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DEFINING SETS IN (PROPER) VERTEX COLORINGS OF THE CARTESIAN PRODUCT OF A CYCLE WITH A COMPLETE GRAPH

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Abstract

In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G, if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by $d(G, c)$.

The $d(G = C_m \times K_n, \chi(G))$ has been studied. In this note we show that the exact value of defining number $d(G = C_m \times K_n, c)$ with $c > \chi(G)$, where $n \geq 2$ and $m \geq 3$, unless the defining number $d(K_3 \times C_{2r}, 4)$, which is given an upper and lower bounds for this defining number. Also some bounds of defining number are introduced.

Keywords: graph coloring, defining set, cartesian product.

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1. Introduction

A *c-coloring* (*proper c-coloring*) of a graph G is an assignment of c different colors to the vertices of G, such that no two adjacent vertices receive the same color. The *vertex chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number c, for which there exists a c-coloring for G. The maximum degree of the vertices in G is $\Delta(G)$ and the minimum degree is $\delta(G)$ and G is *regular* if $\Delta(G) = \delta(G)$. It is *k-regular* graph if the common degree is k (see [9]). In a given graph $G = (V, E)$, a set of vertices S with an assignment of colors to them is said to be a *defining set of the vertex coloring of* G, if there exists a unique extension of the colors of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with the minimum cardinality is called a *minimum defining set* and its cardinality is the *defining number*, denoted by $d(G, c)$. We will use standard notations such as K_n for the complete graph on n vertices, C_m for the cycle of size m and $G \times H$ for cartesian product of G and H. There are some papers on defining set of graphs, especially $d(K_n \times K_n,\ \chi)$ (the critical set of Latin squares of order n), $d(C_m \times K_n,\ \chi)$, $d(G, \chi = k)$ where G is a k-regular graph and defining set on block designs. The interested reader may see [1, 4, 5, 7, 8] and their references.

The following results can be found in [3]:

(1) $d(C_m \times K_3, \chi) = \lfloor \frac{m}{2} \rfloor + 1$,

(2) $m \leq d(C_m \times K_4, \chi) \leq m + 1$,

(3) $d(C_m \times K_5, \chi) = 2m$ for even m and $2m \leq d(C_m \times K_5, \chi) \leq 2m + 1$ for odd m.

The following results can be found in [7]:

(4) $d(C_m \times K_5, \chi) = 2m$, for odd $m (\geq 5)$,

(5) $d(C_m \times K_4, \chi) = m + 1$.

The following results can be found in [6]:

(6) $d(C_m \times K_n, \chi) = m(n - 3)$ for $n \geq 6$,

(7) $d(C_{2n+1} \times K_2, \chi) = n + 1$.

The followings are useful.

Definition A [2]**.** A graph G with n vertices, is called a uniquely 2-list colorable graph, if there exists $S_1, S_2, \cdots S_n$, a list of colors on its vertices, each of size 2, such that there is a unique coloring for G from this list of colors.

Theorem B [2]**.** *A connected graph is uniquely* 2*-list colorable if and only if at least one of its blocks is not a cycle, a complete graph, or a complete bipartite graph.*

Let G be a k-regular graph and vertex colored with k colors. Let C be a cycle in G, then each vertex of C has at least two choice for coloring, in other words C is at least 2-list vertex colorable, if all vertices of $V(G) \setminus V(C)$ have been already colored. So by Theorem B the cycle C is not uniquely 2-list colorable. Now we have

Lemma C [6]**.** *If G is k-regular graph and which is colored with k colors, then every cycle in G has a vertex in defining set of G.*

If $G = C_m \times K_n$ then, each subgraph K_n of G is said to be a *row* and each subgraph C_m of G is said to be a *column*. If $G = K_n \times C_m$ then, each subgraph K_n of G is said to be a column and each subgraph C_m of G is said to be a row.

It is well known that $\chi(C_m \times K_n) = \chi(K_n \times C_m) = n$ for $n > 2$ or for $n = 2$ and even m. Also $\chi(C_{2r+1} \times K_2) = \chi(K_2 \times C_{2r+1}) = 3$.

2. $d(C_m \times K_n,\ n+i)$

In this section we derive $d(C_m \times K_n, n+i)$ for $n,\ m \geq 4$ and $i \geq 0$. We start with the following lemma.

