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THE USE OF EULER'S FORMULA IN $(3,1)^*$-LIST COLORING

Yongqiang Zhao

Department of Mathematics
Shijiazhuang College
Shijiazhuang 050801, P.R. China

e-mail: yqzhao1970@yahoo.com

and

Wenjie He

Applied Mathematics Institute
Hebei University of Technology
Tianjin 300130, P.R. China

e-mail: he_wenjie@yahoo.com

Abstract

A graph G is called $(k,d)^*$-choosable if, for every list assignment L satisfying $|L(v)| = k$ for all $v \in V(G)$, there is an L-coloring of G such that each vertex of G has at most d neighbors colored with the same color as itself. Ko-Wei Lih et al. used the way of discharging to prove that every planar graph without 4-cycles and i-cycles for some $i \in \{5,6,7\}$ is $(3,1)^*$-choosable. In this paper, we show that if G is 2-connected, we may just use Euler's formula and the graph's structural properties to prove these results. Furthermore, for 2-connected planar graph G, we use the same way to prove that, if G has no 4-cycles, and the number of 5-cycles contained in G is at most $11 + \lfloor \sum_{i \geq 5} \frac{5i-24}{4} |V_i| \rfloor$, then G is $(3,1)^*$-choosable; if G has no 5-cycles, and any planar embedding of G does not contain any adjacent 3-faces and adjacent 4-faces, then G is $(3,1)^*$-choosable.

Keywords: list improper coloring, $(L,d)^*$-coloring, $(m,d)^*$-choosable, Euler's formula.

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1. Introduction

All graphs considered in this paper are finite simple graphs. For a plane graph G, we denote its vertex set, edge set, face set, and minimum degree by $V(G)$, $E(G)$, $F(G)$ and $\delta(G)$, respectively. For $x \in V(G) \cup F(G)$, let $d(x)$ denote the degree of x in G. A vertex (or face) of degree k is called a k-vertex (or k-face). Let $N(u)$ denote the set of neighbors of a vertex u in G. Two faces of a plane graph are said to be adjacent if they have at least one common boundary edge. A vertex v and a face f are said to be incident if v lies on the boundary of f. For $x \in V(G) \cup F(G)$, we use $F_k(x)$ to denote the set of all k-faces that are incident or adjacent to x, and $V_k(x)$ to denote the set of all k-vertices that are incident or adjacent to x. For $f \in F(G)$, we write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in the clockwise order. A 3-face $[u_1u_2u_3]$ is called an (m_1, m_2, m_3)-face if $d(u_i) = m_i$ for $i = 1, 2, 3$.

Let $m > 1$ be an integer. A graph G is $(m, d)^*$-colorable, if the vertices of G can be colored with m colors so that each vertex has at most d neighbors colored with the same color as itself. An $(m, 0)^*$-coloring is an ordinary proper m-coloring. A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An $(L, d)^*$-coloring is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors colored with $\phi(v)$. A graph is $(m, d)^*$-choosable, if there exist an $(L, d)^*$-coloring for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. Obviously, $(m, 0)^*$-choosability is the ordinary m-choosability introduced by Erdös, Rubin and Taylor [2], and independently by Vizing [8].

The notion of list improper coloring was introduced independently by Škrekovski [5] and Eaton and Hull [1]. This class of problems has been studied widely [1, 3, 4, 5, 6, 7, 9] since its introduction. Ko-Wei Lih *et al.* [3] used the way of discharging to prove that every planar graph without 4-cycles and i-cycles for some $i \in \{5, 6, 7\}$ is $(3, 1)^*$-choosable. In this paper, we show that if G is 2-connected, we may just use Euler's formula and the graph's structural properties to prove these results. Furthermore, for 2-connected planar graph G, we use the same way to prove that, if G has no 4-cycles, and the number of 5-cycles contained in G is at most $11 + \lfloor \sum_{i \geq 5} \frac{5i-24}{4} |V_i| \rfloor$, then G is $(3, 1)^*$-choosable; if G has no 5-cycles, and any planar embedding of G does not contain any adjacent 3-faces and adjacent 4-faces, then G is $(3, 1)^*$-choosable. In Section 2 we provide some lemmata, and in Section 3 we prove our main results.

