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EXTENSION OF SEVERAL SUFFICIENT CONDITIONS FOR HAMILTONIAN GRAPHS

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Abstract

Let G be a 2-connected graph of order n. Suppose that for all 3-independent sets X in G, there exists a vertex u in X such that $|N(X\backslash\{u\})| + d(u) \geq n - 1$. Using the concept of dual closure, we prove that

1. G is hamiltonian if and only if its 0-dual closure is either complete or the cycle C_7
2. G is nonhamiltonian if and only if its 0-dual closure is either the graph $(K_r \cup K_s \cup K_t) \vee K_2$, $1 \leq r \leq s \leq t$ or the graph $(\frac{n+1}{2})K_1 \vee K_{\frac{n-1}{2}}$.

It follows that it takes a polynomial time to check the hamiltonicity or the nonhamiltonicity of a graph satisfying the above condition. From this main result we derive a large number of extensions of previous sufficient conditions for hamiltonian graphs. All these results are sharp.

Keywords: hamiltonian graph, dual closure, neighborhood closure.

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1. Introduction

We use Bondy and Murty for terminology and notation not defined here and consider simple graphs only $G = (V, E)$. By n, α and κ we denote the order, the independence and the vertex-connectivity number of G. If $A \subset V$, we denote by $G[A]$ the subgraph induced by A.

The *closed neighborhood* and the *degree* of a vertex u are denoted $N[u] = \{u\} \cup N(u)$ and $d(u)$ respectively. For $S \subset V$ and $a \in V \backslash S$, we denote by $N_S(a)$ ($d_S(a)$ resp.) the set (the number resp.) of neighbors of a in S. For $1 \leq k \leq \alpha$, we put $I_k = \{Y \mid Y \text{ is a } k\text{-independent set}\}$. As in [1], for each pair (a, b) of nonadjacent vertices of a graph G we associate

$$\gamma_{ab}(G) := |N_G(a) \cup N_G(b)|, \ \lambda_{ab}(G) := |N_G(a) \cap N_G(b)|,$$
$$T_{ab}(G) := V \backslash (N_G[a] \cup N_G[b]), \ t_{ab} := |T_{ab}|, \overline{\alpha}_{ab} := 2 + t_{ab},$$
$$\delta_{ab} := \min\{d(x) \mid x \in T_{ab}\} \text{ if } T_{ab} \neq \varnothing \text{ and } \delta_{ab} := \delta(G) \text{ otherwise.}$$

For any set $X = \{x_1, x_2, x_3\} \in I_3(G)$, we denote by $\lambda_{\min}(X)$, $\lambda_{med}(X)$ and $\lambda_{\max}(X)$ the smallest, the median and the greatest value in $\{\lambda_{x_1x_2}, \lambda_{x_2x_3}, \lambda_{x_3x_1}\}$ respectively. Moreover we set $X_i = X \backslash \{x_i\}$, $\sigma_X = \sum_{x \in X} d(x)$, $\lambda_X = \lambda_{x_1x_2} + \lambda_{x_2x_3} + \lambda_{x_3x_1}$ and $s_3(X) = |N(x_1) \cap N(x_2) \cap N(x_3)|$. Obviously $s_3 \leq \lambda_{\min} \leq \lambda_{med} \leq \lambda_{\max}$. If no confusion arises, we omit the arguments (G) and (X).

2. Preliminary Results

In [7], Bondy and Chvátal introduced the concept of the *k-closure* for several graph properties. For hamiltonian graphs the *n-closure* generalizes six earlier sufficient degree conditions. In [1], Ainouche and Christofides introduced the 0-*dual closure* $c_0^*(G)$ as an extension of the *n-closure.* Schiermeyer [15] showed that $c_0^*(G)$ is complete whenever G satisfies four more sufficient conditions for hamiltonian graphs. The first author ([3]) improved recently the closure condition given in [1]. In this paper, a relaxation of this strong condition is used. To state it, we need to introduce a binary variable ε_{ab}.

Definition 2.1. Let $\varepsilon_{ab} \in \{0, 1\}$ be a binary variable, associated with a pair (a, b) of nonadjacent vertices. We set $\varepsilon_{ab} = 0$ if and only if

1. $\varnothing \neq T_{ab}$ and all vertices of T_{ab} have the same degree $1 + t_{ab}$,
2. one of the following two local configurations holds
   (a) T_{ab} is a clique (possibly with one element), $\lambda_{ab} \leq 2$ and there exist $u, v \notin T_{ab}$ such that $T_{ab} \subset N(u) \cap N(v)$. Moreover $\lambda_{ab} \leq 1$ if either $\{u, v\} \subset N(a) \backslash N(b)$ or $\{u, v\} \subset N(b) \backslash N(a)$.

(b) T_{ab} is an independent set (with at least two elements), $\lambda_{ab} \leq 1 + t_{ab}$ and either $N(T_{ab}) \subseteq N(a) \cap N(b)$ or there exists a vertex $u \in N(a) \triangle N(b)$ such that $|T_{ab} \backslash N_{T_{ab}}(u)| \leq \max(\lambda_{ab} - 1, 0)$. Moreover T_{ab} is a clique in G^2, the square of G.

Lemma 2.2 (a neighborhood closure condition)**.** *Let G be a 2-connected graph and let (a, b) be a pair of nonadjacent vertices satisfying the condition*

(ncc) $\qquad \overline{\alpha}_{ab} \leq \delta_{ab} + \varepsilon_{ab}$ *(or equivalently $\gamma_{ab} + \delta_{ab} \geq n - \varepsilon_{ab}$).*

Then G is hamiltonian if and only if $(G + ab)$ is hamiltonian.

