, for instance, the Hajós graph H, see Figure 2.

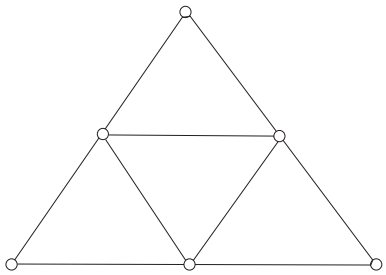

Figure 2. Hajós graph

Then $\rho(H) = 1$ (as H is of diameter 2), $\gamma_t(H) = 2$, but $\gamma^{\{2\}}(H) = 3$.

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