2. Lemmata

We first cite a useful lemma from [3].

Lemma 1 (Lih *et al.* [3]). *Let G be a graph and $d \geq 1$ an integer. If G is not $(k,d)^*$-choosable but every subgraph of G with fewer vertices is, then the following facts hold*:

1. $\delta(G) \geq k$;
2. *If $u \in V(G)$ is a k-vertex and v is a neighbor of u, then $d(v) \geq k + d$.*

The following corollary is obvious.

Corollary 2 (Lih *et al.* [3]). *If G is a plane graph and is not $(3,1)^*$-choosable with the fewest vertices, then the following facts hold*:

1. $\delta(G) \geq 3$;
2. *G does not contain two adjacent 3-vertices*;
3. *G does not contain a $(3,4,4)$-face.*

Corollary 3. *Let G be a 2-connected plane graph without adjacent 3-faces. If G is not $(3,1)^*$-choosable with the fewest vertices, then*

$$|V_3(f)| + |F_3(f)| \leq d(f) \tag{1}$$

for any $f \in F(G)$.

Proof. Suppose that graph G is not $(3,1)^*$-choosable with the fewest vertices. Note that if G is 2-connected, then the boundary of every face of G forms a cycle, and every vertex $v \in V(G)$ is incident to exactly $d(v)$ distinct faces. Let f be a face of G. If $d(f)$=3, then the result is obvious by 2 of Corollary 2 and the assumption. So we suppose $d(f) \geq 4$. If all the faces adjacent to f are 3-faces, then $|F_3(f)| = d(f)$, and $|V_3(f)| = 0$. Otherwise, there will exist two adjacent 3-faces, which contradicts the assumption. By 2 of Corollary 2, it is easy to see that whenever $|F_3(f)|$ lessens 1, $|V_3(f)|$ increases by at most 1. So (1) holds for any $f \in F(G)$. ■

Given a plane graph G, let V_i (F_i, respectively) be the set of all i-vertices (i-faces, respectively) of G, V_3^1 the set of all 3-vertices of G that are not incident to any 3-face, and $V_3^2 = V_3 \setminus V_3^1$.

Lemma 4. *Let G be a 2-connected plane graph that is not $(3,1)^*$-choosable with the fewest vertices.*

1. *If G does not contain 4-cycles, then*

$$3|V_3^1| + 2|V_3^2| + 6|F_3| \le 2|E(G)|. \tag{2}$$

2. *If G contains neither 4-cycles nor 6-cycles, then*

$$3|V_3^1| + 2|V_3^2| + 6|F_3| + 3|F_5| \le 2|E(G)|. \tag{3}$$

3. *If G contains neither 4-cycles nor 7-cycles, then*

$$3|V_3^1| + 2|V_3^2| + 6|F_3| + 2|F_5| + 3|F_6| \le 2|E(G)|. \tag{4}$$

Proof. *Case* 1. Suppose that G does not contain 4-cycles, then G contains neither 4-faces nor adjacent 3-faces. So by (1),

$$\sum_{d(f)\ge 5} |V_3(f)| + \sum_{d(f)\ge 5} |F_3(f)| \le \sum_{d(f)\ge 5} d(f).$$

Since $\sum_{d(f)\ge 5} |V_3(f)| = 3|V_3^1| + 2|V_3^2|$ and $\sum_{d(f)\ge 5} |F_3(f)| = 3|F_3|$, then

$$3|V_3^1| + 2|V_3^2| + 3|F_3| \le \sum_{d(f)\ge 5} d(f),$$

or

$$3|V_3^1| + 2|V_3^2| + 3|F_3| + \sum_{d(f)=3} d(f) \le \sum_{d(f)\ge 5} d(f) + \sum_{d(f)=3} d(f).$$

