# DEFINING SETS IN (PROPER) VERTEX COLORINGS OF THE CARTESIAN PRODUCT OF A CYCLE WITH A COMPLETE GRAPH 

D. Ali Mojdeh<br>Department of Mathematics<br>University of Mazandaran<br>Babolsar, IRAN, P.O. Box 47416-1467<br>e-mail: dmojdeh@umz.ac.ir


#### Abstract

In a given graph $G=(V, E)$, a set of vertices $S$ with an assignment of colors to them is said to be a defining set of the vertex coloring of $G$, if there exists a unique extension of the colors of $S$ to a $c \geq \chi(G)$ coloring of the vertices of $G$. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, c)$.

The $d\left(G=C_{m} \times K_{n}, \chi(G)\right)$ has been studied. In this note we show that the exact value of defining number $d\left(G=C_{m} \times K_{n}, c\right)$ with $c>\chi(G)$, where $n \geq 2$ and $m \geq 3$, unless the defining number $d\left(K_{3} \times C_{2 r}, 4\right)$, which is given an upper and lower bounds for this defining number. Also some bounds of defining number are introduced.


Keywords: graph coloring, defining set, cartesian product.
2000 Mathematics Subject Classification: 05C15, 05C38.

## 1. Introduction

A c-coloring (proper c-coloring) of a graph $G$ is an assignment of $c$ different colors to the vertices of $G$, such that no two adjacent vertices receive the same color. The vertex chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number $c$, for which there exists a $c$-coloring for $G$. The maximum degree of the vertices in $G$ is $\Delta(G)$ and the minimum degree is $\delta(G)$ and $G$ is regular if $\Delta(G)=\delta(G)$. It is $k$-regular graph if the common
degree is $k$ (see [9]). In a given graph $G=(V, E)$, a set of vertices $S$ with an assignment of colors to them is said to be a defining set of the vertex coloring of $G$, if there exists a unique extension of the colors of $S$ to a $c \geq \chi(G)$ coloring of the vertices of $G$. A defining set with the minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, c)$. We will use standard notations such as $K_{n}$ for the complete graph on $n$ vertices, $C_{m}$ for the cycle of size $m$ and $G \times H$ for cartesian product of $G$ and $H$. There are some papers on defining set of graphs, especially $d\left(K_{n} \times K_{n}, \chi\right)$ (the critical set of Latin squares of order $n), d\left(C_{m} \times K_{n}, \chi\right), d(G, \chi=k)$ where $G$ is a $k$-regular graph and defining set on block designs. The interested reader may see $[1,4,5,7,8]$ and their references.
The following results can be found in [3]:
(1) $d\left(C_{m} \times K_{3}, \chi\right)=\left\lfloor\frac{m}{2}\right\rfloor+1$,
(2) $m \leq d\left(C_{m} \times K_{4}, \chi\right) \leq m+1$,
(3) $d\left(C_{m} \times K_{5}, \chi\right)=2 m$ for even $m$ and $2 m \leq d\left(C_{m} \times K_{5}, \chi\right) \leq 2 m+1$ for odd $m$.
The following results can be found in [7]:
(4) $d\left(C_{m} \times K_{5}, \chi\right)=2 m$, for odd $m(\geq 5)$,
(5) $d\left(C_{m} \times K_{4}, \chi\right)=m+1$.

The following results can be found in [6]:
(6) $d\left(C_{m} \times K_{n}, \chi\right)=m(n-3)$ for $n \geq 6$,
(7) $d\left(C_{2 n+1} \times K_{2}, \chi\right)=n+1$.

The followings are useful.
Definition A [2]. A graph $G$ with $n$ vertices, is called a uniquely 2 -list colorable graph, if there exists $S_{1}, S_{2}, \cdots S_{n}$, a list of colors on its vertices, each of size 2 , such that there is a unique coloring for $G$ from this list of colors.

Theorem B [2]. A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.

Let $G$ be a $k$-regular graph and vertex colored with $k$ colors. Let $C$ be a cycle in $G$, then each vertex of $C$ has at least two choice for coloring, in other
words $C$ is at least 2-list vertex colorable, if all vertices of $V(G) \backslash V(C)$ have been already colored. So by Theorem B the cycle $C$ is not uniquely 2-list colorable. Now we have

Lemma C [6]. If $G$ is $k$-regular graph and which is colored with $k$ colors, then every cycle in $G$ has a vertex in defining set of $G$.

If $G=C_{m} \times K_{n}$ then, each subgraph $K_{n}$ of $G$ is said to be a row and each subgraph $C_{m}$ of $G$ is said to be a column. If $G=K_{n} \times C_{m}$ then, each subgraph $K_{n}$ of $G$ is said to be a column and each subgraph $C_{m}$ of $G$ is said to be a row.

