KERNELS IN MONOCHROMATIC PATH DIGRAPHS

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Abstract

We call the digraph D an m-coloured digraph if its arcs are coloured with m colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike.

Let D be an m-coloured digraph. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the following two conditions:

  (i) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and

 (ii) for each vertex $x \in (V(D) - N)$ there is a vertex $y \in N$ such that there is an xy-monochromatic directed path.

In this paper is defined the monochromatic path digraph of D, $MP(D)$, and the inner m-colouration of $MP(D)$. Also it is proved that if D is an m-coloured digraph without monochromatic directed cycles, then the number of kernels by monochromatic paths in D is equal to the

number of kernels by monochromatic paths in the inner m-colouration of $MP(D)$. A previous result is generalized.

Keywords: kernel, line digraph, kernel by monochromatic paths, monochromatic path digraph, edge-coloured digraph.

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1.   Introduction

For general concepts we refer the reader to [1]. Let $D = (V(D), A(D))$ be a digraph, a set $K \subseteq V(D)$ is said to be a kernel if it is both independent (a vertex in K has no successor in K) and absorbing (a vertex not in K has a successor in K). This concept was introduced by Von Neumann [13] and it has found many applications (see for example [1, 2]). Several authors have been investigating sufficient conditions for the existence of kernels in digraphs, namely, Von Neumann and Morgenstern [13], Duchet [3], Duchet and Meyniel [4] and Galeana-Sánchez and Neumann-Lara [5].

In [12] M. Harminc considered the existence of kernels in the line digraph of a given digraph D, and he proved the following Theorem 1.1.

Theorem 1.1 [12]. *The number of kernels of a digraph D is equal to the number of kernels in its line digraph.*

An extension of Theorem 1.1 for semikernels, quasikernels and Grundy functions (concepts closely related to those of kernel) was considered in [6], where it was proved that: If D is a digraph such that $\delta_D^-(x) \geq 1$ for each $x \in V(D)$, then the number of semikernels (quasikernels) of a digraph D is less than or equal to the number of semikernels (quasikernels) of its line digraph; and the number of Grundy functions of D is equal to the number of Grundy functions of its line digraph. Another extension of Theorem 1.1 for (k, ℓ)-kernels (a concept which generalizes that of kernel) was proved in [8].

In [10] edge-coloured digraphs were considered and the following result similar to Theorem 1.1 was proved:

Let D be an m-coloured digraph without monochromatic directed cycles. The number of kernels by monochromatic paths of D is equal to the number of kernels by monochromatic paths in the inner m-colouration of its line digraph $L(D)$.

The main result of this paper (announced in the abstract) generalizes Theorem 1.1.

Definition 1.1 [9]**.** Let D be an m-coloured digraph. A set $N \subseteq V(D)$ is independent by monochromatic paths if for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them. The set $N \subseteq V(D)$ is absorbant by monochromatic paths if for every vertex $x \in (V(D) - N)$, there is a vertex $y \in N$ such that there exists an xy-monochromatic directed path. And $N \subseteq V(D)$ is a kernel by monochromatic paths if N is both independent and absorbant by monochromatic paths.

This concept was introduced in [9]. The existence of kernels by monochromatic paths in edge-coloured digraphs was studied primarily by Sauer, Sands and Woodrow in [14], where they proved that any 2-coloured digraph has a kernel by monochromatic paths. Sufficient conditions for the existence of kernels by monochromatic paths in m-coloured digraphs have been studied in [7, 9, 11, 15].

The monochromatic path digraph of D, is the digraph $MP(D) = (V(MP(D)), A(MP(D)))$, whose vertex set is the set of monochromatic directed paths of D of length at least one; and for $h, k \in V(MP(D))$, $(h, k) \in A(MP(D))$ if and only if the terminal endpoint of h is the initial endpoint of k. The inner m-colouration of $MP(D)$ is the edge-colouration of $MP(D)$ defined as follows: If h is a monochromatic directed path of D coloured c, then any arc of the form (x, h) in $MP(D)$ is also coloured c.

Throughout this paper we write mdp insted of monochromatic directed path of length at least one, and kmp instead of kernel by monochromatic paths. In what follows we denote the mdp $h = (x_0, x_1, \ldots, x_n)$, and the vertex $h \in V(MP(D))$ by the same symbol. If H is a subset of $\Pi = \{P | P$ is a mdp in $D\}$ it is also a set of vertices of $MP(D)$; when we want to emphasize our interest in $H \subseteq \Pi$ as a set of vertices of $MP(D)$, we use the symbol H_{MP} instead of H.

As usual we denote by $V(D)$ (resp. $A(D)$) the set of vertices (resp. arcs) of D; a sequence $(x_0, x_1, \ldots, x_n)$ such that $(x_i, x_{i+1}) \in A(D)$ for each $0 \leq i \leq n - 1$ will be called a directed walk; when $x_i \neq x_j$ for $i \neq j$, $\{i, j\} \subseteq \{0, 1, \ldots, n\}$, it is a directed path; and a directed cycle is a directed walk $(x_0, x_1, \ldots, x_n, x_0)$ such that $x_i \neq x_j$ for $i \neq j$, $\{i, j\} \subseteq \{0, 1, \ldots, n\}$.