Lemma 2.1. *If $G = C_m \times K_n$ is colored with $n+i$ colors for $0 \leq i \leq 3$, then for each row, there exist at least, $n+i-3$ vertices in defining set.*

Proof. Assume that, there exists a row for which the defining set contains $k < n+i-3$ vertices and all other rows are completely colored. The induced subgraph of the non colored vertices of this row is a complete graph and cannot be uniquely colored by Theorem B. ■

In the following arrays the non indexed labels denote the colors of the vertices in the defining set of the graph $C_m \times K_n$, the indexed labels denote the colors of the vertices out of defining set and the indices denote the ordering of the coloring of these vertices.

Theorem 2.1. *For $n, m \geq 4$, $d(C_m \times K_n, n+1) = m(n-2)$.*

Proof. Let $G = C_m \times K_n$. From Lemma 2.1 we obtain $d(G, n+1) \geq m(n-2)$. To show equality we give a defining set S of size $m(n-2)$ as in following arrays.

(1) For $m \geq 4$ and $n = 4$, consider the arrays

$$\begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 4 & 1 & 2_5 & 5_6 \\ 2_8 & 5_7 & 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 3_5 & 1 & 4_6 & 5 \\ 4 & 3_7 & 2 & 1_8 \\ 2_9 & 5 & 3 & 4_{10} \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 4 & 1 & 3_6 & 2_5 \\ 3_7 & 2_8 & 1 & 5 \\ 2 & 4 & 5_9 & 3_{10} \\ 4_{12} & 5_{11} & 3 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 4_1 & 5_2 \\ 2_4 & 4_3 & 5 & 3 \\ 5_6 & 1 & 3_5 & 4 \\ 3 & 5_7 & 2 & 1_8 \\ 5_{10} & 4 & 1_9 & 3 \\ 2 & 3_{12} & 5 & 1_{11} \\ 5_{13} & 1_{14} & 3 & 4 \end{bmatrix}$$

for $C_4 \times K_4$, $C_5 \times K_4$, $C_6 \times K_4$ and $C_7 \times K_4$ respectively with $5 = 4 + 1$ colors.

(2) For $m \geq 4$ and $n = 5$, consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 6_2 & 4_1 \\ 2_4 & 3_3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6_6 & 4_5 \\ 2_8 & 3_7 & 6 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4_1 & 5_2 \\ 3_3 & 1_4 & 4 & 5 & 6 \\ 2 & 4_5 & 5 & 3 & 1_6 \\ 5_8 & 6 & 1_7 & 4 & 3 \\ 3_{10} & 1 & 2 & 6 & 4_9 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4_2 & 5_1 \\ 2_4 & 3_3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6_5 & 4_6 \\ 1_8 & 2_7 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5_9 & 6_{10} \\ 3_{12} & 1_{11} & 5 & 6 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 6_2 & 4_1 \\ 3_3 & 1_4 & 4 & 5 & 6 \\ 2 & 4_5 & 5 & 3 & 1_6 \\ 5_8 & 6 & 1_7 & 4 & 3 \\ 4_9 & 1 & 2 & 6 & 5_{10} \\ 3 & 4_{11} & 5 & 1_{12} & 2 \\ 4_{13} & 6 & 2 & 3_{14} & 5 \end{bmatrix}$$

for $C_4 \times K_5$, $C_5 \times K_5$, $C_6 \times K_5$ and $C_7 \times K_5$ respectively with $6 = 5 + 1$ colors.

(3) For $m \geq 4$ and $n \geq 6$, consider the following arrays,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 5_7 & 4_8 & 6 & 7 & 8 & \cdots & n-1 & 2 & n+1 & 3 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ (n+1)_9 & 5 & 6 & 7 & 8 & \cdots & n-1 & 2_{10} & 3 & 4 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ 5 & 1 & 6 & 7 & 8 & \cdots & n-1 & 2 & n_9 & 4 & (n+1)_{10} \\ 2_{12} & 4_{11} & 5 & 6 & 7 & \cdots & n-2 & n+1 & 3 & n-1 & n \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & \cdots & n-4 & n-3 & n-2 & (n+1)_2 & (n-1)_1 \\ 2_4 & 3_3 & 4 & 5 & 6 & \cdots & n-3 & n-2 & n-1 & n & n+1 \\ 4 & 1 & 5 & 6 & 7 & \cdots & n-2 & 3 & n_5 & 2 & (n-1)_6 \\ 2_8 & 6 & 7 & 8 & 9 & \cdots & n & 1 & n+1 & 5_7 & 3 \\ 1_9 & 5 & 6 & 7 & 8 & \cdots & n-1 & 3 & 2_{10} & 4 & n \\ n+1 & 6_{11} & 7 & 8 & 9 & \cdots & n & 2 & 3 & 5_{12} & 1 \\ 3_{13} & 4 & 5 & 6 & 7 & \cdots & n-2 & n-1 & 2 & 1_{14} & n \end{bmatrix}$$