It is well known that $\chi\left(C_{m} \times K_{n}\right)=\chi\left(K_{n} \times C_{m}\right)=n$ for $n>2$ or for $n=2$ and even $m$. Also $\chi\left(C_{2 r+1} \times K_{2}\right)=\chi\left(K_{2} \times C_{2 r+1}\right)=3$.

$$
\text { 2. } d\left(C_{m} \times K_{n}, n+i\right)
$$

In this section we derive $d\left(C_{m} \times K_{n}, n+i\right)$ for $n, m \geq 4$ and $i \geq 0$. We start with the following lemma.

Lemma 2.1. If $G=C_{m} \times K_{n}$ is colored with $n+i$ colors for $0 \leq i \leq 3$, then for each row, there exist at least, $n+i-3$ vertices in defining set.

Proof. Assume that, there exists a row for which the defining set contains $k<n+i-3$ vertices and all other rows are completely colored. The induced subgraph of the non colored vertices of this row is a complete graph and cannot be uniquely colored by Theorem B.

In the following arrays the non indexed labels denote the colors of the vertices in the defining set of the graph $C_{m} \times K_{n}$, the indexed labels denote the colors of the vertices out of defining set and the indices denote the ordering of the coloring of these vertices.

Theorem 2.1. For $n, m \geq 4, d\left(C_{m} \times K_{n}, n+1\right)=m(n-2)$.
Proof. Let $G=C_{m} \times K_{n}$. From Lemma 2.1 we obtain $d(G, n+1) \geq$ $m(n-2)$. To show equality we give a defining set $S$ of size $m(n-2)$ as in following arrays.
(1) For $m \geq 4$ and $n=4$, consider the arrays

$$
\begin{aligned}
& {\left[\begin{array}{llll}
1 & 2 & 4_{1} & 5_{2} \\
2_{4} & 4_{3} & 5 & 3 \\
4 & 1 & 2_{5} & 5_{6} \\
2_{8} & 5_{7} & 3 & 4
\end{array}\right],\left[\begin{array}{llll}
1 & 2 & 4_{1} & 5_{2} \\
2_{4} & 4_{3} & 5 & 3 \\
3_{5} & 1 & 4_{6} & 5 \\
4 & 3_{7} & 2 & 1_{8} \\
2_{9} & 5 & 3 & 4_{10}
\end{array}\right]} \\
& {\left[\begin{array}{llll}
1 & 2 & 4_{1} & 5_{2} \\
2_{4} & 4_{3} & 5 & 3 \\
4 & 1 & 3_{6} & 2_{5} \\
3_{7} & 2_{8} & 1 & 5 \\
2 & 4 & 59 & 3_{10} \\
4_{12} & 5_{11} & 3 & 1
\end{array}\right] \text { and }\left[\begin{array}{llll}
1 & 2 & 4_{1} & 5_{2} \\
2_{4} & 4_{3} & 5 & 3 \\
5_{6} & 1 & 3_{5} & 4 \\
3 & 5_{7} & 2 & 1_{8} \\
5_{10} & 4 & 1_{9} & 3 \\
2 & 3_{12} & 5 & 1_{11} \\
5_{13} & 1_{14} & 3 & 4
\end{array}\right]}
\end{aligned}
$$

for $C_{4} \times K_{4}, C_{5} \times K_{4}, C_{6} \times K_{4}$ and $C_{7} \times K_{4}$ respectively with $5=4+1$ colors.
(2) For $m \geq 4$ and $n=5$, consider the arrays

$$
\begin{aligned}
& {\left[\begin{array}{lllll}
1 & 2 & 3 & 6_{2} & 4_{1} \\
2_{4} & 3_{3} & 4 & 5 & 6 \\
3 & 1 & 2 & 6_{6} & 4_{5} \\
2_{8} & 3_{7} & 6 & 4 & 5
\end{array}\right],\left[\begin{array}{lllll}
1 & 2 & 3 & 4_{1} & 5 \\
3_{3} & 1_{4} & 4 & 5 & 6 \\
2 & 4_{5} & 5 & 3 & 1_{6} \\
5_{8} & 6 & 1_{7} & 4 & 3 \\
3_{10} & 1 & 2 & 6 & 4_{9}
\end{array}\right],} \\
& {\left[\begin{array}{lllll}
1 & 2 & 3 & 4_{2} & 51 \\
2_{4} & 3_{3} & 4 & 5 & 6 \\
3 & 1 & 2 & 6_{5} & 4_{6} \\
1_{8} & 2_{7} & 6 & 4 & 5 \\
2 & 3 & 1 & 59 & 6_{10} \\
3_{12} & 1_{11} & 5 & 6 & 4
\end{array}\right] \text { and }\left[\begin{array}{lllll}
1 & 2 & 3 & 6_{2} & 4_{1} \\
33 & 1_{4} & 4 & 5 & 6 \\
2 & 4_{5} & 5 & 3 & 1_{6} \\
58 & 6 & 1_{7} & 4 & 3 \\
4_{9} & 1 & 2 & 6 & 510 \\
3 & 4_{11} & 5 & 1_{12} & 2 \\
4_{13} & 6 & 2 & 3_{14} & 5
\end{array}\right]}
\end{aligned}
$$

