Discussiones Mathematicae
Graph Theory 26 (2006) 41–48

A CHARACTERIZATION OF PLANAR MEDIAN GRAPHS

Iztok Peterin *

Faculty of Electrical Engineering and Computer Science
University of Maribor
Smetanova ulica 17, 2000 Maribor, Slovenia

e-mail: iztok.peterin@uni-mb.si

Abstract

Median graphs have many interesting properties. One of them is—in connection with triangle free graphs—the recognition complexity. In general the complexity is not very fast, but if we restrict to the planar case the recognition complexity becomes linear. Despite this fact, there is no characterization of planar median graphs in the literature. Here an additional condition is introduced for the convex expansion procedure that characterizes planar median graphs.

Keywords: median graphs, planar graphs, expansion.

2000 Mathematics Subject Classification: 05C10, 05C75.

1. Introduction and Preliminaries

Partial cubes are isometric subgraphs of hypercubes and have been largely investigated, see the book [5] and the references therein. Most important subclass of partial cubes are median graphs. There are over 50 characterizations of median graphs, see the survey [7]. Both classes are also interesting from recognition point of view. In particular the recognition complexity for median graphs is closely connected with the recognition complexity of triangle free graphs, see [6, 5]. For planar median graphs the time complexity is linear [6]. Thus we can recognize for a given graph very fast whether it is

*This work was supported in part by the Ministry of Science of Slovenia under the grant P1-0297.

planar median or not. Despite this fact no characterization of planar median graphs is known.

Here we give a characterization of planar median graphs. For this we use the famous Mulder's convex expansion theorem [9, 10] and a special condition on it, which assures planarity. The same condition is not enough any more in the case of other graph classes that can be obtain by some other expansion procedure. For more about these classes of graphs we recommend [2] and the references therein.

The *distance* $d_G(u,v)$ between two vertices u and v in a graph G is defined as the number of edges on a shortest u,v-path. A subgraph H of G is called *isometric*, if $d_H(u,v) = d_G(u,v)$ for all $u,v \in V(H)$ and H is *convex* if for every $u,v \in V(H)$ all shortest u,v-paths belong to H. Convex subgraphs are clearly isometric.

The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ where the vertex (a,x) is adjacent to (b,y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. *Hypercubes* or *n-cubes* Q_n are Cartesian products of n copies of K_2. Isometric subgraphs of hypercubes are called *partial cubes*.

A graph G is a *median graph* if there exists a unique vertex x to every triple of vertices u,v, and w of G such that x lies on a shortest u,v-path, on a shortest u,w-path, and on a shortest v,w-path. Trees and n-cubes are median graphs.

Let G^1 and G^2 be a cover of a graph G with nonempty intersection $G^1 \cap G^2 = G'$. Note that there is no edge from $G^1 \backslash G'$ to $G^2 \backslash G'$. Graph H is an *expansion of G with respect to* G^1 and G^2 as follows. Take disjoint copies of G^1 and G^2 and connect every vertex from G' in G^1 with the same vertex of G' in G^2 with an edge. Such pairs of vertices will be called *expansions neighbors*. Expansion is *peripheral* if $G^1 = G$ (or $G^2 = G$). In that case $G' = G^2$ and we say that H is a peripheral expansion of G with respect to G'. We say that expansion is *convex* (*isometric, connected, arbitrary*) if G' is convex (isometric, connected, arbitrary). It is not hard to see that copies of G' in G^1 and in G^2 and new edges between those two copies form the Cartesian product $G' \Box K_2$.

In [9, 10] Mulder has shown that G is median if and only if it can be obtained from K_1 by a sequence of convex expansions, in [11] he has shown that we can restrict to pheripheral expansions, and in [3] Chepoi has shown that G is a partial cube if and only if it can be obtained from K_1 by a sequence of arbitrary expansions (if G_1 and G_2 are isometric subgraphs).