The 0-dual neighborhood closure $nc_0^*(G)$ is the graph obtained from G by successively joining (a, b) satisfying the condition (ncc) until no such pair remains. It is easy to see that $nc_0^*(G)$ is well defined. Moreover, it is shown in ([3]) that it takes a polynomial time to construct $nc_0^*(G)$ and to exhibit a longest cycle in G whenever a longest cycle is known in $nc_0^*(G)$.

For simplicity we sometimes say *neighborhood closure* instead of 0-*dual neighborhood closure*. As a direct consequence of Lemma 2.2 we have:

Corollary 2.3. *Let G be a 2-connected graph. Then G is hamiltonian if and only if $nc_0^*(G)$ is complete.*

To get an idea of the strength of Corollary 2.3, we describe two infinite families of hamiltonian graphs for which $nc_0^*(G)$ is complete. To our knowledge there is no known theorem from which we can draw the same conclusion. Let p, q be nonnegative integers.

Definition 2.4. Let $p \geq 2$ and $q \geq 2$ be integers, $A := \{a_0, a_1, ..a_{p+1}\}$, $X := \{x_1, .., x_p\}$ be two distinct independent sets and $B := \{b_0, b_1, ..b_q\}$ be a clique. A graph G is in $\mathcal{A}_1(p, q)$ if it is constructed from A, B, X as follows: $N[a_0] = A$, $G[\{a_1, .., a_p\}, X] = K_{p,p}$, $N(a_{p+1}) = \{a_0, b_1\} \cup \{x_1, .., x_{p-1}\}$ and $N(x_p) = \{a_1, .., a_p\} \cup \{b_q\}$.

It is easy to check that $n = 2p + q + 3$, $\alpha(G) = p + 2$ and $\kappa(G) = 2$. To construct the closure, we start with (a_0, b_q). Indeed $T_{a_0 b_q} = X \backslash \{x_p\}$ and (ncc) holds for this pair of nonadjacent vertices. Then choose (x_i, b_q), $i = 1, 2, .., p - 1$, as next pairs. It is now easy to check that $nc_0^*(G) = K_n$.

Definition 2.5. Let $p \geq 2$ and $q \geq 1$ be integers. Let $A := \{a_0, a_1, ..a_p\}$, $X := \{x_1, .., x_q\}$, $B := \{b_0, b_1, ..b_p\}$ be three distinct sets. A graph G is in $\mathcal{A}_2(p, q)$ if it is constructed from A, B, X as follows: $\{a_1, ..a_p\}, \{b_1, ..b_p\}$ are independent sets, $X \cup \{a_1, b_1\}$ is a clique, $N[a_0] = A$, $N[b_0] = B$, $G[\{a_2, .., a_p\}, \{b_2, .., b_p\}] = K_{p-1,p-1}$.

For this graph $n = 2(p+1) + q$, $\alpha(G) = p + 1$ and $\kappa(G) = 2$. To construct the closure, it is necessary to start with (a_i, b_1) or (b_i, a_1), $i = 2, .., p$.

Ainouche and Schiermeyer [4] proved that for a larger spectra of sufficient conditions for Hamiltonian graphs, the corresponding neighborhood closure $nc_0^*(G)$ is complete. In particular, the following results are obtained.

Theorem 2.6. *Let G be a 2-connected graph of order $n \geq 3$. Then $nc_0^*(G)$ is complete if*

$$X \in I_3(G) \Rightarrow \sigma_X \geq n + \lambda_{\min}(X). \tag{2.6}$$

Note that (2.6) is equivalent to the condition: for any $X \in I_3(G)$ there exists $u \in X$ such that $|N(X \backslash \{u\})| + d(u) \geq n$.

Theorem 2.7. *Let G be a 2-connected graph of order $n \geq 3$. Then $nc_0^*(G)$ is complete if*

$$X = \{x_1, x_2, x_3\} \in I_3(G) \Rightarrow \sum_{i=1}^{3} |N(X_i)| > 2(n-2). \tag{2.7}$$

Theorem 2.8. *Let G be a 2-connected graph of order $n \geq 3$. Then $nc_0^*(G)$ is complete if*

$$ab \notin E \Rightarrow 3\gamma_{ab} + \max\{2, \lambda_{ab}\} > 2(n-1). \tag{2.8}$$

In this paper, we go a step further by relaxing the condition of Theorem 2.6 by one unit. We prove that for a large spectra of conditions satisfied by a graph G, its 0-*dual neighborhood closure* is either complete or a maximal nonhamiltonian graph. For graphs satisfying these various sufficient conditions, the hamiltonian problem becomes polynomial.

3. Main Results

Our first result provides a common generalization of Theorems 2.7 and 2.8.

Theorem 3.1. *Let* $X = \{x_1, x_2, x_3\} \in I_3(G)$. *If*

$$\sum_{i=1}^{3} |N(X_i)| + \max\{2, \lambda_{\max}\} > 2(n-1) \tag{3.1}$$

then (2.6) *holds unless* $\lambda_{\min} = \lambda_{\max} = 1$, $\sigma_X = n$ *and* G *is constructed from* 3 *complete subgraphs* $G_i, i = 1, 2, 3$ *and a cycle* $C_6 := x_1, a_3, x_2, a_1, x_3, a_2, x_1$ *as follows: each vertex of* $\{x_i, a_{i+1}, a_{i+2}\}$, *where the addition is modulo* 3, *is joined to each vertex of* G_i. *In any case* $nc_0^*(G)$ *is complete if* (3.1) *holds for all* $X \in I_3(G)$.

To state our new results, we define the nonhamiltonian graphs $G(r, s, t)$ and G_1 as respectively the graphs $(K_r \cup K_s \cup K_t) \vee K_2$, $1 \le r \le s \le t$ and $\overline{K}_{(\frac{n+1}{2})} \vee K_{(\frac{n-1}{2})}$, where $\overline{K}_{(\frac{n+1}{2})} = (\frac{n+1}{2})K_1$. Moreover the graph C_7 is the cycle on 7 vertices.