for $C_{4} \times K_{5}, C_{5} \times K_{5}, C_{6} \times K_{5}$ and $C_{7} \times K_{5}$ respectively with $6=5+1$ colors.
(3) For $m \geq 4$ and $n \geq 6$, consider the following arrays,

$$
\begin{aligned}
& {\left[\begin{array}{llllllllllll}
1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_{2} & (n-1)_{1} \\
2_{4} & 3_{3} & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\
4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_{5} & 2 & (n-1)_{6} \\
5 & 4_{8} & 6 & 7 & 8 & \cdots & n-1 & 2 & n+1 & 3 & n
\end{array}\right]} \\
& {\left[\begin{array}{llllllllllll}
1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_{2} & (n-1)_{1} \\
2_{4} & & 3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\
4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_{5} & 2 & (n-1)_{6} \\
2_{8} & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_{7} & 3 \\
(n+1)_{9} & 5 & 6 & 7 & 8 & \cdots & n-1 & 2_{10} & 3 & 4 & n \\
1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_{2} & (n-1)_{1} \\
2_{4} & 3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\
4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_{5} & 2 & (n-1)_{6} \\
2_{8} & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5 & 3 \\
5 & 1 & 6 & 7 & 8 & \cdots & n-1 & 2 & n_{9} & 4 & (n+1)_{10} \\
2_{12} & 4_{11} & 5 & 6 & 7 & \cdots & n-2 & n+1 & 3 & n-1 & n
\end{array}\right]}
\end{aligned}
$$

and

$$
\left[\begin{array}{lllllllllll}
1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_{2} & (n-1)_{1} \\
2_{4} & 3_{3} & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\
4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_{5} & 2 & (n-1)_{6} \\
2_{8} & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_{7} & 3 \\
1_{9} & 5 & 6 & 7 & 8 & \cdots & n-1 & 3 & 2_{10} & 4 & n \\
n+1 & 6_{11} & 7 & 8 & 9 & \cdots & n & 2 & 3 & 5_{12} & 1 \\
3_{13} & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & 2 & 1_{14} & n
\end{array}\right]
$$

for $C_{4} \times K_{n}, C_{5} \times K_{n}, C_{6} \times K_{n}$ and $C_{7} \times K_{n}$ respectively with $n+1$ colors $(n \geq 6)$. The above arrays show that $d\left(C_{m} \times K_{n}, n+1\right)=m(n-2)$ for $(4 \leq m \leq 7)$ and $n \geq 4$.

To obtain a defining set for $C_{m} \times K_{n}$, with $m \geq 8$, one can write $m=4 t+r$ where $4 \leq r \leq 7$ and $t \geq 1$ are integers. We successively treat the $t$ above arrays for $C_{4} \times K_{n}$ and then treat to with the one for $C_{r} \times K_{n}$. So $d\left(C_{m} \times K_{n}, n+1\right)=m(n-2)$ for $n, m \geq 4$.

Theorem 2.2. For $n, m \geq 4, d\left(C_{m} \times K_{n}, n+2\right)=m(n-1)$.

Proof. Let $G=C_{m} \times K_{n}$. From Lemma 2.1 we obtain $d(G, n+2) \geq$ $m(n-1)$. To show equality we give a defining set, $S$ of size $m(n-1)$ as in following arrays.
(1) For $m \geq 4$ and $n=4$, consider the arrays

$$
\begin{aligned}
& {\left[\begin{array}{llll}
1 & 2 & 3 & 6_{1} \\
4 & 3_{2} & 6 & 5 \\
6_{3} & 1 & 2 & 3 \\
5 & 6 & 1_{4} & 4
\end{array}\right],\left[\begin{array}{llll}
1 & 2 & 3 & 6_{1} \\
2 & 4 & 6_{2} & 5 \\
3 & 6_{3} & 1 & 2 \\
24 & 5 & 4 & 1 \\
6 & 3_{5} & 1 & 4
\end{array}\right]} \\
& {\left[\begin{array}{llll}
1 & 2 & 3 & 6_{1} \\
2 & 4 & 6_{2} & 5 \\
3 & 6_{3} & 1 & 2 \\
2_{4} & 5 & 4 & 1 \\
6 & 4 & 1 & 2 \\
2_{6} & 3 & 5 & 4
\end{array}\right] \text { and }\left[\begin{array}{llll}
1 & 2 & 3 & 6_{1} \\
2 & 4 & 6_{2} & 5 \\
3 & 6_{3} & 1 & 2 \\
6_{4} & 5 & 4 & 1 \\
2 & 6_{5} & 1 & 4 \\
4 & 3 & 66 & 2 \\
6 & 1_{7} & 5 & 4
\end{array}\right]}
\end{aligned}
$$