for $C_4 \times K_n$, $C_5 \times K_n$, $C_6 \times K_n$ and $C_7 \times K_n$ respectively with $n+1$ colors ($n \geq 6$). The above arrays show that $d(C_m \times K_n, n+1) = m(n-2)$ for ($4 \leq m \leq 7$) and $n \geq 4$.

To obtain a defining set for $C_m \times K_n$, with $m \geq 8$, one can write $m = 4t + r$ where $4 \leq r \leq 7$ and $t \geq 1$ are integers. We successively treat the t above arrays for $C_4 \times K_n$ and then treat to with the one for $C_r \times K_n$. So $d(C_m \times K_n, n+1) = m(n-2)$ for $n,\ m \geq 4$. ■

Theorem 2.2. *For $n, m \geq 4$, $d(C_m \times K_n, n+2) = m(n-1)$.*

Proof. Let $G = C_m \times K_n$. From Lemma 2.1 we obtain $d(G, n+2) \geq m(n-1)$. To show equality we give a defining set, S of size $m(n-1)$ as in following arrays.

(1) For $m \geq 4$ and $n = 4$, consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 4 & 3_2 & 6 & 5 \\ 6_3 & 1 & 2 & 3 \\ 5 & 6 & 1_4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 2_4 & 5 & 4 & 1 \\ 6 & 3_5 & 1 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 2_4 & 5 & 4 & 1 \\ 6 & 4_5 & 1 & 2 \\ 2_6 & 3 & 5 & 4 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 6_1 \\ 2 & 4 & 6_2 & 5 \\ 3 & 6_3 & 1 & 2 \\ 6_4 & 5 & 4 & 1 \\ 2 & 6_5 & 1 & 4 \\ 4 & 3 & 6_6 & 2 \\ 6 & 1_7 & 5 & 4 \end{bmatrix}$$

for $C_4 \times K_4$, $C_5 \times K_4$, $C_6 \times K_4$ and $C_7 \times K_4$ respectively with $6 = 4 + 2$ colors.

(2) For $m \geq 4$ and $n = 5$, consider the arrays

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 1_4 & 6 & 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 5_4 & 6 & 2 & 1 \\ 4_5 & 3 & 2 & 6 & 5 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 7 & 5_4 & 6 & 2 & 1 \\ 5_5 & 3 & 1 & 6 & 4 \\ 2 & 4 & 6_6 & 7 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 3 & 4 & 7_1 \\ 2 & 3 & 5 & 7_2 & 6 \\ 3 & 4 & 7_3 & 1 & 2 \\ 5 & 7_4 & 6 & 3 & 1 \\ 7_5 & 2 & 4 & 6 & 3 \\ 1 & 5_6 & 6 & 3 & 4 \\ 2 & 7 & 4_7 & 1 & 5 \end{bmatrix}$$

for $C_4 \times K_5$, $C_5 \times K_5$, $C_6 \times K_5$ and $C_7 \times K_5$ respectively with $7 = 5 + 2$ colors.

(3) For $m \geq 4$ and $n \geq 6$, consider the following arrays,

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5_4 & 6 & 7 & \cdots & n-1 & n+1 & n+2 & 3 & 2 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 2 & 6 & 7 & \cdots & n-1 & n & 4_4 & 3 & 5 & 1 \\ 4_5 & 5 & 6 & \cdots & n-2 & n-1 & n+2 & n+1 & 3 & n \end{bmatrix},$$