Theorem 3.2. *Let* G *be a* 2*-connected graph. If*

$$X \in I_3(G) \Rightarrow \sigma_X \ge n - 1 + \lambda_{\min}(X) \tag{3.2}$$

then $nc_0^*(G) \in \{C_7, K_n, G(r, s, t), G_1\}$.

An immediate consequence of Theorem 3.2 is:

Corollary 3.3. *Let* G *be a* 1*-tough graph satisfying the condition* (3.2). *Then* $nc_0^*(G) \in \{C_7, K_n\}$.

The next corollary can be considered as an equivalent statement of Theorem 3.2.

Corollary 3.4. *Let* G *be a* 2*-connected graph of order* $n \ge 3$ *satisfying the condition* (3.2). *Then* G *is hamiltonian if and only if* $nc_0^*(G) \in \{C_7, K_n\}$ *and* G *is nonhamiltonian if and only if* $nc_0^*(G) \in \{G(r, s, t), G_1\}$.

4. Corollaries

It happens that Theorem 3.2 covers a large spectra of new results. In particular, it generalizes all the 16 following sufficient conditions.

Corollary 4.1. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow \sum_{i=1}^{3} |N(X_i)| + \lambda_{\max} > 2(n-2) \tag{4.1}$$

then $nc_0^*(G) \in \{C_7, K_n, G(r,s,t), G_1\}$.

Under the condition $\sum_{i=1}^{3} |N(X_i)| + \lambda_{\max} > 2(n-1)$, G is hamiltonian and $nc_0^*(G)$ is complete. This is a new condition.

Corollary 4.2. *Let G be a 2-connected graph of order $n \geq 3$. Assume $d(x_1) \leq d(x_2) \leq d(x_3)$ for all $X = \{x_1, x_2, x_3\} \in I_3(G)$. If*

$$\gamma_{x_1x_2} + d(x_3) \geq n-1 \tag{4.2}$$

then $nc_0^*(G) \in \{K_n, G(r,s,t), G_1\}$.

Under the condition $X \in I_3(G) \Rightarrow \gamma_{x_1x_2} + d(x_3) \geq n$, $nc_0^*(G)$ is complete. This is a new condition.

Corollary 4.3. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow \sum_{i=1}^{3} |N(X_i)| + \sigma_X > 3(n-2) \tag{4.3}$$

then $nc_0^*(G) \in \{C_7, K_n, G(r,s,t), G_1\}$.

Under the condition $\sum_{i=1}^{3} |N(X_i)| + \sigma_X > 3(n-1)$, G is hamiltonian [2] and $nc_0^*(G)$ is complete [4].

Corollary 4.4. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow |N(X)| + \lambda_{\max} \geq n-1 \tag{4.4}$$

then $nc_0^*(G) \in \{K_n, G(r,s,t), G_1\}$.

Under the condition $|N(X)| + \lambda_{\max} \geq n$, $nc_0^*(G) = K_n$. This is a new condition.

Corollary 4.5. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow |N(X)| + \sigma_X > 2(n-2) \tag{4.5}$$

then $nc_0^*(G) \in \{K_n, G(r,s,t), G_1\}$.

Under the condition $|N(X)| + \sigma_X > 2(n-1)$, G is hamiltonian [2] and $nc_0^*(G)$ is complete [4].

Corollary 4.6. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow \sum_{i=1}^{3} |N(X_i)| \geq 2(n-2) \tag{4.6}$$

then $nc_0^*(G) \in \{C_7, K_n, G(r,s,t)\}$.

Under the condition $\sum_{i=1}^{3} |N(X_i)| > 2(n-2)$, G is hamiltonian [2] and $nc_0^*(G)$ is complete [4]. The condition $\sum_{i=1}^{3} |N(X_i)| > 2(n-1)$ implying hamiltonicity appeared in [14].

Corollary 4.7. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow 3\gamma_{ab} + \max\{2, \lambda_{ab}\} > 2(n-2) \tag{4.7}$$

then $nc_0^*(G) \in \{K_n, G(r,s,s), G_1\}$ with $s \in \{r, r+1\}$.

Under the condition $3\gamma_{ab} + \max\{2, \lambda_{ab}\} > 2(n-1)$, G is hamiltonian [12] and $nc_0^*(G)$ is complete [4].

Corollary 4.8. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow \gamma_{ab} + \delta_{ab} \geq n-1 \tag{4.8}$$

then $nc_0^*(G) \in \{K_n, G(r,s,t), G_1\}$.

Under the condition $\gamma_{ab} + \delta_{ab} \geq n$, G is hamiltonian [1] and $nc_0^*(G)$ is complete [4].

Corollary 4.9. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow \gamma_{ab} + \max\{d(a), d(b)\} \geq n - 1 \tag{4.9}$$

then $nc_0^*(G) \in \{K_n, G(r,s,s), G_1\}$.

Under the condition $\gamma_{ab} + \max\{d(a), d(b)\} \geq n$, G is hamiltonian [12] and $nc_0^*(G)$ is complete [4].

Corollary 4.10. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow 3\gamma_{ab} \geq 2(n-2) \tag{4.10}$$

then $nc_0^*(G) \in \{K_n, G(r,r,r)\}$.

Under the condition $3\gamma_{ab} > 2(n-2)$, G is hamiltonian [2] and $nc_0^*(G)$ is complete [4]. This is an improvement of the condition $3\gamma_{ab} > 2(n-1)$ given in [13] and in [11].

Corollary 4.11. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$X \in I_3(G) \Rightarrow 2\sigma_X > 3(n-2) \tag{4.11}$$

then $nc_0^*(G) \in \{K_n, G(1,s,t), G_1\}$ with $3 \leq s+t \leq 4$.