for $C_{4} \times K_{4}, C_{5} \times K_{4}, C_{6} \times K_{4}$ and $C_{7} \times K_{4}$ respectively with $6=4+2$ colors.
(2) For $m \geq 4$ and $n=5$, consider the arrays

$$
\begin{aligned}
& {\left[\begin{array}{lllll}
1 & 2 & 3 & 4 & 7_{1} \\
2 & 3 & 5 & 7_{2} & 6 \\
3 & 4 & 7_{3} & 1 & 2 \\
7 & 1_{4} & 6 & 3 & 5
\end{array}\right],\left[\begin{array}{lllll}
1 & 2 & 3 & 4 & 7_{1} \\
2 & 3 & 5 & 7_{2} & 6 \\
3 & 4 & 7_{3} & 1 & 2 \\
7 & 5_{4} & 6 & 2 & 1 \\
4_{5} & 3 & 2 & 6 & 5
\end{array}\right]} \\
& {\left[\begin{array}{lllll}
1 & 2 & 3 & 4 & 7_{1} \\
2 & 3 & 5 & 7_{2} & 6 \\
3 & 4 & 7_{3} & 1 & 2 \\
7 & 5_{4} & 6 & 2 & 1 \\
5_{5} & 3 & 1 & 6 & 4 \\
2 & 4 & 6 & 7 & 5
\end{array}\right] \text { and }\left[\begin{array}{lllll}
1 & 2 & 3 & 4 & 7_{1} \\
2 & 3 & 5 & 7_{2} & 6 \\
3 & 4 & 7_{3} & 1 & 2 \\
5 & 7_{4} & 6 & 3 & 1 \\
7_{5} & 2 & 4 & 6 & 3 \\
1 & 5_{6} & 6 & 3 & 4 \\
2 & 7 & 4_{7} & 1 & 5
\end{array}\right]}
\end{aligned}
$$

for $C_{4} \times K_{5}, C_{5} \times K_{5}, C_{6} \times K_{5}$ and $C_{7} \times K_{5}$ respectively with $7=5+2$ colors.
(3) For $m \geq 4$ and $n \geq 6$, consider the following arrays,

$$
\begin{aligned}
& {\left[\begin{array}{llllllllll}
1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_{1} \\
2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_{2} & n+1 \\
4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_{3} & 1 & 2 \\
54 & 6 & 7 & \cdots & n-1 & n+1 & n+2 & 3 & 2 & n
\end{array}\right],} \\
& {\left[\begin{array}{llllllllll}
1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_{1} \\
2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_{2} & n+1 \\
4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_{3} & 1 & 2 \\
2 & 6 & 7 & \cdots & n-1 & n & 4_{4} & 3 & 5 & 1 \\
45 & 5 & 6 & \cdots & n-2 & n-1 & n+2 & n+1 & 3 & n
\end{array}\right],} \\
& {\left[\begin{array}{llllllllll}
1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_{1} \\
2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_{2} & n+1 \\
4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_{3} & 1 & 2 \\
5 & 6 & 7 & \cdots & n-1 & n & (n+2)_{4} & 3 & 2 & 4 \\
4 & 7 & 8 & \cdots & n & 5 & 1 & (n+1)_{5} & 6 & 2 \\
56 & 6 & 7 & \cdots & n-1 & n+1 & 2 & n+2 & 3 & n
\end{array}\right]}
\end{aligned}
$$

and

$$
\left[\begin{array}{llllllllll}
1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_{1} \\
2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_{2} & n+1 \\
4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_{3} & 1 & 2 \\
5 & 6 & 7 & \cdots & n-1 & n & (n+2)_{4} & 3 & 2 & 4 \\
4 & 7 & 8 & \cdots & n & n+1 & 1 & (n+2)_{5} & 6 & 2 \\
(n+2)_{6} & 6 & 7 & \cdots & n-1 & n & 2 & 5 & n+1 & 1 \\
3 & 5 & 6 & \cdots & n-2 & n+2 & 1 & 2 & 4_{7} & n
\end{array}\right]
$$

for $C_{4} \times K_{n}, C_{5} \times K_{n}, C_{6} \times K_{n}$ and $C_{7} \times K_{n}$ respectively with $n+2$ colors where $n \geq 6$. The above arrays show that $d\left(C_{m} \times K_{n}, n+2\right)=m(n-1)$ for $(4 \leq m \leq 7)$ and $n \geq 4$.