We say that a bipartite graph G satisfies *the quadrangle property* if for any vertices u, w, x, y of G with $d(u, x) = d(u, y) = k = d(u, w) - 1$ and w is a common neighbor of x and y, there exists a common neighbor v of x and y with $d(u, v) = k - 1$. Median graphs are precisely connected bipartite graphs that fulfill the quadrangle property and contain no induced $K_{2,3}$, cf. [7].

Graph G is *planar* if it can be drawn in the plane such that any two edges cross only in an endvertex (if they are incident with the same endvertex). Such drawings are called *plane drawings* of G. Any plane drawing of G divides the plane into regions which are called *faces*. One of those faces is unbounded and is called the *exterior* or the *outer* face, the others are *interior* or *inner* faces. Vertices that lie on an outer face are called *outer vertices* and other are *inner vertices*. Note that the boundary of every face of some plane drawing can be boundary of an outer face of some other plane drawing of the same graph.

A graph G is *outerplanar* if it is planar and embeddable into the plane so that all vertices lie on the outer face of the embedding. Such an embedding is called an *outerplanar embedding* of G. In [1] Behzad and Mahmoodian have shown that G is outerplanar if and only if $G \Box K_2$ is planar. For more information about planar graphs (or more general graphs on surfaces) we recommend [8].

2. Face Expansions

Vertex u of a graph G is a *cut vertex* if $G - u$ has more components as G, while edge e is a *bridge* if $G - e$ has more components as G. (We remove only edge e without endvertices.)

Let G be a planar graph. We construct graph G^- as follows. First delete all bridges from G. Let u be a cut vertex in the obtained graph. We delete u, add copies of u back to all components incident with u in the natural way and denote this graph with G_u^-. With G^- we denote the graph that remains from G after this procedure is executed for all cut vertices of G. For a tree T on n vertices we have totally disconnected graph on n vertices for T^-. Another example is on Figure 1.

We say that a face F is (*non*)*induced*, if the cycle that contain edges of F is (non)induced in G. Vertices of any induced cycle are clearly in the same component of G^-.

With this terminology we can write a simple lemma for drawings of a graph. The rather technical proof is omitted.

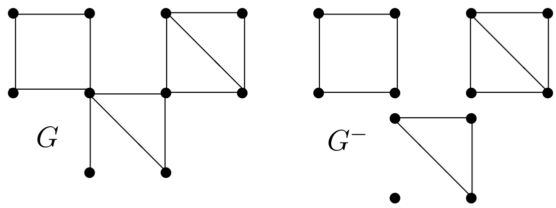

Figure 1. Graphs G and G^-.

Lemma 1. *Let $\mathcal{D}$ be a planar drawing of a graph G and F some noninduced face with boundary $u_0, u_1, \ldots, u_k$. Suppose that there exists edge u_0u_i, $i \in \{2, \ldots, k-1\}$. Then there exists a planar drawing $\mathcal{D}'$ of G with a face F' on vertices $u_0, u_i, u_{i+1}, \ldots, u_k$.*

Let H be an expansion of a planar graph G with respect to G^1 and G^2. Then H is a *face expansion* of G if all vertices of $G' = G^1 \cap G^2$ are on one face of some plane drawing of G. We need another lemma before our main result.

Lemma 2. *Let H be a peripheral expansion of a planar graph G with respect to G'. Suppose that among any two incident edges $e = uv$ and $f = vw$ from G' at least one is a bridge in G or v is a cut vertex and e and f are in different connected components of G^-. Then H is a peripheral face expansion of G with respect to G'.*

Proof. Clearly G' does not contain any cycle and is a tree. In each component C_i of G^- is at most one edge of G'. If there is such an edge e_i in C_i, choose any face F_i in C_i that contains e_i and draw C_i in such a way that F_i is an outer face of C_i. Draw all others components in any planar way. The only thing to consider is that every vertex, that is adjacent with a vertex in G' with a bridge, is drawn on an outer face. All edges of G' that are still missing in G^- are bridges. Return those bridges to G^- and join all copies of cut vertices of G'. Clearly G' is on outer face of this drawing. We add all other bridges of G to this drawing and join all other cut vertices of G that have been disjoint. The outer face of the obtained drawing of G still includes G' and thus H is a peripheral face expansion. ∎