Under the condition $2\sigma_X > 3(n-1)$, G is hamiltonian [6] and $nc_0^*(G)$ is complete [4].

Corollary 4.12. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow 2\gamma_{ab} + d(a) + d(b) > 2(n-2) \tag{4.12}$$

then $nc_0^*(G) \in \{K_n, G(r,s,s), G_1\}$ with $s \in \{r, r+1\}$.

Under the condition $2\gamma_{ab} + d(a) + d(b) > 2(n-1)$, G is hamiltonian [8] and $nc_0^*(G)$ is complete [4].

Corollary 4.13. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow d(a) + d(b) \geq n - 1 \tag{4.13}$$

then $nc_0^*(G) \in \{K_n, G_1\}$.

Under the condition $d(a) + d(b) \geq n$, G is hamiltonian [16] and $nc_0^*(G)$ is complete [4].

Corollary 4.14. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$ab \notin E \Rightarrow \gamma_{ab} + \delta(G) \geq n - 1 \tag{4.14}$$

then $nc_0^*(G) \in \{K_n, G(r, r, r), G_1\}$.

Under the condition $\gamma_{ab} + \delta(G) \geq n$, G is hamiltonian [1], [10] and $nc_0^*(G)$ is complete [4].

Corollary 4.15. *Let G be a 2-connected graph of order $n \geq 3$. If*

$$\delta(G) \geq \frac{n-1}{2} \tag{4.15}$$

then $nc_0^*(G) \in \{K_n, G_1\}$.

Under the condition $\delta(G) \geq \frac{n}{2}$, G is hamiltonian [9] and $nc_0^*(G)$ is complete.

Corollary 4.16. *Let G be a κ-connected graph, $\kappa \geq 2$, of order $n \geq 3$. If*

$$ab \notin E \Rightarrow \gamma_{ab} + \kappa \geq n - 1 \tag{4.16}$$

then $nc_0^*(G) \in \{K_n, G_1\}$.

Remark 4.17. All the above results are sharp. Each one of the conditions (4.1) to (4.16), once relaxed by one unit, is satisfied either by the Petersen graph or the graph $(mK_1 \cup K_2) \vee K_m$, $m \geq 3$.

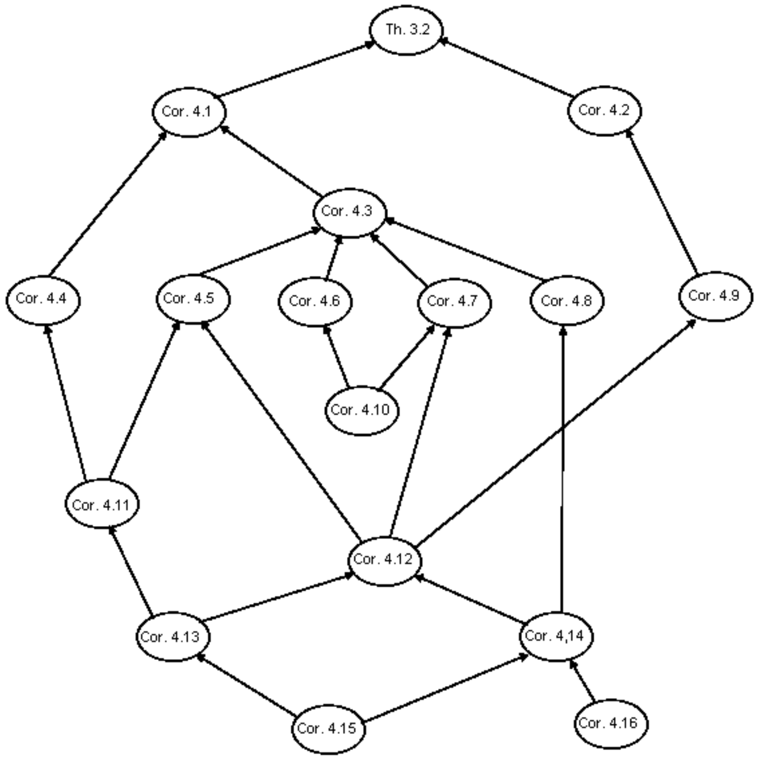

Figure 1. Hierarchy among the sufficient conditions considered in this paper.

5. Proofs

***Proof of Theorem* 3.1.**

Case 1. $\lambda_{\max} \geq 2$.
If $\sum_{i=1}^{3} |N(X_i)| + \max\{2, \lambda_{\max}(X)\} > 2(n-1)$ then $2\sigma_X - \lambda_X + \lambda_{\max}(X) > 2(n-1)$ and hence $\sigma_X \geq n + \lambda_{\min}$ since $\lambda_{\min} \leq \lambda_{med}$.

Case 2. $\lambda_{\max} \leq 1$ and $\lambda_X \leq 2$.
Now $\sum_{i=1}^{3} |N(X_i)| > 2(n-2)$ is equivalent to $2\sigma_X > 2(n-2) + \lambda_X$. If $\lambda_X = 0$ then $\sigma_X = |N(X)| \geq n-1$, a contradiction since $|N(X)| \leq n-3$. If $\lambda_X = 1$ then $\sigma_X = |N(X)| + 1 \geq n-1$, a contradiction since $|N(X)| + 1 \leq n-2$. If $\lambda_X = 2$ then necessarily $\lambda_{\min} = 0$ and $\sigma_X \geq n = n + \lambda_{\min}$.