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5 & 6 & 7 & \cdots & n-1 & n & (n+2)_4 & 3 & 2 & 4 \\ 4 & 7 & 8 & \cdots & n & 5 & 1 & (n+1)_5 & 6 & 2 \\ 5_6 & 6 & 7 & \cdots & n-1 & n+1 & 2 & n+2 & 3 & n \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 & \cdots & n-5 & n-4 & n-3 & n-2 & n-1 & (n+2)_1 \\ 2 & 3 & 4 & \cdots & n-4 & n-3 & n-2 & n & (n+2)_2 & n+1 \\ 4 & 5 & 6 & \cdots & n-2 & n-1 & n+1 & (n+2)_3 & 1 & 2 \\ 5 & 6 & 7 & \cdots & n-1 & n & (n+2)_4 & 3 & 2 & 4 \\ 4 & 7 & 8 & \cdots & n & n+1 & 1 & (n+2)_5 & 6 & 2 \\ (n+2)_6 & 6 & 7 & \cdots & n-1 & n & 2 & 5 & n+1 & 1 \\ 3 & 5 & 6 & \cdots & n-2 & n+2 & 1 & 2 & 4_7 & n \end{bmatrix}$$

for $C_4 \times K_n$, $C_5 \times K_n$, $C_6 \times K_n$ and $C_7 \times K_n$ respectively with $n+2$ colors where $n \geq 6$. The above arrays show that $d(C_m \times K_n, n+2) = m(n-1)$ for $(4 \leq m \leq 7)$ and $n \geq 4$.

To obtain a defining set for $C_m \times K_n$, with $m \geq 8$, one can write $m = 4t + r$ where $4 \leq r \leq 7$ and $t \geq 1$ are integers. We successively treat the t above arrays for $C_4 \times K_n$ and then treat to with the one for $C_r \times K_n$. So $d(C_m \times K_n, n+2) = m(n-1)$ for $n,\ m \geq 4$. ■

Lemma 2.2. *Let $G = (V, E)$ be a graph with $c \geq \Delta(G) + 2$. Then $d(G, c) =\mid V \mid$.*

Proof. Let S be a defining set of G and v be a vertex for which $v \notin S$. So if all of the neighbors of vertex v are colored then the vertex v has at least two choices for coloring. ■

Theorem 2.3. *For $n, m \geq 4$, $d(C_m \times K_n, n + i) = mn$ where $i \geq 3$.*

Proof. The degree of any vertex in $C_m \times K_n$ is $n+1$, $\mid V(C_m \times K_n) \mid= mn$ and for $i \geq 3$, $n + i \geq \Delta(C_m \times K_n) + 2$. Now use the Lemma 2.2. ■

3. $d(K_3 \times C_m, c > \chi)$

Note that $\chi(K_3 \times C_m) = 3$.

Lemma 3.1. *Let $G = K_3 \times C_r$. Then $d(G, 4) \geq r + 1$.*

Proof. On the contrary assume that $d(G, 4) \leq r$. If S is a defining set of G with cardinality at most r and V is the set of vertices of G then the induced subgraph $\langle V \setminus S \rangle$ of G has $3r - d(G, 4)$ vertices and has at least $6r - 4d(G, 4)$ edges. Since $r - d(G, 4) \geq 0$ we have $6r - 4d(G, 4) \geq 3r - d(G, 4)$. Therefore $\langle V \setminus S \rangle$ has a cycle and we use Lemma C. ■

Theorem 3.1. *Let $G = K_3 \times C_r$. Then $d(G, 4) = r + 1$ for even r and $r + 1 \leq d(G, 4) \leq r + 2$ for odd r.*

Proof. Let $G = K_3 \times C_r$. From Lemma 3.1 we obtain $d(G, 4) \geq r + 1$. We give a defining set S of size $r + 1$ for even r and a defining set S of size $r + 2$ for odd r.

Let $v_1, v_2, \cdots, v_r$ are the vertices of first row, $u_1, u_2, \cdots, u_r$ the vertices of the second row and $w_1, w_2, \cdots, w_r$ the vertices of the third row.

If $r = 2n$ then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1 (\text{mod } 6), \\ 2 & \text{if} \quad m \equiv 3 (\text{mod } 6), \\ 3 & \text{if} \quad m \equiv 5 (\text{mod } 6) \end{cases}$$

except for $m = 2n - 1$ when $2n \equiv 2 (\text{mod } 6)$,

$$c(u_m) = \begin{cases} 3 & \text{if} \quad m \equiv 2 (\text{mod } 6), \\ 1 & \text{if} \quad m \equiv 4 (\text{mod } 6), \\ 2 & \text{if} \quad m \equiv 6 (\text{mod } 6) \end{cases}$$

except for $m = 2n$ when $2n \equiv 2(\text{mod } 6)$. In this case we set $c(u_{2n}) = 1$ when $2n \equiv 2(\text{mod } 6)$.