Theorem 3. *A graph G is a planar median graph if and only if G can be obtained from K_1 by a sequence of convex peripheral face expansions.*

Proof. Suppose that G can be obtained from K_1 by a sequence of convex peripheral face expansions. Then G is median by Mulder's convex expansion theorem. We will show that face expansions preserve planarity by induction on the number of expansions. Let $H_0 = K_1$ and denote with H_k the graph obtained after k peripheral expansions with corresponding subgraph H'_k for the next peripheral expansion. Suppose that H_k is planar and that it is drawn in such a way that H_{k+1} can be obtained from H_k with a face expansion. Denote this face with F. Then H'_k is outerplanar and $H'_k \Box K_2$ is planar by the result of Behzad and Mahmoodian. Draw H_k and H'_k so that F is an outerface for both of them and that drawings do not intersect. Now just connect by an edge every vertex of H'_k in the drawing of H_k with the same vertex in the drawing of H'_k. Clearly this can be done so that a new drawing of H_{k+1} is planar. Hence G is planar.

Suppose now that G is planar and median. Then G can be obtain by a sequence of convex peripheral expansions from K_1 by Mulder's theorem. Assume that one of this expansions, say H_k to H_{k+1} with respect to H'_k, is not a face expansion for every drawing of a graph H_k. Choose index k to be the smallest of all such expansions and fix one drawing $\mathcal{D}$.

By Lemma 2 there are at least two incident edges in H'_k that are in the same component of H_k^-. We distinguish three cases:

Case 1. In H'_k are three edges that are all incident with vertex u_1 and are in the same component of H_k^-.

Let u_1 be a vertex in H'_k with at least three neighbors $u_2, u_3, u_4 \in H'_k$ in the same connected component C_j of H_k^-. Clearly these three edges are all not on the same face of $\mathcal{D}$.

We will use the following notation. Let $x_1, x_2, \ldots, x_p$, $y_1, y_2, \ldots, y_q$ and $z_1, z_2, \ldots, z_r$ be neighbors of u_1 in C_j such that x_i, y_i and z_i lie on $\mathcal{D}$ between u_2 and u_3, between u_3 and u_4, and between u_4 and u_2, respectively. Denote with u_5 the expansion neighbor of u_1. We claim that $\{u_1, u_2, u_3, u_4, u_5\}$ form a subdivision of K_5—a contradiction with planarity of G.

Vertex u_1 is a neighbor of u_2, u_3, u_4, and u_5. Let u'_2, u'_3, and u'_4 be expansions neighbors of u_2, u_3, and u_4 in H_{k+1}, respectively. Then $u_5u'_2u_2$, $u_5u'_3u_3$, and $u_5u'_4u_4$ are edge disjoint paths from u_5 to u_2, u_3, and u_4 respectively. Even more, none of edges on those paths are in H_k.

For $u_2 = x_0$ and $u_3 = x_{p+1}$ the path $x_iu_1x_{i+1}$ is on the same face F_i^{23}, for $i \in \{0, 1, \ldots, p\}$. Then $F_0 \backslash \{u_2u_1u_3\}$ is a u_2, u_3-path if $p = 0$ and the symmetric sum

$$F_0^{23} \backslash \{u_2 u_1\} \oplus F_p^{23} \backslash \{u_3 u_1\} \oplus_{i=1}^{p-1} F_i^{23}$$

forms a u_2, u_3-path if $p > 0$. Analogously for $u_3 = y_0$ and $u_4 = y_{q+1}$ the path $y_i u_1 y_{i+1}$ is on the same face F_i^{34}, for $i \in \{0, 1, \ldots, q\}$. Then $F_0^{34} \backslash \{u_3 u_1 u_4\}$ is a u_3, u_4-path if $q = 0$ and the symmetric sum