Since $\sum_{d(f)=3} d(f) = 3|F_3|$ and that G does not contain any 4-face, then

$$3|V_3^1| + 2|V_3^2| + 6|F_3| \le 2E(G).$$

Case 2. Suppose that G is a plane graph without 4-cycles and 6-cycles, then any 3-face is not adjacent to a 5-face in G. So

$$\sum_{d(f)=5} |V_3(f)| + \sum_{d(f)=5} |F_3(f)| = \sum_{d(f)=5} |V_3(f)| \le 2|F_5|.$$

By (1),

$$\sum_{d(f)\geq 7}|V_3(f)|+\sum_{d(f)\geq 7}|F_3(f)|\leq \sum_{d(f)\geq 7}d(f).$$

Combining the two equalities above,

$$\sum_{d(f)\geq 7}|V_3(f)|+\sum_{d(f)=5}|V_3(f)|+\sum_{d(f)\geq 7}|F_3(f)|+\sum_{d(f)=5}|F_3(f)|\leq \sum_{d(f)\geq 7}d(f)+2|F_5|.$$

By the same cases used in the proof of Case 1, we have

$$3|V_3^1|+2|V_3^2|+3|F_3|\leq \sum_{d(f)\geq 7}d(f)+2|F_5|,$$

and

$$3|V_3^1|+2|V_3^2|+6|F_3|+3|F_5|\leq \sum_{d(f)\geq 7}d(f)+5|F_5|+3|F_3|=2|E(G)|.$$

Case 3. Suppose that G contains neither 4-cycles nor 7-cycles, then any 5-face is adjacent to at most one 3-face in G. So

$$\sum_{d(f)=5}|V_3(f)|+\sum_{d(f)=5}|F_3(f)|\leq 3|F_5|,$$

and

$$\sum_{d(f)=6}|V_3(f)|+\sum_{d(f)=6}|F_3(f)|\leq 3|F_6|.$$

By (1),

$$\sum_{d(f)\geq 8}|V_3(f)|+\sum_{d(f)\geq 8}|F_3(f)|\leq \sum_{d(f)\geq 8}d(f).$$

Combining these three equalities above, we have

$$\sum_{d(f)\geq 5}|V_3(f)|+\sum_{d(f)\geq 5}|F_3(f)|\leq \sum_{d(f)\geq 8}d(f)+3|F_5|+3|F_6|.$$

Furthermore,

$$3|V_3^1| + 2|V_3^2| + 3|F_3| \leq \sum_{d(f)\geq 8} d(f) + 3|F_5| + 3|F_6|.$$

So

$$\begin{aligned} &3|V_3^1| + 2|V_3^2| + 6|F_3| + 2|F_5| + 3|F_6| \\ &\leq \sum_{d(f)\geq 8} d(f) + 3|F_3| + 5|F_5| + 6|F_6| = 2|E(G)|. \end{aligned}$$

The proof is complete. ■

Lemma 5. *If G is a plane graph without adjacent 3-faces and is not $(3,1)^*$-choosable with the fewest vertices, then*

$$|V_3^2| \leq \frac{1}{2}\sum_{i\geq 5} i|V_i|. \tag{5}$$

Proof. By 2 and 3 of Corollary 2, if v is a 3-vertex of G incident to a 3-face, then v must be adjacent to a vertex whose degree is at least 5. So for a vertex $v \in V(G)$, $d(v) \geq 5$, let

$$V_3^*(v) = \{u | u \in N(v) \cap V_3^2, \ \text{ and } \ uv \ \text{ is a triangle's edge}\},$$

then $V_3^2 = \bigcup_{d(v)\geq 5} V_3^*(v)$. Since G does not contain adjacent 3-faces and adjacent 3-vertices, then $|V_3^*(v)| \leq \frac{1}{2}d(v)$. Therefore

$$|V_3^2| \leq \sum_{d(v)\geq 5} |V_3^*(v)| \leq \frac{1}{2}\sum_{d(v)\geq 5} d(v) = \frac{1}{2}\sum_{i\geq 5} i|V_i|.$$

■

3. Main Results

In this section we just use Euler's formula and the lemmata provided in the previous section to prove the theorems.