Finally, let $c(w_1) = 2$ if $2n \equiv 0$ or $4(\text{mod } 6)$ and if $2n \equiv 2(\text{mod } 6)$, we set $c(w_1) = 2$ and $c(w_{2n-1}) = 3$. In each case we have $d(G, 4) = r + 1$ if r is even.

If $r = 2n + 1$ then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1 (\text{mod } 6), \\ 2 & \text{if} \quad m \equiv 3 (\text{mod } 6), \\ 3 & \text{if} \quad m \equiv 5 (\text{mod } 6) \end{cases}$$

except for $m = 2n + 1$ when $2n + 1 \equiv 1(\text{mod } 6)$. In this case we set $c(v_{2n+1}) = 4$ when $2n + 1 \equiv 1(\text{mod } 6)$,

$$c(u_m) = \begin{cases} 3 & \text{if} \quad m \equiv 2 (\text{mod } 6), \\ 1 & \text{if} \quad m \equiv 4 (\text{mod } 6), \\ 2 & \text{if} \quad m \equiv 6 (\text{mod } 6) \end{cases}$$

and if $2n + 1 \equiv 3$ or $5(\text{mod } 6)$ we set $c(w_1) = 2$ and $c(w_{2n}) = 4$.

Finally we set $c(w_1) = 2$ and $c(w_{2n}) = 3$ if $2n + 1 \equiv 1(\text{mod } 6)$. Thus $r + 1 \leq d(G, 4) \leq r + 2$ when r is odd. ■

We have the following

Conjecture. $d(K_3 \times C_r,\ 4) = r + 2$ for odd r.

Lemma 3.2. *Let $G = (V, E)$ be a graph. Let S be a defining set of G with $c = \Delta(G) + 1$. If v is a vertex and $deg(v) \leq \Delta(G) - 1$ then $v \in S$ and if $deg(v) = \Delta(G)$ then $v \in S$ or all neighbors of v are in S.*

Proof. If v is a vertex with $deg(v) \leq \Delta(G) - 1$ and $v \notin S$ then there exists at least two choices of colors for v eventually all of neighbors are colored. If $deg(v) = \Delta(G)$, vertex u is a neighbor of v, $(u, v \notin S)$ and all the other neighbors of v are in S then we have two choices of colors for u and v. ■

Theorem 3.2. *Let $G = K_3 \times C_r$. Then $d(G, 5) = 2r$.*

Proof. Let $G = K_3 \times C_r$. From Lemma 3.2 we obtain $d(G, 5) \geq 2r$. To show equality we give a defining set, S of size $2r$.

Let $v_1, v_2, \cdots, v_r$ are the vertices of first row, $u_1, u_2, \cdots, u_r$ the vertices of the second row and $w_1, w_2, \cdots, w_r$ the vertices of the third row.

If $r = 2n$ then we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1 (\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 3 (\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 5 (\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 7 (\text{mod } 10), \\ 5 & \text{if} \quad m \equiv 9 (\text{mod } 10) \end{cases}$$

and $c(v_{2n}) = 5$ when $2n \equiv 2$ or $8(\text{mod } 10)$,

$$c(u_m) = \begin{cases} 5 & \text{if} \quad m \equiv 2 (\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 4 (\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 6 (\text{mod } 10), \\ 1 & \text{if} \quad m \equiv 8 (\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 0 (\text{mod } 10) \end{cases}$$

for $2 \leq m \leq 2n$, except $m \neq 2n$ when $2n \equiv 2$ or $8(\text{mod } 10)$. In this case we set $c(u_{2n}) = 2$ when $2n \equiv 2(\text{mod } 10)$ and $c(u_{2n}) = 3$ when $2n \equiv 8(\text{mod } 10)$,

$$c(w_m) = \begin{cases} 1 & \text{if} \quad m \equiv 3 \text{ or } 5 (\text{mod } 10), \\ 5 & \text{if} \quad m \equiv 4 \text{ or } 6 (\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 2 \text{ or } 7 (\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 0 \text{ or } 8 (\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 1 \text{ or } 9 (\text{mod } 10) \end{cases}$$

for $m \neq 1, 2, 2n-1$ and $2n$. Finally, the following cases conclude the even case.