$$F_0^{34} \backslash \{u_3 u_1\} \oplus F_q^{34} \backslash \{u_4 u_1\} \oplus_{i=1}^{q-1} F_i^{34}$$

forms a u_3, u_4-path if $q > 0$. And finally for $u_4 = z_0$ and $u_2 = z_{r+1}$ the path $z_i u_1 z_{i+1}$ is on the same face F_i^{42}, for $i \in \{0, 1, \ldots, r\}$. Then $F_0^{42} \backslash \{u_4 u_1 u_2\}$ is a u_4, u_2-path if $r = 0$ and the symmetric sum

$$F_0^{42} \backslash \{u_4 u_1\} \oplus F_r^{42} \backslash \{u_2 u_1\} \oplus_{i=1}^{r-1} F_i^{42}$$

forms a u_4, u_2-path if $r > 0$. Clearly all these paths are disjoint and we have a subdivision of K_5 in H_{k+1}^- and thus in G, which is impossible.

Case 2. H_k' has two incident edges that are in the same connected component of H_k^- and are not on a four cycle.

We will show that no two such edges of H_k' lie on the same face. Assume contrary that edges $e = uv$ and $f = vw$ that are in the same component C_j of H_k^- are on the same face F of $\mathcal{D}$. Then there exists some u, w-path other than uvw. Among all such paths choose the shortest one and denoted with P. Clearly P has even length and cycle $wvuP$ is isometric. Let x be a middle vertex of P. For $|P| = 2$ vertices $uvwx$ form a four cycle and if $|P| > 2$ there exist a vertex z so that $uvwz$ form a four cycle by the quadrangle property for x, u, w, v. In each case H_k' contains a cycle, since the expansion is convex, contrary to the assumption.

Thus there are two edges $e = u_1 u_2$ and $f = u_1 u_3$ in H_k' which are not on the same face in $\mathcal{D}$. Denote with $x_1, x_2, \ldots, x_p$ and $y_1, y_2, \ldots, y_q$ neighbors of u_1 in C_j in H_k^-, where x_i are on the one side and y_j on the other side of the path $u_2 u_1 u_3$ on drawing $\mathcal{D}$. Since u_2 and u_3 are not on the same face of $\mathcal{D}$, we have $p \geq 1$ and $q \geq 1$. Let $v_1 = x_1$, $v_2 = y_1$, and v_3 be the expansion neighbor of u_1. We will show that $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ form a partition of subdivision of $K_{3,3}$.

Vertex u_1 is a neighbor of v_1, v_2, and v_3. Let u_2' and u_3' be expansions neighbors of u_2 and u_3 in H_{k+1}, respectively. Then $v_3 u_2' u_2$ and $v_3 u_3' u_3$ are edge disjoint paths from v_3 to u_2 and u_3, respectively. Even more, none of edges on those paths are in H_k.

For $u_2 = x_0$ and $u_3 = x_{p+1}$ path $x_i u_1 x_{i+1}$ is on the same face F_i of C_j, for $i \in \{0, 1, \ldots, p\}$. The path $F_0 \backslash \{u_2 u_1 v_1\}$ is a v_1, u_2-path, the path $F_1 \backslash \{u_2 u_1 v_1\}$ is a v_1, u_3-path if $p = 1$, and the symmetric sum

$$F_1 \backslash \{u_1 v_1\} \oplus F_p \backslash \{u_1 u_3\} \oplus_{i=2}^{p-1} F_i$$

is a v_1, u_3-path if $p > 1$. Analogously for $u_3 = y_0$ and $u_2 = y_{q+1}$ the path $y_i u_1 y_{i+1}$ is on the same face E_i, for $i \in \{0, 1, \ldots, q\}$. The path $E_0 \backslash \{u_3 u_1 v_2\}$ is a v_2, u_3-path, the path $E_1 \backslash \{u_3 u_1 v_2\}$ is a v_2, u_2-path if $q = 1$, and the symmetric sum

$$E_1 \backslash \{u_1 v_2\} \oplus E_q \backslash \{u_1 u_2\} \oplus_{i=2}^{q-1} E_i$$

is a v_2, u_2-path.