2. Kernels by Monochromatic Paths

Lemma 2.1. *Let D be an m-coloured digraph without monochromatic directed cycles, and let $h, k \in V(MP(D))$, $h \neq k$. Suppose there exists an*

hk-mdp in the inner m-colouration of $MP(D)$. Then there exists a mdp in D from the terminal endpoint of h to the initial endpoint of k whose colour is equal to those of k, and the terminal endpoint of h is different from the terminal endpoint of k.

Proof. Let $(h = t_0, t_1, \ldots, t_n = k)$ be an hk-mdp coloured (say) c in the inner m-colouration of $MP(D)$. It follows from the definition of $MP(D)$ that k is coloured c in D, and the terminal endpoint of t_i is the initial endpoint of t_{i+1} for each i, $0 \leq i \leq n-1$. Hence $t_1 \cup t_2 \cup \cdots \cup t_{n-1}$ is a monochromatic directed walk coloured c from the terminal endpoint of h to the initial endpoint of k, moreover $t_1 \cup t_2 \cup \cdots \cup t_{n-1}$ is a mdp (as D has no monochromatic directed cycles). Now $t_1 \cup t_2 \cup \cdots \cup t_n$ is a mdp in D coloured c (as D has no monochromatic directed cycles) from the terminal endpoint of h to the terminal endpoint of k, thus the terminal endpoint of h is different from the terminal endpoint of k. ∎

Definition 2.1. Let $D = (V(D), A(D))$ be a digraph. We denote by $\mathcal{P}(X)$ the set of all the subsets of the set X; $f: \mathcal{P}(V(D)) \to \mathcal{P}(V(MP(D)))$ will denote the function defined as follows: for each $Z \subseteq V(D)$, $f(Z) = \{t = (x_1, \ldots, x_n) \in V(MP(D)) | x_n \in Z\}$ (the set of monochromatic directed paths of D whose terminal endpoint are in Z). Also we denote by $g: \mathcal{P}(V(MP(D))) \to \mathcal{P}(V(D))$ the function defined as follows: for each $H \subseteq V(MP(D))$, $g(H) = C(H) \cup D(H)$ where $C(H) = \{x_m \in V(D) | \exists t = (x_0, \ldots, x_m) \in H\}$ (the set of the terminal endpoints of the monochromatic directed paths which are in H), and $D(H) = \{x \in V(D) | \delta_D^-(x) = 0$ and there is no mdp from x to $C(H)\}$.

Lemma 2.2. *Let D be an m-coloured digraph without monochromatic directed cycles. If $Z \subseteq V(D)$ is independent by monochromatic paths in D, then $f(Z)_{MP}$ is independent by monochromatic paths in the inner m-colouration of $MP(D)$.*

Proof. We proceed by contradiction. Let D and $Z \subseteq V(D)$ be as in the hypothesis and assume (by contradiction) that $f(Z)_{MP}$ is not independent by monochromatic paths in the inner m-colouration of $MP(D)$. Thus there exist $h, k \in f(Z)_{MP}$, $h \neq k$ and an hk-mdp in the inner m-colouration of $MP(D)$. It follows from Lemma 2.1 that the terminal endpoint of h is different from the terminal endpoint of k, and there exists a mdp say t from the terminal endpoint of h to the initial endpoint of k, whose colour is equal

to those of k; since D has no monochromatic directed cycles it follows that $t \cup k$ is a mdp from the terminal endpoint of h to the terminal endpoint of k, so we have a mdp between two vertices of Z (as $\{h, k\} \subseteq f(Z)_{TM}$), a contradiction. ∎

Theorem 2.1. *Let $D = (V(D), A(D))$ be an m-coloured digraph without monochromatic directed cycles. The number of kernels by monochromatic paths of D is equal to the number of kernels by monochromatic paths in the inner m-colouration of $MP(D)$.*

Proof. Denote by $\mathcal{K}$ the set of all the kernels by monochromatic paths of D and by $\mathcal{K}^*$ the set of all the kernels by monochromatic paths of the inner m-colouration of $MP(D)$.

(1) If $Z \in \mathcal{K}$, then $f(Z)_{MP} \in \mathcal{K}^*$.
Since $Z \in \mathcal{K}$, we have that Z is independent by monochromatic paths and Lemma 2.2 implies $f(Z)_{MP}$ is independent by monochromatic paths. Now we will prove that $f(Z)_{MP}$ is absorbant by monochromatic paths. Let $k = (k_0, k_1, \ldots, k_m) \in (V(MP(D)) - f(Z)_{MP})$; it follows from Definition 2.1 that $k_m \in (V(D) - Z)$. Since Z is a kmp of D it follows that there exists $z \in Z$ and a $k_m z$-mdp say h in D. Thus (k, h) is a mdp in the inner m-colouration of $MP(D)$ with $h \in f(Z)_{MP}$ (as z is the terminal endpoint of h and $z \in Z$).