Case 3. $\lambda_{\min} = \lambda_{\max} = 1$ (i.e., $\lambda_X = 3$). In this case, we have $\sigma_X = n$ for otherwise (2.6) holds. For this particular configuration, we proved in [4] that $nc_0^*(G)$ is complete. ■

***Proof of Theorem* 3.2.** Set $H := nc_0^*(G)$ and assume $H \neq K_n$. We note that if (3.2) holds for G it also holds for H. An independent triple S whose degree sum σ_S is minimum will be called a *suitable set.* Moreover a pair (a, b) of nonadjacent vertices is *critical* if $\overline{\alpha}_{ab} = \delta_{ab} + 1$ and $\varepsilon_{ab} = 0$. Consider a suitable set $S = \{a, b, x\}$ and assume without loss of generality, $\lambda_{ab} = \lambda_{\min}(S)$. By hypothesis we have $\sigma_S \geq n + \lambda_{ab} - 1 \Leftrightarrow \gamma_{ab} + d(x) \geq n - 1$. By the choice of S we must have $d(x) = \delta_{ab}$. Thus $\gamma_{ab} + \delta_{ab} \geq n - 1$. On the other hand, $ab \notin E(H) \Rightarrow \gamma_{ab} + \delta_{ab} \leq n - \varepsilon_{ab} - 1$. It follows that $\varepsilon_{ab} = 0$, $\gamma_{ab} + \delta_{ab} = n - 1$ and hence

(1) (a, b) is *critical* and T_{ab} satisfies the conditions of Definition 2.1.

For convenience we set $A := N(a)\backslash N(b)$, $B := N(b)\backslash N(a)$, $D := N(a) \cap N(b)$ where $|D| = \lambda_{ab}$, $T := T_{ab}$ and $t = |T|$. Since (a, b) is *critical*, T is either a clique or an independent set. Throughout the proof, x denotes an arbitrary vertex of T. Also and for any critical pair (y, z) of vertices we denote by respectively $u(y, z)$ and $v(y, z)$ the vertices u, v mentioned in Definition 2.1 (2.a) if T_{yz} is a clique.

Claim 5.1. If $N(T) \cap (A \cup B) = \varnothing$ then either $H = (K_r \cup K_s \cup K_t) \vee K_2$ or $H = \overline{K}_{(\frac{n+1}{2})} \vee K_{(\frac{n-1}{2})}$.

For all $x \in T$, $N(x) \cap N(a) \cap N(b) = D$ since $\lambda_{ab} \leq \min\{\lambda_{xa}, \lambda_{xb}\}$ and $N(T)\backslash T \subseteq D$. If T is a clique then $D = \{u, v\}$ where $u := u(a, b)$, $v := v(a, b) \notin T$. Moreover $\lambda_{bx} = \lambda_{ax} = \lambda_{ab}$ and replacing (a, b) by respectively (b, x) and (a, x) we obtain that $A \cup \{a\}$ and $B \cup \{b\}$ must be cliques and $\{u, v\} = \{u(b, x), v(b, x)\} = \{u(a, x), v(a, x)\}$. Moreover $uv \in E(H)$ since $T_{uv} = \varnothing$ in which case (ncc) holds. By setting $\{r, s\} = \{|A| + 1, |B| + 1\}$, we clearly have $H = (K_r \cup K_s \cup K_t) \vee K_2$.

Suppose next that T is an independent set with $t \geq 2$ (T is a clique if $t = 1$). Now $N(x) = D$ is true for any $x \in T$ since $\lambda_{ab} \leq \min\{\lambda_{ax}, \lambda_{bx}\}$. Also $A = B = \varnothing$ since by the choice of $\{a, b, x\}$ we cannot have for instance $d(a) > d(x)$ in which case $\sigma_{\{x_1, x_2, b\}} < \sigma_{\{a, b, x\}}$. To finish the proof in this case we show that $|D| = \frac{n-1}{2}$ and D is a clique in H. Since $d(x) = |D| = 1 + t$ and $n = 3 + 2t$, it follows that $|D| = \frac{n-1}{2}$ and $d(u) \geq \frac{n+1}{2}$ is true for all $u \in D$.

It is then easy to see that (ncc) holds for every pair $\{d, d'\}$ of nonadjacent vertices of D in H. Thus $H = \overline{K}_{(\frac{n+1}{2})} \vee K_{(\frac{n-1}{2})} = G_1$.

Claim 5.2. If $N(T) \cap (A \cup B) \neq \varnothing$ then there exists $u \in A \cup B$ such that $N_T(u) = T$ and (u, b) (resp. (u, a)) is critical if $u \in A$ (resp. $u \in B$).

Without loss of generality, assume $A \neq \varnothing$ and let $u \in A$ be a vertex satisfying the condition 2.b of Definition 2.1. This vertex exists since $\varepsilon_{ab} = 0$. Considering (u, b) we have:

$$\overline{\alpha}_{ub} \leq |\{b\}| + |A \setminus \{N(u)\}| + |T \setminus N_T(u)| \leq 1 + d(a) - \lambda_{ab} - d_A(u) + |T \setminus N_T(u)| .$$

On the other hand $\overline{\alpha}_{ub} \geq \delta_{ub} + \varepsilon_{ub} + 1$ by Lemma 2.2 since $ub \notin E(H)$. It follows that

$$d_A(u) + \lambda_{ab} + \delta_{ub} + \varepsilon_{ub} \leq d(a) + \max(0, \lambda_{ab} - 1) . \tag{2}$$

By the choice of S, $d(a) \leq d(a')$ is true for all $a' \in A$ for otherwise we set $S := \{a', b, x\}$ instead of $\{a, b, x\}$. Therefore $d(a) \leq \delta_{ub}$ and (2) leads to a number of obvious observations:

$$\left\{ \begin{array}{ll} \text{(i)} & \varepsilon_{ub} = 0 \text{ and } (u, b) \text{ is } \textit{critical}, \\ \text{(ii)} & N_T(u) = T \text{ and } N_A(u) = \varnothing, \\ \text{(iii)} & \lambda_{ab} = \lambda_{\min}(S) = 0 \text{ (ie } D = \varnothing), \\ \text{(iv)} & d(a') = d(a) \ \forall a' \in T_{ub} = A \setminus \{u\}. \end{array} \right. \tag{3}$$

Set $A := \{a_1, a_2, .., a_r\}$, $B := \{b_1, .., b_s\}$, $T := \{x_1, .., x_t\} = T_{ab}$ where $r = d(a)$, $s = d(b)$, $t \geq 2$. Also assume throughout $N_T(a_1) = T$ with $a_1 = u$ and set $A_i := A \setminus \{a_i\}$, $B_i := B \setminus \{b_i\}$ and $X_i := T \setminus \{x_i\}$.