All these paths are edge disjoint and we have a subdivision of $K_{3,3}$ in H_{k+1}^- and thus in G, which is impossible.

Case 3. H_k' has a four cycle $C_4 = uvxy$.

Suppose that this cycle is not a boundary of any face on any drawing of G. We claim that at most two incident edges of H_k' are on the same face. Indeed, if there are three edges of H_k' on the face F, F is not induced. By Lemma 1 there exists a drawing $\mathcal{D}'$ where H_k' is a boundary of a face. We do the same if two nonincident edges lie on the same face. However this is not enough. We have to show also that any two incident edges of H_k' are not on the same face. Suppose they are. Without loss of generality we may assume that there exist an u, x-path P and a v, y-path Q. Denote with P' the shortest u, x-path that does not contain vertices v and y. P' must have even length. If $P' = uwx$ has length two, vertices u, v, x, y, w form a $K_{2,3}$ which is impossible. So let $|P'| > 2$. Then there exists a common neighbor z of u and x by the quadrangle property for vertices u, x, v, and for the middle vertex of P'. Again u, v, x, y, z form a $K_{2,3}$, a contradiction for a median graph.

As in Case 2 we thus have two edges $e = u_1 u_2$ and $f = u_1 u_3$ in H_k' which are not on the same face in $\mathcal{D}'$. We proceed as in Case 2 and the proof is complete. ■

As already mentioned the same argument does not hold for graphs obtainable from K_1 by a sequence of isometric (connected, any) expansions. Counterexample due to Klavžar for isometric expansion is a graph H that is an isometric expansion of the cube Q_3 with respect to two graphs Q_3^-, so that Q_3' is isometric C_6. (Q_3^- is a cube Q_3 minus a vertex.) Clearly H is planar but the mentioned expansion is not a face expansion and also can not be obtained from face expansions.

Acknowledgment

The author wish to thanks to Sandi Klavžar for several discussions and useful comments to the text.

REFERENCES

[1] M. Behzad and E.S. Mahmoodian, *On topological invariants of the product of graphs*, Canad. Math. Bull. **12** (1969) 157–166.

[2] B. Brešar, W. Imrich, S. Klavžar, H.M. Mulder and R. Škrekovski, *Tiled partial cubes*, J. Graph Theory **40** (2002) 91–103.

[3] V.D. Chepoi, *Isometric subgraphs of Hamming graphs and d-convexity*, (Russian) Kibernetika (Kiev) (1988) 6–9; translation in Cybernetics **24** (1988) 6–11.

[4] D. Djoković, *Distance preserving subgraphs of hypercubes*, J. Combin. Theory (B) **14** (1973) 263–267.

[5] W. Imrich and S. Klavžar, Product Graphs: Structure and Recognition (John Wiley & Sons, New York, 2000).

[6] W. Imrich, S. Klavžar and H.M. Mulder, *Median graphs and triangle-free graphs*, SIAM J. Discrete Math. **12** (1999) 111–118.

[7] S. Klavžar and H.M. Mulder, *Median graphs: characterization, location theory and related structures*, J. Combin. Math. Combin. Comp. **30** (1999) 103–127.

[8] B. Mohar and C. Thomassen, Graphs on Surfaces (Johns Hopkins University Press, Baltimore, MD, 2001).

[9] H.M. Mulder, *The structure of median graphs*, Discrete Math. **24** (1978) 197–204.

[10] H.M. Mulder, The Interval Function of a Graph (Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980).

[11] H.M. Mulder, The expansion procedure for graphs, Contemporary methods in graph theory 459–477 Bibliographisches Inst. Mannheim, 1990.

[12] P.M. Winkler, *Isometric embedding in products of complete graphs*, Discrete Appl. Math. **7** (1984) 221–225.

Received 7 October 2004
Revised 21 April 2005