Theorem 6 (Lih *et al.* [3])**.** *If G is a 2-connected planar graph without 4-cycles and i-cycles for some $i \in \{5,6,7\}$, then G is $(3,1)^*$-choosable.*

Proof. Suppose that G is a counterexample with the fewest vertices, and we consider the planar embeddings of G. By Euler's formula

$$|V(G)| + |F(G)| = |E(G)| + 2$$

or

$$\sum_{i\geq 3} |V_i| + \sum_{i\geq 3} |F_i| = |E(G)| + 2,$$

we have

$$\frac{1}{4}|V_3| + \frac{1}{4}\sum_{i\geq 3} i|V_i| - \frac{1}{4}\sum_{i\geq 5}(i-4)|V_i| + \frac{3}{6}|F_3|$$

$$+\frac{2}{6}|F_4| + \frac{1}{6}|F_5| + \frac{1}{6}\sum_{i\geq 3} i|F_i| \geq |E(G)| + 2,$$

i.e.,

$$\frac{1}{4}|V_3| + \frac{2|E(G)|}{4} - \frac{1}{4}\sum_{i\geq 5}(i-4)|V_i| + \frac{1}{2}|F_3|$$

$$+\frac{1}{3}|F_4| + \frac{1}{6}|F_5| + \frac{2|E(G)|}{6} \geq |E(G)| + 2$$

or

$$3|V_3| - 3\sum_{i\geq 5}(i-4)|V_i| + 6|F_3| + 4|F_4| + 2|F_5| \geq 2|E(G)| + 24. \tag{6}$$

Case 1. G has no 4-cycles and 5-cycles. By (2) and (6),

$$3|V_3| - 3\sum_{i\geq 5}(i-4)|V_i| + 6|F_3| \geq 3|V_3^1| + 2|V_3^2| + 6|F_3| + 24,$$

i.e.,

$$|V_3^2| - 3\sum_{i\geq 5}(i-4)|V_i| \geq 24.$$

By (5),

$$\frac{1}{2}\sum_{i\geq 5} i|V_i| - 3\sum_{i\geq 5}(i-4)|V_i| \geq 24,$$

i.e.,

$$\sum_{i\geq 5}\left(12-\frac{5}{2}i\right)|V_i| \geq 24,$$

which is impossible, since $12-\frac{5}{2}i<0$ when $i\geq 5$.

Case 2. G has no 4-cycles and 6-cycles. By (3) and (6),

$$3|V_3|-3\sum_{i\geq 5}(i-4)|V_i|+6|F_3|+2|F_5| \geq 3|V_3^1|+2|V_3^2|+6|F_3|+3|F_5|+24,$$

i.e.,

$$|V_3^2|-3\sum_{i\geq 5}(i-4)|V_i| \geq |F_5|+24.$$

By (5),

$$\frac{1}{2}\sum_{i\geq 5} i|V_i|-3\sum_{i\geq 5}(i-4)|V_i| \geq |F_5|+24,$$

i.e.,

$$\sum_{i\geq 5}\left(12-\frac{5}{2}i\right)|V_i| \geq |F_5|+24,$$

which is impossible.

