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HIGHLY CONNECTED COUNTEREXAMPLES TO A CONJECTURE ON α-DOMINATION *

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Abstract

An infinite class of counterexamples is given to a conjecture of Dahme *et al.* [Discuss. Math. Graph Theory, 24 (2004) 423–430.] concerning the minimum size of a dominating vertex set that contains at least a prescribed proportion of the neighbors of each vertex not belonging to the set.

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1. Introduction

Let α be a fixed real number, $0 < \alpha \leq 1$. In a graph $G = (V, E)$, an *α-dominating set* is a vertex subset $D \subseteq V$ such that each $v \in V \setminus D$ is adjacent to at least $\alpha \cdot d(v)$ vertices of D. (As usual, $d(v)$ denotes the degree of v.) The *α-domination number*, denoted $\gamma_\alpha(G)$, is the minimum cardinality of

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an α-dominating set in G. For every natural number k, let us define

$$\gamma_\alpha(n,k) = \max\{\gamma_\alpha(G) : |V(G)| = n,\ G \text{ is } k\text{-connected}\}.$$

For simplicity, we shall write $\gamma_\alpha(n)$ for $\gamma_\alpha(n,1)$. Since the α-domination number of a disconnected graph is just the sum of those of its components, we disregard the case $k = 0$ in the present context.

The graph invariant γ_α was introduced by Dunbar *et al.*, who proved among other results that $\gamma_{1/2}(n) = \lfloor n/2 \rfloor$ ([2, Corollary 10]). Dahme, Rautenbach and Volkmann [1] remarked that this can be extended for every $\alpha < 1$ to the more general inequality

$$\gamma_\alpha(n) \le n \cdot \left(1 - \frac{1}{\lceil \frac{1}{1-\alpha} \rceil}\right) \tag{1}$$

that follows from a result of Cowen and Emerson (Theorem 5.1 in [3]); a self-contained proof is given in Theorem 2.1 of [1].

The authors of [1] noted (Observation 2.4) that the general upper bound (1) is essentially tight, and formulated the open problem (Conjecture 2.5) that a substantial improvement to $\gamma_\alpha(G) \le \lceil \alpha \cdot (n-1) \rceil$ is possible whenever G is supposed to be 2-vertex-connected, for all $0 < \alpha < 1$.

The goal of the present note is to point out that the conjecture is false in a general sense. We prove in Theorem 1 that for almost all α, the — exact or asymptotic — upper bound αn cannot hold, no matter how high connectivity is assumed. In fact, all the positive exceptions are covered in Equation (1), namely those values where $\frac{1}{1-\alpha}$ is an integer.

The 2-connected case is discussed in Section 2, and then in Section 3 we show how to extend the ideas for higher connectivity. Some related comments and open problems are given in Section 4.

2. The 2-Connected Construction

Let $k \ge 3$ be any integer. Consider the following k-regular 2-connected graph, denoted $G_{m,k}$, for each $m \ge 2$. Begin with m vertex-disjoint copies $K^1, \ldots, K^m$ of $K_{k+1} - e$, i.e., one edge deleted from the complete graph of order $k + 1$. Let u_i, v_i denote the two nonadjacent vertices of K^i. We join these subgraphs in a cyclic manner, with the edges u_1v_2, u_2v_3, $\ldots$, $u_{m-1}v_m$, u_mv_1.

The graphs $G_{m,k}$ provide 2-connected counterexamples to the conjecture of [1] for almost all values of α. This fact will be deduced from the following observation.

Proposition 1. *If $\alpha > \frac{j}{k}$ for some $0 \le j \le k-2$, then* $\gamma_\alpha(G_{m,k}) \ge \frac{j+1}{k+1}\,|V(G_{m,k})|$.

Proof. Let D be an α-dominating set in $G = G_{m,k}$. The proof will be done if we show that each K^i contains at least $j+1$ vertices of D. Since G is k-regular, each vertex $v \notin D$ should be adjacent to at least $\lceil \alpha k \rceil \ge j+1$ vertices of D. Thus, if $|V(K^i) \cap D| \le j$, then $K^i - D \subseteq \{u_i, v_i\}$ should hold, which implies that D contains at least $\min(k-1, j+1) = j+1$ vertices of K^i. ■

From the above, we deduce

Theorem 1. *If α is not of the form $1 - \frac{1}{\ell}$ for some natural number ℓ, then there exists a constant $c = c(\alpha) > 0$ such that $\gamma_\alpha(n, 2) > (\alpha + c) \cdot n$ for every sufficiently large n.*

Proof. We consider the infinite class of graphs $G_{m,k}$, for $m \ge 2$ and $k \ge 2$. The assertion for values α in the open interval $\left(\frac{k-2}{k}, \frac{k-1}{k+1}\right)$ follows from Proposition 1, taking $j = k-2$. The values uncovered so far are of the form $\alpha = \frac{2\ell-1}{2\ell+1}$. For them, we observe

$$\frac{2\ell^2 - 1}{2\ell^2 + 2\ell} < \frac{2\ell - 1}{2\ell + 1} < \frac{2\ell^2}{2\ell^2 + 2\ell + 1}\,.$$

Thus, the proof can be completed by Proposition 1 with $k = 2\ell^2 + 2\ell$ and $j = 2\ell^2 - 1$. ■

3. Higher Connectivity

In this section we show how graphs of arbitrarily high connectivity can be constructed, to obtain counterexamples to the conjecture of [1]. Our goal will be to build an infinite class of k-regular k-connected graphs whose α-domination number is relatively large. Throughout this section, we restrict our attention to the case of k even.