For the remainder of the proof we may assume that $\lambda_{\min}(S) = 0$ is true for any suitable set S.

Claim 5.3. We may assume that (a_r, x_t) is critical.

Suppose first $N_T(a_i) = T$ for all $i \geq 1$. It is then clear that for all $i \geq 1$, (a_i, b) is critical and hence by (3), $d(a_i) = d(a)$ and $N_A(a_i) = \varnothing$, that is A is an independent set. If T is a clique then necessarily $a_1 = u(a, b)$, $a_2 = v(a, b)$ and $r = 2 = d(a) = d(a_1) = d(a_2)$. This leads to the conclusion that $t = 1$ and H is disconnected. If T is an independent set then $d(a) \leq d(x)$ for otherwise setting $S := \{x_1, x_2, b\}$ we have a contradiction. On the other hand $d(a_1) = d(a) \geq d(x)$ since $N(a_1) \supseteq \{a\} \cup T$ and hence $d(a_1) \geq 1 + t = d(x)$. Therefore $d(a) = d(x)$. It follows that for all $i \geq 1$, $N(a_i) = \{a\} \cup T$ and $N(x_i) = A$. Again H is disconnected. Assume for the remaining $a_r x_t \notin E(H)$. Clearly $\{a_r, x_t, b\}$ is a suitable set since $d(a_r) = d(a)$ and $d(x_t) = d(x)$. To prove the claim it suffices to show that $\lambda_{a_r x_t} = 0$. Otherwise choose any $y \in N(a_r) \cap N(x_t)$. If $y \in B$ then $\lambda_{\min}(\{a_r, x_t, b\}) \geq 1$, a contradiction to our assumption. If $y \in A$ then $y \in A_1 = T_{a_1 b}$ and A_1 must be a clique. In that case $a = u(a_1, b)$ and $x_t = v(a_1, b)$, implying $N(x_t) \supset A_1$, a contradiction since $a_r x_t \notin E(H)$. It remains to assume $y \in T$, in which case T must be a clique. In that case, $a_1 = u(a, b)$ and $a_r = v(a, b)$, implying $N(a_r) \supset T$, a contradiction since $a_r x_t \notin E(H)$. The proof of the claim is now complete.

Claim 5.4. *A_1 must be a clique.*

By contradiction suppose that A_1 is an independent set and $|A_1| \geq 2$, that is $r \geq 3$. By (3.iv), $d(a_i) = d(a)\ \forall a_i \in A_1$. Also $d(a_i) = d(a) \geq \max\{d(b), d(x)\}$ for if $d(a) < d(b)$, setting $S' := \{a_2, a_3, x\}$ we get a contradiction since then $\sigma_{S'} < \sigma_S$. A similar contradiction is obtained if $d(a) < d(x)$. As a first step, assume that T is an independent set with $t \geq 2$. Then $d(a) = d(x)$ for if $d(a) > d(x)$ we obtain a contradiction by setting $S := \{x_1, x_2, b\}$. Moreover if $d(b) = d(a) = d(x)$ then $\{x_1, x_2, a\}$ would be suitable with $\lambda_{\min\{x_1, x_2, a\}} \geq 1$, a contradiction. Therefore we are left with $d(b) < d(a) = d(x)$. This is a contradiction to the fact that $x_1, b \in T_{a_r x_t}$ and $\varepsilon_{a_r x_t} = 0$ by Claim 5.3. As (a, b) is critical we have to admit that T is a clique. Then we may assume $a_1 = u(a_r, x_t)$ and set $v := v(a_r, x_t)$. Now, $N_{A \cup B}(x_t) = \{a_1, v\}$. If $N_B(x_t) = \varnothing$ then $\{a_r, x_t, b\}$ is suitable and $\lambda_{x_t b} = 0$. It follows that (x_t, b) is critical and $T_{x_t b}$ must be a clique since $a a_r$ is an edge in $H[T_{x_t b}]$. Then we may assume $a_1 = u(x_t, b)$, in which case $a a_1 \in E(H) \Rightarrow a_r a_1 \in E(H)$, a contradiction to (3.ii). Thus we are left with $v \in B$. Set $v := b_1$. As a_i, $1 < i < r$ and b are in $T_{a_r x_t}$, we conclude that $d(a) = d(a_i) = d(b)$ since (a_r, x_t) is critical. Thus $|A| = |B|$, $N(a_r) = \{a\} \cup B$ and more precisely $N_B(a_r) = B \backslash \{b_1\}$ for if $a_r b_1$ then $\{a_r, x_t, b\}$ would be a suitable set with $\lambda_{\min\{a_r, x_t, b\}} \geq 1$. This conclusion must clearly hold for any vertex a_i, $1 < i < r$. If $a_1 b_1 \in E$ then we recognize that $H \in \mathcal{A}_2(p, q)$ by setting $p := r = s$ and $q := t$. It is easy to check that $H = K_n$ (for instance (ncc) applies to each pair $(a_i, b_1), i > 1$ and $(b_j, a_1), j > 1$). If $a_1 b_1 \notin E$ then $N(a_1) = \{a\} \cup T$. By the choice of S, $d(a) \leq d(a_1) = 1 + t = d(x)$. Thus $d(a) = d(b) = d(x)$ and a contradiction arises since $\{a_1, a_2, a_3\}$ is a suitable set with $\lambda_{\min\{a_1, a_2, a_3\}} \geq 1$. With this last contradiction we assume for the remainder of the proof that A_1 is a clique, possibly with one vertex.