To obtain a defining set for $C_{m} \times K_{n}$, with $m \geq 8$, one can write $m=4 t+r$ where $4 \leq r \leq 7$ and $t \geq 1$ are integers. We successively treat the $t$ above arrays for $C_{4} \times K_{n}$ and then treat to with the one for $C_{r} \times K_{n}$. So $d\left(C_{m} \times K_{n}, n+2\right)=m(n-1)$ for $n, m \geq 4$.

Lemma 2.2. Let $G=(V, E)$ be a graph with $c \geq \Delta(G)+2$. Then $d(G, c)=|V|$.

Proof. Let $S$ be a defining set of $G$ and $v$ be a vertex for which $v \notin S$. So if all of the neighbors of vertex $v$ are colored then the vertex $v$ has at least two choices for coloring.

Theorem 2.3. For $n, m \geq 4, d\left(C_{m} \times K_{n}, n+i\right)=m n$ where $i \geq 3$.
Proof. The degree of any vertex in $C_{m} \times K_{n}$ is $n+1,\left|V\left(C_{m} \times K_{n}\right)\right|=m n$ and for $i \geq 3, n+i \geq \Delta\left(C_{m} \times K_{n}\right)+2$. Now use the Lemma 2.2.

$$
\text { 3. } \quad d\left(K_{3} \times C_{m}, c>\chi\right)
$$

Note that $\chi\left(K_{3} \times C_{m}\right)=3$.
Lemma 3.1. Let $G=K_{3} \times C_{r}$. Then $d(G, 4) \geq r+1$.
Proof. On the contrary assume that $d(G, 4) \leq r$. If $S$ is a defining set of $G$ with cardinality at most $r$ and $V$ is the set of vertices of $G$ then the induced subgraph $\langle V \backslash S\rangle$ of $G$ has $3 r-d(G, 4)$ vertices and has at least $6 r-4 d(G, 4)$ edges. Since $r-d(G, 4) \geq 0$ we have $6 r-4 d(G, 4) \geq 3 r-d(G, 4)$. Therefore $\langle V \backslash S\rangle$ has a cycle and we use Lemma C.

Theorem 3.1. Let $G=K_{3} \times C_{r}$. Then $d(G, 4)=r+1$ for even $r$ and $r+1 \leq d(G, 4) \leq r+2$ for odd $r$.

Proof. Let $G=K_{3} \times C_{r}$. From Lemma 3.1 we obtain $d(G, 4) \geq r+1$. We give a defining set $S$ of size $r+1$ for even $r$ and a defining set $S$ of size $r+2$ for odd $r$.

Let $v_{1}, v_{2}, \cdots, v_{r}$ are the vertices of first row, $u_{1}, u_{2}, \cdots, u_{r}$ the vertices of the second row and $w_{1}, w_{2}, \cdots, w_{r}$ the vertices of the third row.

If $r=2 n$ then we determine the defining set with their colors as follows:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 6) \\
2 & \text { if } & m \equiv 3(\bmod 6) \\
3 & \text { if } & m \equiv 5(\bmod 6)
\end{array}\right.
$$

except for $m=2 n-1$ when $2 n \equiv 2(\bmod 6)$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
3 & \text { if } & m \equiv 2(\bmod 6) \\
1 & \text { if } & m \equiv 4(\bmod 6) \\
2 & \text { if } & m \equiv 6(\bmod 6)
\end{array}\right.
$$

except for $m=2 n$ when $2 n \equiv 2(\bmod 6)$. In this case we set $c\left(u_{2 n}\right)=1$ when $2 n \equiv 2(\bmod 6)$.

Finally, let $c\left(w_{1}\right)=2$ if $2 n \equiv 0$ or $4(\bmod 6)$ and if $2 n \equiv 2(\bmod 6)$, we set $c\left(w_{1}\right)=2$ and $c\left(w_{2 n-1}\right)=3$. In each case we have $d(G, 4)=r+1$ if $r$ is even.

If $r=2 n+1$ then we determine the defining set with their colors as follows:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 6) \\
2 & \text { if } & m \equiv 3(\bmod 6) \\
3 & \text { if } & m \equiv 5(\bmod 6)
\end{array}\right.
$$

except for $m=2 n+1$ when $2 n+1 \equiv 1(\bmod 6)$. In this case we set $c\left(v_{2 n+1}\right)=4$ when $2 n+1 \equiv 1(\bmod 6)$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
3 & \text { if } & m \equiv 2(\bmod 6) \\
1 & \text { if } & m \equiv 4(\bmod 6) \\
2 & \text { if } & m \equiv 6(\bmod 6)
\end{array}\right.
$$

and if $2 n+1 \equiv 3$ or $5(\bmod 6)$ we set $c\left(w_{1}\right)=2$ and $c\left(w_{2 n}\right)=4$.
Finally we set $c\left(w_{1}\right)=2$ and $c\left(w_{2 n}\right)=3$ if $2 n+1 \equiv 1(\bmod 6)$. Thus $r+1 \leq d(G, 4) \leq r+2$ when $r$ is odd.