The local $v \to K$ replacement. Let G be a k-connected graph, and v a vertex of degree k. We denote the neighbors of v with $v_1, \ldots, v_k$. The local replacement deletes v and inserts a subgraph $K \cong K_{k+1} - \frac{k}{2}K_2$, a maximum matching removed from the complete graph of order $k+1$. We shall adopt the convention that the edges $x_1x_2, x_3x_4, \ldots, x_{k-1}x_k$ have been removed from K_{k+1}, and its last vertex is x. We join this graph to $G - v$ with the k edges x_iv_i, $1 \le i \le k$.

The structure of the graph after this local replacement $v \to K$ may depend on the order how the neighbors of v are labeled. The next assertion ensures that a suitable labeling exists.

Lemma 1. *There is a way to perform the $v \to K$ replacement to yield a k-connected graph, whenever the initial graph G is k-connected.*

Proof. By assumption, $G - v$ has connectivity at least $k - 1 \ge k/2$. Let us split the ex-neighbors of v into two sets, say A and B, of size $|A| = |B| = k/2$. As a consequence of Menger's theorem, there exist $k/2$ disjoint paths from A to B. We label the vertices in $A \cup B$ in such a way that the ith path $(i = 1, \ldots, k/2)$ joins vertex v_{2i-1} with v_{2i}; and then join each x_i of K to v_i of $G - v$.

We are going to show that the graph remains k-connected after this modification, i.e., that there exist k internally disjoint paths between any two vertices. This property easily follows by the k-connectivity of G when at least one of the two vertices involved, say w, is outside K. Indeed, there are k disjoint paths from w to $\{x_1, \ldots, x_k\}$, that correspond to the original k paths from w to v in G. If the vertex to be reached from w is x, these k paths directly extend to the w–x paths required. And if it is some x_i, say x_1, then $k-2$ of the paths can be completed with the edges $x_1x_3, x_1x_4, \ldots, x_1x_k$, one with the path x_1xx_2, and one ends in x_1 itself.

Between x and x_1 there are $k - 2$ paths xx_ix_1 $(3 \le i \le k)$, the edge xx_1, moreover xx_2 completed with the v_1–v_2 path inside $G - v$. Similarly, between x_1 and x_2 there are $k - 1$ paths of length 2 inside K, and one path extending the v_1–v_2 path of $G - v$.

Finally, consider two adjacent vertices of $K - x$, say x_1 and x_3. They are joined by an edge, and by $k - 3$ paths of length 2 inside K. To obtain two further paths, we complete the v_1–v_2 path and the v_3–v_4 path of $G - v$ with the edge v_2v_3 and v_1v_4, respectively. By the assumption on the labeling, these paths are disjoint. ∎

We say that a $v \to K$ replacement is feasible if it keeps the graph k-connected.

The $G \to G^+$ construction. Starting with any k-regular k-connected graph G, a graph G^+ is obtained by subsequently applying a *feasible* $v \to K$ replacement for each vertex v of G. By the repeated application of the previous lemma, this can be done for any G.

Proposition 2. *Let G^+ be a k-regular k-connected graph obtained by the $G \to G^+$ construction. If $\alpha > \frac{j}{k}$, where $j \leq k-2$ is an even integer, then $\gamma_\alpha(G^+) \geq \frac{j+1}{k+1}|V(G^+)|$.*

Proof. Let D be an α-dominating set of G^+, and consider the local $v \to K$ replacement that has been performed at any vertex v of G. It will suffice to prove that the set $D_K := D \cap V(K)$ has at least $j+1$ elements.

Suppose for a contradiction that $|D_K| \leq j$. Then $x \in D_K$, because x has no neighbors outside K. Assume next $x_1 \notin D_K$. Since x_1 has just one neighbor outside K, it must have j neighbors in D_K. We obtain, in particular, that $x_2 \notin D_K$. It follows that a nonadjacent vertex pair of K is either completely inside D or completely outside of D. This leads to the contradiction that $|D_K \setminus \{x\}| = j-1$ should be even. ■

4. Concluding Remarks

(1) We have characterized the values α such that $\gamma_\alpha(n,2) \leq \alpha n$ for all n. It remains an open problem, however, to determine the exact or asymptotic value of $\gamma_\alpha(n,k)$ as a function of n, for $k \geq 2$. General lower bounds can be obtained from the constructions presented in Sections 2 and 3.

(2) The assertion of Proposition 1 does not hold true for $j = k-1$. Indeed, γ_α never exceeds the number of vertices *minus the independence number*, for any graph. This means, e.g. for m even, that an α-dominating set of $G_{m,k}$ is obtained by deleting u_i and v_i for each i odd, and one vertex of $K^i - u_i - v_i$ for each i even. In this way we obtain that the ratio of γ_α and the number of vertices in $G_{m,k}$ is $\frac{2k-1}{2k+2} < \frac{k-1}{k}$ for $k > 2$.

(3) Though stated in [2], it is not true that for $\alpha > 0$ every α-dominating set is a dominating set. Indeed, the latter must contain all isolated vertices, while the former need not. This is one reason why general upper bounds like (1) can be formulated without excluding isolated vertices.

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