If $2n \equiv 4$ or $6(\text{mod } 10)$ we set $c(w_1) = 3, c(w_2) = 4, c(w_{2n-1}) = 1$ and $c(w_{2n}) = 5$.

If $2n \equiv 2(\text{mod } 10)$ we set $c(w_1) = 3, c(w_2) = 4$ and $c(w_{2n-1}) = 4$.

If $2n \equiv 8(\text{mod } 10)$ we set $c(w_1) = 4, c(w_2) = 3$ and $c(w_{2n-1}) = 1$.

If $2n \equiv 0(\text{mod } 10)$ we set $c(w_1) = 4, c(w_2) = 3, c(w_{2n}) = 2$ and $c(w_{2n-1}) = 4$.

For $r = 2n+1$ we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1(\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 3(\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 5(\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 7(\text{mod } 10), \\ 5 & \text{if} \quad m \equiv 9(\text{mod } 10) \end{cases}$$

for $1 \leq m \leq 2n+1$ and $m \neq 2n+1$ when $2n+1 \equiv 1(\text{mod } 10)$. And we set $c(v_{2n+1}) = 2$ when $2n+1 \equiv 1(\text{mod } 10)$,

$$c(u_m) = \begin{cases} 5 & \text{if} \quad m \equiv 2(\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 4(\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 6(\text{mod } 10), \\ 1 & \text{if} \quad m \equiv 8(\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 0(\text{mod } 10) \end{cases}$$

for $1 \leq m \leq 2n$.

Furthermore let $c(u_{2n+1}) = 4$ when $2n+1 \equiv 1, 3$ or $9(\text{mod } 10)$, let $c(u_{2n+1}) = 2$ when $2n+1 \equiv 5(\text{mod } 10)$ and let $c(u_{2n+1}) = 3$ when $2n+1 \equiv 7(\text{mod } 10)$

$$c(w_m) = \begin{cases} 1 & \text{if} \quad m \equiv 3 \text{ or } 5(\text{mod } 10), \\ 5 & \text{if} \quad m \equiv 4 \text{ or } 6(\text{mod } 10), \\ 3 & \text{if} \quad m \equiv 2 \text{ or } 7(\text{mod } 10), \\ 2 & \text{if} \quad m \equiv 0 \text{ or } 8(\text{mod } 10), \\ 4 & \text{if} \quad m \equiv 1 \text{ or } 9(\text{mod } 10) \end{cases}$$

for $m \neq 1, 2, 2n$ and $2n+1$. Again some special cases completes the proof.

If $2n+1 \equiv 1$ or $3(\text{mod } 10)$ we set $c(w_1) = 3, c(w_2) = 4, c(w_{2n}) = 1$.

If $2n+1 \equiv 9(\text{mod } 10)$ we set $c(w_1) = 3, c(w_2) = 4$ and $c(w_{2n}) = 2$.

If $2n+1 \equiv 5(\text{mod } 10)$ we set $c(w_1) = 4, c(w_2) = 3$ and $c(w_{2n}) = 5$.

If $2n+1 \equiv 7(\text{mod } 10)$ we set $c(w_1) = 2, c(w_2)$ and $c(w_{2n}) = 5$. ■

4. $d(K_2 \times C_m, c > \chi)$

Note that $\chi(K_2 \times C_m) = 3$ if m is odd and $\chi(K_2 \times C_m) = 2$ if m is even.

Lemma 4.1. *Let $G = K_2 \times C_r$. Then $d(G,3) \geq \lfloor \frac{r}{2} \rfloor + 1$.*

Proof. On the contrary, assume that $d(G,3) \leq \lfloor \frac{r}{2} \rfloor$. If S is a defining set of G with cardinality at most $\lfloor \frac{r}{2} \rfloor$ and V is the set of vertices of G then the induced subgraph $\langle V \setminus S \rangle$ of G has $2r - d(G,3)$ vertices and has at least $3r - 3d(G,3)$ edges. Since $\lfloor \frac{r}{2} \rfloor - d(G,3) \geq 0$ we have $r \geq 2\lfloor \frac{r}{2} \rfloor \geq 2d(G)$ and $3r - 3d(G,3) \geq 2r - d(G,3)$. Therefore $\langle V \setminus S \rangle$ has a cycle and we use Lemma C. ■

Theorem 4.1. *Let $G = K_2 \times C_{2n}$. Then $d(G,3) = n + 1$.*

Proof. Let $G = K_2 \times C_{2n}$. From Lemma 4.1 we obtain $d(G,3) \geq n+1$. To show equality we give a defining set, S of size $n+1$.