We have the following
Conjecture. $d\left(K_{3} \times C_{r}, 4\right)=r+2$ for odd $r$.

Lemma 3.2. Let $G=(V, E)$ be a graph. Let $S$ be a defining set of $G$ with $c=\Delta(G)+1$. If $v$ is a vertex and $\operatorname{deg}(v) \leq \Delta(G)-1$ then $v \in S$ and if $\operatorname{deg}(v)=\Delta(G)$ then $v \in S$ or all neighbors of $v$ are in $S$.

Proof. If $v$ is a vertex with $\operatorname{deg}(v) \leq \Delta(G)-1$ and $v \notin S$ then there exists at least two choices of colors for $v$ eventually all of neighbors are colored. If $\operatorname{deg}(v)=\Delta(G)$, vertex $u$ is a neighbor of $v,(u, v \notin S)$ and all the other neighbors of $v$ are in $S$ then we have two choices of colors for $u$ and $v$.

Theorem 3.2. Let $G=K_{3} \times C_{r}$. Then $d(G, 5)=2 r$.
Proof. Let $G=K_{3} \times C_{r}$. From Lemma 3.2 we obtain $d(G, 5) \geq 2 r$. To show equality we give a defining set, $S$ of size $2 r$.

Let $v_{1}, v_{2}, \cdots, v_{r}$ are the vertices of first row, $u_{1}, u_{2}, \cdots, u_{r}$ the vertices of the second row and $w_{1}, w_{2}, \cdots, w_{r}$ the vertices of the third row.

If $r=2 n$ then we determine the defining set with their colors as follows:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 10) \\
2 & \text { if } & m \equiv 3(\bmod 10) \\
3 & \text { if } & m \equiv 5(\bmod 10) \\
4 & \text { if } & m \equiv 7(\bmod 10) \\
5 & \text { if } & m \equiv 9(\bmod 10)
\end{array}\right.
$$

and $c\left(v_{2 n}\right)=5$ when $2 n \equiv 2$ or $8(\bmod 10)$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
5 & \text { if } \quad m \equiv 2(\bmod 10) \\
4 & \text { if } & m \equiv 4(\bmod 10) \\
2 & \text { if } \quad m \equiv 6(\bmod 10) \\
1 & \text { if } & m \equiv 8(\bmod 10) \\
3 & \text { if } \quad m \equiv 0(\bmod 10)
\end{array}\right.
$$

for $2 \leq m \leq 2 n$, except $m \neq 2 n$ when $2 n \equiv 2$ or $8(\bmod 10)$. In this case we set $c\left(u_{2 n}\right)=2$ when $2 n \equiv 2(\bmod 10)$ and $c\left(u_{2 n}\right)=3$ when $2 n \equiv 8(\bmod 10)$,

$$
c\left(w_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 3 \text { or } 5(\bmod 10) \\
5 & \text { if } & m \equiv 4 \text { or } 6(\bmod 10) \\
3 & \text { if } & m \equiv 2 \text { or } 7(\bmod 10) \\
2 & \text { if } & m \equiv 0 \text { or } 8(\bmod 10) \\
4 & \text { if } & m \equiv 1 \text { or } 9(\bmod 10)
\end{array}\right.
$$

for $m \neq 1,2,2 n-1$ and $2 n$. Finally, the following cases conclude the even case.

If $2 n \equiv 4$ or $6(\bmod 10)$ we set $c\left(w_{1}\right)=3, c\left(w_{2}\right)=4, c\left(w_{2 n-1}\right)=1$ and $c\left(w_{2 n}\right)=5$.

If $2 n \equiv 2(\bmod 10)$ we set $c\left(w_{1}\right)=3, c\left(w_{2}\right)=4$ and $c\left(w_{2 n-1}\right)=4$.
If $2 n \equiv 8(\bmod 10)$ we set $c\left(w_{1}\right)=4, c\left(w_{2}\right)=3$ and $c\left(w_{2 n-1}\right)=1$.
If $2 n \equiv 0(\bmod 10)$ we set $c\left(w_{1}\right)=4, c\left(w_{2}\right)=3, c\left(w_{2 n}\right)=2$ and $c\left(w_{2 n-1}\right)=4$.