Claim 5.5. *$H = G = C_7$.*

Since $T_{a_1b} = A_1$, we set $a := u(a_1, b)$ and $v := v(a_1, b)$. Clearly $v \neq x_t$ since $a_r x_t \notin E(H)$ by Claim 5.3. If $v = x_j$ for some $j < t$ then necessarily $t \geq 2$ and T must be an independent set. This is true for otherwise T would be a clique since (a, b) is critical and $x_j a_r \in E(H) \Rightarrow x_t a_r \in E(H)$. Therefore $v \in B$ and we may set $v = b_s$. Suppose first that T is an independent set with $t \geq 2$. By Claim 5.3, (a_r, x_t) is critical. Thus $d(x) = d(b)$ since $x_1, b \in T_{a_r x_t}$. But now we have a contradiction since $\{x_1, x_2, a\}$ is a suitable set with $\lambda_{\min\{x_1,x_2,a\}} \geq 1$. To finish the proof, we suppose that T is a clique. Thus we may assume $a_1 := u(a, b)$ and $b_1 := v(a, b)$. Clearly $v(a, b) \neq b_s$ for otherwise $\{x_1, a_r, b\}$ is a suitable set with $\lambda_{\min\{x_1,a_r,b\}} \geq 1$. Since $N_T(b_1) = T$ then by symmetry with a_1 we obtain that (a, b_1) is critical, $N_{B_1}(b_1) = \varnothing$ and B_1 must be a clique. But now $T_{b_s x_t} = \{a\}$ and hence $d(a) = 2$ by (ncc). By symmetry we have $d(b) = 2$. Furthermore $T_{b x_t} = \{b_1\}$ and hence $d(b_1) = 2$ by (ncc). This means that $N(b_1) = \{b, x\}$ with $T = \{x\}$. We have just proved that $a a_2 b_2 b b_1 x a_1 a = C_7$. The proof of Theorem 3.2 is now complete. ■

***Proof of the Corollaries* 4.1 *to* 4.16.** Set $H := nc_0^*(G)$. We first note that for the graph C_7 only the conditions (3.2), (4.1), (4.3), and (4.6) are satisfied. Moreover $H \neq G_1$ if G satisfies condition (4.6) or (4.10) unless $H = 3K_1 \vee K_2 = G(1,1,1)$.The restrictions on r, s, t in $G(r, s, t)$ appearing in some Corollaries are easily obtained by a simple counting argument. The remainder of the proof consists on establishing a number of implications among the conditions (3.2) and (4.1) to (4.16).

Claim 5.6. (4.9) $\Rightarrow$ (4.2) $\Rightarrow$ (3.2); (4.16) $\Rightarrow$ (4.14) $\Rightarrow$ (4.8) $\wedge$ (4.12), (4.10) $\Rightarrow$ (4.7) $\wedge$ (4.6), (4.13) $\Rightarrow$ (4.11) $\wedge$ (4.12), (4.12) $\Rightarrow$ (4.5) $\wedge$ (4.7) $\wedge$ (4.9) and (4.15) $\Rightarrow$ (4.13) $\wedge$ (4.14).

The proofs are straithforward. For the remaining we set $X = \{x_1, x_2, x_3\}$ if $X \in I_3(G)$ and $X = \{a, b\}$ if $X \in I_2(G)$.

Claim 5.7. (4.3) $\vee$ (4.4) $\Rightarrow$ (4.1) $\Rightarrow$ (3.2).

(i) $\sum_{i=1}^{3}|N(X_i)| + \lambda_{\max} > 2(n-2) \Leftrightarrow 2\sigma_X - \lambda_X + \lambda_{\max} > 2(n-2)$. This leads to $\sigma_X \geq n-1+\lambda_{\min}$ since $\lambda_X - \lambda_{\max} = \lambda_{\min} + \lambda_{med} \geq 2\lambda_{\min}$. Therefore (4.1) $\Rightarrow$ (3.2).

(ii) By contradiction suppose that (4.3) $\not\Rightarrow$ (4.1). Thus $\sum_{i=1}^{3}|N(X_i)| + \sigma_X > 3(n-2) \Leftrightarrow 3\sigma_X - \lambda_X > 3(n-2)$ while $\sum_{i=1}^{3}|N(X_i)| + \lambda_{\max} \leq 2(n-2)$ or equivalently $2\sigma_X - \lambda_X + \lambda_{\max} \leq 2(n-2)$. We obtain $2\lambda_{\max} < \lambda_{\min} + \lambda_{med}$, a contradiction.

(iii) To prove (4.4) $\Rightarrow$ (4.1), suppose by contradiction $|N(X)| + \lambda_{\max} > n-2$ while $\sum_{i=1}^{3}|N(X_i)| + \lambda_{\max} \leq 2(n-2)$. Then $\sum_{i=1}^{3}|N(X_i)| - |N(X)| < n-2)$. Expressing $\sum_{i=1}^{3}|N(X_i)|$ and $|N(X)|$ in terms of $s_i := |\{u \notin X \mid |N_X(u)| = i\}|$ (see [4]) we obtain $s_1 + 2s_2 + 2s_3 = \sigma_X - s_3 < n-2$. Thus $\sigma_X < n-2+\lambda_{\min}$ since obviously $s_3 \leq \lambda_{\min}$. This is a contradiction since $|N(X)| + \lambda_{\max} > n-2 \Leftrightarrow \sigma_X > n-2+\lambda_{\min}+\lambda_{med} > n-2+\lambda_{\min}$.