If $v_1, v_2, \cdots, v_{2n}$ are the vertices of first row and $u_1, u_2, \cdots, u_{2n}$ the vertices of the second row. We determine the defining set with their colors as in following tables:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m = 1 \quad \text{and} \quad m = 2n-2, \\ 2 & \text{if} \quad m \equiv 0 (\text{mod } 4) \quad \text{and} \quad 1 \leq m \leq 2n-3 \end{cases}$$

also

$$c(u_m) = \; 2 \quad \text{if} \quad m \equiv 2 (\text{mod } 4), \quad (m \leq 2n-3) \quad \text{and} \quad m = 2n-1.$$

For $2n = 4$ we say $c(v_1) = c(u_3) = 1$ and $c(v_4) = 2$. ■

Theorem 4.2. *If $G = K_2 \times C_r$ then $d(G,4) = 2\lceil \frac{r}{2} \rceil$.*

Proof. Let $G = K_2 \times C_r$. From Lemma 3.2 we obtain $d(G,4) \geq 2\lceil \frac{r}{2} \rceil$. To show equality we give a defining set, S of size $2\lceil \frac{r}{2} \rceil$.

Let $v_1, v_2, \cdots, v_r$ are the vertices of first row, $u_1, u_2, \cdots, u_r$ the vertices of the second row.

If $r = 2n$ we determine the defining set with their colors as follows:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1 (\text{mod } 8), \\ 2 & \text{if} \quad m \equiv 3 (\text{mod } 8), \\ 3 & \text{if} \quad m \equiv 5 (\text{mod } 8), \\ 4 & \text{if} \quad m \equiv 7 (\text{mod } 8) \end{cases}$$

except $m = 2n - 1$ when $n \equiv 1 (\text{mod } 4)$. In this case we set $c(v_{2n-1}) = 2$,

$$c(u_m) = \begin{cases} 3 & \text{if} \quad m \equiv 2 (\text{mod } 8), \\ 4 & \text{if} \quad m \equiv 4 (\text{mod } 8), \\ 1 & \text{if} \quad m \equiv 6 (\text{mod } 8), \\ 2 & \text{if} \quad m \equiv 0 (\text{mod } 8) \end{cases}$$

except $m = 2n - 2,\ m = 2n$ when $n \equiv 1(\text{mod } 4)$ and $m = 2n$ when $n \equiv 3(\text{mod } 4)$, in this case we say $c(u_{2n}) = 4,\ c(u_{2n-2}) = 3$ when $n \equiv 1(\text{mod } 4)$ and we say $c(u_{2n}) = 2$ when $n \equiv 3(\text{mod } 4)$.

If $r = 2n + 1$ we determine the defining set with their colors as in following tables:

$$c(v_m) = \begin{cases} 1 & \text{if} \quad m \equiv 1 (\text{mod } 8), \\ 2 & \text{if} \quad m \equiv 3 (\text{mod } 8), \\ 3 & \text{if} \quad m \equiv 5 (\text{mod } 8), \\ 4 & \text{if} \quad m \equiv 7 (\text{mod } 8) \end{cases}$$

except $m = 2n + 1$ when $n \equiv 0(\text{mod } 4)$. In this case we set $c(v_{2n+1}) = 2$,

$$c(u_m) = \begin{cases} 3 & \text{if} \quad m \equiv 2 (\text{mod } 8), \\ 4 & \text{if} \quad m \equiv 4 (\text{mod } 8), \\ 1 & \text{if} \quad m \equiv 6 (\text{mod } 8), \\ 2 & \text{if} \quad m \equiv 0 (\text{mod } 8) \end{cases}$$

except $m = 2n$ when $n \equiv 0, (\text{mod } 4)$ in this case $c(u_{2n}) = 3$ and $c(u_{2n+1}) = 4$ when $n \equiv 0$ or $1(\text{mod } 4)$, $c(u_{2n+1}) = 2$, when $n \equiv 2$ or $3(\text{mod } 4)$. ■

Corollary 4.3. $d(K_2 \times C_r, 5) = 2r$.

Proof. By Lemma 2.2, each of column has at least 2 vertices in defining set. Therefore all vertices are in defining set. ■

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