For $r=2 n+1$ we determine the defining set with their colors as follows:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 10) \\
2 & \text { if } & m \equiv 3(\bmod 10) \\
3 & \text { if } & m \equiv 5(\bmod 10) \\
4 & \text { if } & m \equiv 7(\bmod 10) \\
5 & \text { if } & m \equiv 9(\bmod 10)
\end{array}\right.
$$

for $1 \leq m \leq 2 n+1$ and $m \neq 2 n+1$ when $2 n+1 \equiv 1(\bmod 10)$. And we set $c\left(v_{2 n+1}\right)=2$ when $2 n+1 \equiv 1(\bmod 10)$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
5 & \text { if } & m \equiv 2(\bmod 10), \\
4 & \text { if } & m \equiv 4(\bmod 10), \\
2 & \text { if } & m \equiv 6(\bmod 10), \\
1 & \text { if } & m \equiv 8(\bmod 10), \\
3 & \text { if } & m \equiv 0(\bmod 10)
\end{array}\right.
$$

for $1 \leq m \leq 2 n$.
Furthermore let $c\left(u_{2 n+1}\right)=4$ when $2 n+1 \equiv 1,3$ or $9(\bmod 10)$, let $c\left(u_{2 n+1}\right)=2$ when $2 n+1 \equiv 5(\bmod 10)$ and let $c\left(u_{2 n+1}\right)=3$ when $2 n+1 \equiv$ $7(\bmod 10)$

$$
c\left(w_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 3 \text { or } 5(\bmod 10), \\
5 & \text { if } & m \equiv 4 \operatorname{or} 6(\bmod 10), \\
3 & \text { if } & m \equiv 2 \text { or } 7(\bmod 10), \\
2 & \text { if } & m \equiv 0 \operatorname{or} 8(\bmod 10), \\
4 & \text { if } & m \equiv 1 \operatorname{or} 9(\bmod 10)
\end{array}\right.
$$

for $m \neq 1,2,2 n$ and $2 n+1$. Again some special cases completes the proof.
If $2 n+1 \equiv 1$ or $3(\bmod 10)$ we set $c\left(w_{1}\right)=3, c\left(w_{2}\right)=4, c\left(w_{2 n}\right)=1$.
If $2 n+1 \equiv 9(\bmod 10)$ we set $c\left(w_{1}\right)=3, c\left(w_{2}\right)=4$ and $c\left(w_{2 n}\right)=2$.
If $2 n+1 \equiv 5(\bmod 10)$ we set $c\left(w_{1}\right)=4, c\left(w_{2}\right)=3$ and $c\left(w_{2 n}\right)=5$.
If $2 n+1 \equiv 7(\bmod 10)$ we set $c\left(w_{1}\right)=2, c\left(w_{2}\right)$ and $c\left(w_{2 n}\right)=5$.
4. $d\left(K_{2} \times C_{m}, c>\chi\right)$

Note that $\chi\left(K_{2} \times C_{m}\right)=3$ if $m$ is odd and $\chi\left(K_{2} \times C_{m}\right)=2$ if $m$ is even.

Lemma 4.1. Let $G=K_{2} \times C_{r}$. Then $d(G, 3) \geq\left\lfloor\frac{r}{2}\right\rfloor+1$.
Proof. On the contrary, assume that $d(G, 3) \leq\left\lfloor\frac{r}{2}\right\rfloor$. If $S$ is a defining set of $G$ with cardinality at most $\left\lfloor\frac{r}{2}\right\rfloor$ and $V$ is the set of vertices of $G$ then the induced subgraph $\langle V \backslash S\rangle$ of $G$ has $2 r-d(G, 3)$ vertices and has at least $3 r-3 d(G, 3)$ edges. Since $\left\lfloor\frac{r}{2}\right\rfloor-d(G, 3) \geq 0$ we have $r \geq 2\left\lfloor\frac{r}{2}\right\rfloor \geq 2 d(G)$ and $3 r-3 d(G, 3) \geq 2 r-d(G, 3)$. Therefore $\langle V \backslash S\rangle$ has a cycle and we use Lemma C.

Theorem 4.1. Let $G=K_{2} \times C_{2 n}$. Then $d(G, 3)=n+1$.
Proof. Let $G=K_{2} \times C_{2 n}$. From Lemma 4.1 we obtain $d(G, 3) \geq n+1$. To show equality we give a defining set, $S$ of size $n+1$.

If $v_{1}, v_{2}, \cdots, v_{2 n}$ are the vertices of first row and $u_{1}, u_{2}, \cdots, u_{2 n}$ the vertices of the second row. We determine the defining set with their colors as in following tables:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m=1 \quad \text { and } \quad m=2 n-2 \\
2 & \text { if } & m \equiv 0(\bmod 4) \quad \text { and } \quad 1 \leq m \leq 2 n-3
\end{array}\right.
$$

also

$$
c\left(u_{m}\right)=2 \quad \text { if } \quad m \equiv 2(\bmod 4), \quad(m \leq 2 n-3) \quad \text { and } \quad m=2 n-1
$$

For $2 n=4$ we say $c\left(v_{1}\right)=c\left(u_{3}\right)=1$ and $c\left(v_{4}\right)=2$.
Theorem 4.2. If $G=K_{2} \times C_{r}$ then $d(G, 4)=2\left\lceil\frac{r}{2}\right\rceil$.
Proof. Let $G=K_{2} \times C_{r}$. From Lemma 3.2 we obtain $d(G, 4) \geq 2\left\lceil\frac{r}{2}\right\rceil$. To show equality we give a defining set, $S$ of size $2\left\lceil\frac{r}{2}\right\rceil$.