Case 3. G has no 4-cycles and 7-cycles. By (4) and (6),

$$3|V_3|-3\sum_{i\geq 5}(i-4)|V_i|+6|F_3|+2|F_5|$$

$$\geq 3|V_3^1|+2|V_3^2|+6|F_3|+2|F_5|+3|F_6|+24,$$

i.e.,

$$|V_3^2|-3\sum_{i\geq 5}(i-4)|V_i| \geq 3|F_6|+24.$$

By (5),

$$\frac{1}{2}\sum_{i\geq 5} i|V_i|-3\sum_{i\geq 5}(i-4)|V_i| \geq 3|F_6|+24,$$

i.e.,

$$\sum_{i\geq 5}\left(12-\frac{5}{2}i\right)|V_i| \geq 3|F_6|+24,$$

which is impossible. The proof is complete. ∎

Theorem 7. *Let G be a 2-connected planar graph.*

1. *If G has no 4-cycles, and the number of 5-cycles contained in G is at most $11 + \lfloor \sum_{i\geq 5} \frac{5i-24}{4}|V_i| \rfloor$, then G is $(3,1)^*$-choosable.*
2. *If G has no 5-cycles, and any planar embedding of G does not contain any adjacent 3-faces and adjacent 4-faces, then G is $(3,1)^*$-choosable.*

Proof. 1. Suppose that G is a 2-connected planar graph without 4-cycles and is not $(3,1)^*$-choosable with the fewest vertices. We consider the planar embeddings of G. By (2) and (6),

$$3|V_3| - 3\sum_{i\geq 5}(i-4)|V_i| + 6|F_3| + 2|F_5| \geq 3|V_3^1| + 2|V_3^2| + 6|F_3| + 24,$$

i.e.,

$$|V_3^2| - 3\sum_{i\geq 5}(i-4)|V_i| + 2|F_5| \geq 24.$$

By (5),

$$\frac{1}{2}\sum_{i\geq 5} i|V_i| - 3\sum_{i\geq 5}(i-4)|V_i| + 2|F_5| \geq 24$$

or

$$|F_5| \geq 12 + \sum_{i\geq 5}\frac{5i-24}{4}|V_i|,$$

a contradiction.

2. Suppose that G is a counterexample with the fewest vertices, and we consider the planar embeddings of G. Since G has no 5-cycles, then there is no 3-face adjacent to a 4-face in any planar embedding of G. By 2 of Corollary 2, G does not contain adjacent 3-vertices. So for any $f \in F(G)$, we have

$$|V_3(f)| + |F_3(f)| + |F_4(f)| \leq d(f). \tag{7}$$

When $d(f) = 3, 4$, (7) trivially holds. So we suppose $d(f) \geq 6$. If $F_i(f) = \phi$ for $i \geq 5$, then $|F_3(f)| + |F_4(f)| = d(f)$ and $|V_3(f)| = 0$ by the conditions of the theorem. It is easy to see that whenever $|F_3(f)| + |F_4(f)|$ lessens 1, $|V_3(f)|$ increases by at most 1. So (7) holds for any $f \in F(G)$.

By (7),

$$\sum_{d(f)\geq 4}|V_3(f)| + \sum_{d(f)\geq 4}|F_3(f)| + \sum_{d(f)\geq 4}|F_4(f)| \leq \sum_{d(f)\geq 4} d(f)$$

or

$$3|V_3^1| + 2|V_3^2| + 3|F_3| + 4|F_4| \leq \sum_{i\geq 4} i|F_i|.$$

Therefore

$$3|V_3^1| + 2|V_3^2| + 6|F_3| + 4|F_4| \leq 2|E(G)|. \tag{8}$$

By (6) and (8),

$$3|V_3| - 3\sum_{i\geq 5}(i-4)|V_i| + 6|F_3| + 4|F_4| \geq 3|V_3^1| + 2|V_3^2| + 6|F_3| + 4|F_4| + 24,$$

i.e.,

$$|V_3^2| - 3\sum_{i\geq 5}(i-4)|V_i| \geq 24.$$

By (5),

$$\frac{1}{2}\sum_{i\geq 5} i|V_i| - 3\sum_{i\geq 5}(i-4)|V_i| \geq 24.$$

i.e.,

$$\sum_{i\geq 5}\left(12 - \frac{5}{2}i\right)|V_i| \geq 24,$$

which is impossible, since $12 - \frac{5}{2}i < 0$ when $i \geq 5$. ■

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