Claim 5.8. (4.5) $\vee$ (4.8) $\vee$ (4.7) $\vee$ (4.6) $\Rightarrow$ (4.3).

(i) (4.5) $\Rightarrow$ (4.3). Otherwise suppose $|N(X)| + \sigma_X > 2(n-2)$ while $\sum_{i=1}^{3}|N(X_i)| + \sigma_X \leq 3(n-2) \Leftrightarrow 3\sigma_X \leq 3(n-2) + \lambda_X$. As $|N(X)| = \sigma_X - \lambda_X + s_3(X)$ we obtain $3s_3(X) > \lambda_X$, a contradiction.

(ii) (4.6) $\Rightarrow$ (4.3). By contradiction suppose $\sum_{i=1}^{3}|N(X_i)| \geq 2(n-2)$ but $\sum_{i=1}^{3}|N(X_i)| + \sigma_X \leq 3(n-2)$. Then equivalently $(2s_1+3s_2+3s_3) \geq 2(n-2)$ but $(2s_1+3s_2+3s_3) + (s_1+2s_2+3s_3) \leq 3(n-2)$. From these inequalities we obtain $s_2 = s_3 = 0$ and hence $\sum_{i=1}^{3}|N(X_i)| = 2|N(X)| \geq 2(n-2) \Rightarrow |N(X)| \geq n-2$, a contradiction.

(iii) (4.7) $\Rightarrow$ (4.3). Suppose first $\lambda_{\min} \geq 1$.Then using (4.7) as in (ii), we get the required implication. Next suppose $0 = \lambda_{x_1x_2} \leq 1 \leq \lambda_{x_2x_3} \leq \lambda_{x_3x_1}$. Then $2\gamma_{x_1x_2} + 2\sigma_{x_1x_2} \geq 2(n-2)$, $2\gamma_{x_2x_3} + 2\sigma_{x_2x_3} > 2(n-2)$ and $2\gamma_{x_3x1} + 2\sigma_{x_3x_1} > 2(n-2)$. Adding these inequalities we get $2\sum_{i=1}^{3}|N(X_i)| + 2\sigma_X \geq 6n-10$, that is $\sum_{i=1}^{3}|N(X_i)| + \sigma_X \geq 3n-5$. Again we have (4.7) $\Rightarrow$ (4.3). For the next case, suppose $0 = \lambda_{x_1x_2} = \lambda_{x_2x_3} \leq 1 \leq \lambda_{x_3x_1}$. Now $2\sum_{i=1}^{3}|N(X_i)| + 2\sigma_X \geq 6n-11$, that is $\sum_{i=1}^{3}|N(X_i)| + \sigma_X \geq 3n-5$ and (4.7) $\Rightarrow$ (4.3). As a last case suppose $0 = \lambda_{x_1x_2} = \lambda_{x_2x_3} = \lambda_{x_3x_1}$. Now $\sum_{i=1}^{3}|N(X_i)| + \sigma_X = 3|N(X)| \geq 3n-6$. Therefore $|N(X)| \geq n-2$, a contradiction since obviously $|N(X)| \leq n-3$.

(iv) Using (4.8) for successively (x_1, x_2), (x_2, x_3), (x_3, x_1) and adding we directly prove the implication (4.8) $\Rightarrow$ (4.3).

Claim 5.9. (4.11) $\Rightarrow$ (4.4) $\wedge$ (4.5).

(i) (4.11) $\Rightarrow$ (4.4). Otherwise $2\sigma_X = 2(s_1 + 2s_2 + 3s_3) > 3(n-2)$ but $|N(X)| + \lambda_{\max} = s_1 + s_2 + s_3 + \lambda_{\max} \leq n-2$. Moreover $\lambda_{\max} \geq \lambda_X/3 \geq s_2/3 + s_3$. We reach a contradiction by getting on one hand $\frac{2}{3}s_1 + \frac{4}{3}s_2 + 2s_3 > n-2$ and on the other hand $s_1 + \frac{4}{3}s_2 + 2s_3 \leq n-2$.

(ii) (4.11) $\Rightarrow$ (4.5). Otherwise $\frac{2}{3}s_1 + \frac{4}{3}s_2 + 2s_3 > n-2$ but $|N(X)| + \sigma_X = 2s_1 + 3s_2 + 4s_3 \leq 2(n-2)$, that is $s_1 + \frac{3}{2}s_2 + 2s_3 \leq n-2$. Again we have a contradiction. ∎

6. Open Problems

These open problems are motivated by the two following results.

Theorem 6.1 [12]. *A 2-connected graph G of order $n \geq 3$ and satisfying the condition*

$$X \in I_3(G) \Rightarrow \sigma_X \geq n + s_3(X) \tag{6.1}$$

is hamiltonian.

Obviously (2.7) $\Rightarrow$ (6.1) since $s_3(X) \leq \lambda_{\min}(X)$.

Theorem 6.2 [5]. *Let G be a 2-connected non hamiltonian graph of order n. If*

$$X \in I_3(G) \Rightarrow \sigma_X \geq n - 1 + s_3(X) \tag{6.2}$$

then $nc_0^*(G) \in \{G_1, G(r,s,t)\}$.

Note that (3.2) $\Rightarrow$ (6.2).

Problem 6.3. Let G be a 2-connected graph satisfying (6.1). Then $nc_0^*(G)$ is complete.

Problem 6.3 is suggested by Lemma 2.2 and Theorem 6.1.

Problem 6.4. Let G be a 2-connected graph satisfying (6.2). Then $nc_0^*(G) \in \{C_7, K_n, G_1, G(r,s,t)\}$, $1 \leq r \leq s \leq t$.

Problem 6.4 is suggested by Theorems 3.2 and 6.2.

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