Let $v_{1}, v_{2}, \cdots, v_{r}$ are the vertices of first row, $u_{1}, u_{2}, \cdots, u_{r}$ the vertices of the second row.

If $r=2 n$ we determine the defining set with their colors as follows:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 8) \\
2 & \text { if } & m \equiv 3(\bmod 8) \\
3 & \text { if } & m \equiv 5(\bmod 8) \\
4 & \text { if } & m \equiv 7(\bmod 8)
\end{array}\right.
$$

except $m=2 n-1$ when $n \equiv 1(\bmod 4)$. In this case we set $c\left(v_{2 n-1}\right)=2$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
3 & \text { if } & m \equiv 2(\bmod 8), \\
4 & \text { if } & m \equiv 4(\bmod 8), \\
1 & \text { if } & m \equiv 6(\bmod 8), \\
2 & \text { if } & m \equiv 0(\bmod 8)
\end{array}\right.
$$

except $m=2 n-2, m=2 n$ when $n \equiv 1(\bmod 4)$ and $m=2 n$ when $n \equiv$ $3(\bmod 4)$, in this case we say $c\left(u_{2 n}\right)=4, c\left(u_{2 n-2}\right)=3$ when $n \equiv 1(\bmod 4)$ and we say $c\left(u_{2 n}\right)=2$ when $n \equiv 3(\bmod 4)$.

If $r=2 n+1$ we determine the defining set with their colors as in following tables:

$$
c\left(v_{m}\right)=\left\{\begin{array}{lll}
1 & \text { if } & m \equiv 1(\bmod 8), \\
2 & \text { if } & m \equiv 3(\bmod 8), \\
3 & \text { if } & m \equiv 5(\bmod 8), \\
4 & \text { if } & m \equiv 7(\bmod 8)
\end{array}\right.
$$

except $m=2 n+1$ when $n \equiv 0(\bmod 4)$. In this case we set $c\left(v_{2 n+1}\right)=2$,

$$
c\left(u_{m}\right)=\left\{\begin{array}{lll}
3 & \text { if } & m \equiv 2(\bmod 8), \\
4 & \text { if } & m \equiv 4(\bmod 8), \\
1 & \text { if } & m \equiv 6(\bmod 8), \\
2 & \text { if } & m \equiv 0(\bmod 8)
\end{array}\right.
$$

except $m=2 n$ when $n \equiv 0,(\bmod 4)$ in this case $c\left(u_{2 n}\right)=3$ and $c\left(u_{2 n+1}\right)=4$ when $n \equiv 0$ or $1(\bmod 4), c\left(u_{2 n+1}\right)=2$, when $n \equiv 2$ or $3(\bmod 4)$.

Corollary 4.3. $d\left(K_{2} \times C_{r}, 5\right)=2 r$.
Proof. By Lemma 2.2, each of column has at least 2 vertices in defining set. Therefore all vertices are in defining set.

## References

[1] J. Cooper, D. Donovan and J. Seberry, Latin squares and critical sets of minimal size, Austral. J. Combin. 4 (1991) 113-120.
[2] M. Mahdian and E.S. Mahmoodian, A characterization of uniquely 2-list colorable graph, Ars Combin. 51 (1999) 295-305.
[3] M. Mahdian, E.S. Mahmoodian, R. Naserasr and F. Harary, On defining sets of vertex colorings of the cartesian product of a cycle with a complete graph, Combinatorics, Graph Theory and Algorithms (1999) 461-467.
[4] E.S. Mahmoodian and E. Mendelsohn, On defining numbers of vertex coloring of regular graphs, 16th British Combinatorial Conference (London, 1997). Discrete Math. 197/198 (1999) 543-554.
[5] E.S. Mahmoodian, R. Naserasr and M. Zaker, Defining sets in vertex colorings of graphs and Latin rectangles, Discrete Math. (to appear).
[6] E. Mendelsohn and D.A. Mojdeh, On defining spectrum of regular graph, (submitted).
[7] D.A. Mojdeh, On conjectures of the defining set of (vertex) graph colourings, Austral. J. Combin. (to appear).
[8] A.P. Street, Defining sets for block designs; an update, in: C.J. Colbourn, E.S. Mahmoodian (eds), Combinatorics advances, Mathematics and its applications (Kluwer Academic Publishers, Dordrecht, 1995) 307-320.
[9] D.B. West, Introduction to Graph Theory (Second Edition) (Prentice Hall, USA, 2001).

Received 6 November 2004
Revised 13 September 2005