(2) The function $f' : \mathcal{K} \to \mathcal{K}^*$, where f' is the restriction of f to $\mathcal{K}$ is an injective function.

Let $Z_1, Z_2 \in \mathcal{K}$, $Z_1 \neq Z_2$. Let us suppose w.l.o.g. that $Z_1 - Z_2 \neq \emptyset$, and take $v \in (Z_1 - Z_2)$. Since Z_2 is a kernel by monochromatic paths of D, it follows that there exists $u \in Z_2$ and a vu-mdp say h, and from Definition 2.1 we have that $h \in f(Z_2)_{MP}$. Since $v \in Z_1$, Z_1 is independent by monochromatic paths and h is a vu-mdp, it follows $u \notin Z_1$ and then $h \notin f(Z_1)_{MP}$. We conclude $h \in (f(Z_2)_{MP} - f(Z_1)_{MP})$ and thus $f(Z_1)_{MP} \neq f(Z_2)_{MP}$.

(3) If $H_{MP} \in \mathcal{K}^*$ then $g(H_{MP}) \in \mathcal{K}$.

(3.1) If $H_{MP} \in \mathcal{K}^*$ then $g(H_{MP})$ is independent by monochromatic paths. Suppose $H_{MP} \in \mathcal{K}^*$, and let $u, v \in g(H_{MP})$, $u \neq v$; we will prove that there is no uv-mdp in D. We will analyze several cases:

Case 3.1.a. $u, v \in C(H_{MP})$.

In this case we proceed by contradiction. Suppose (by contradiction) that there exists an uv-mdp say ℓ in D. Since $u, v \in C(H_{MP})$, u (resp. v) is the terminal endpoint of a mdp h (resp. k) with $h, k \in H_{MP}$; $\ell \notin H_{MP}$, otherwise we get a contradiction as $(h, \ell) \in A(MP(D))$, $h \in H_{MP}$ and H_{MP} is independent by monochromatic paths. Since H_{MP} is absorbant by monochromatic paths and $\ell \notin H_{MP}$ it follows that there exist $p \in H_{MP}$ and a ℓp-mdp in the inner colouration of $MP(D)$. It follows from Lemma 2.1 that there exists a mdp say s from v (the terminal endpoint of ℓ) to the initial endpoint of p whose colour is equal to that of p, and the terminal endpoint of ℓ is different from the terminal endpoint of p; now $k \neq p$ (notice that v is the terminal endpoint of ℓ and also of k; so k and p have different terminal endpoints). We conclude that (k, s, p) is a mdp in the inner colouration of $MP(D)$, with $k, p \in H_{MP}$, a contradiction (as H_{MP} is independent by monochromatic paths).

Case 3.1.b. $u \in C(H_{MP})$, $v \in D(H_{MP})$.
In this case there is no uv-mdp in D, as $\delta_D^-(v) = 0$.

Case 3.1.c. $u \in D(H_{MP})$, $v \in C(H_{MP})$.
Now, there is no uv-mdp in D, as there is no mdp in D from u to $C(H)$.

Case 3.1.d. $u, v \in D(H_{MP})$.
There is no uv-mdp in D because $\delta_D^-(v) = 0$ (as $v \in D(H_{MP})$).

(3.2) If $H_{MP} \in \mathcal{K}^*$, then $g(H_{MP})$ is absorbant by monochromatic paths. Let $u \in (V(D) - g(H_{MP}))$. Since $u \notin (C(H)_{MP} \cup D(H_{MP}))$, we have that there is no mdp in H whose terminal endpoint is u, and at least one of the two following conditions holds: $\delta_D^-(u) > 0$ or there exists a mdp from u to $C(H_{MP})$. We will analyze the two possible cases:

Case 1.   There is no mdp in H whose terminal endpoint is u and $\delta_D^-(u) > 0$.
The hypothesis in this case implies that there exists a mdp say k such that u is the terminal endpoint of k and $k \notin H_{MP}$. Since $H_{MP} \in \mathcal{K}^*$, we have that H_{MP} is absorbant by monochromatic paths, hence there exists $h \in H_{MP}$ and a mdp from k to h in the inner m-colouration of $MP(D)$. It follows from Lemma 2.1 that there exists a mdp say ℓ in D from the terminal endpoint of k to the initial endpoint of h whose colour is equal to that of h, and the terminal endpoint of k is different from the terminal endpoint of h.

Now $\ell \cup h$ is a mdp of D (as D has no monochromatic directed cycles) from u to the terminal endpoint of h (say) v, and it follows from Definition 2.1 that $v \in g(H_{MP})$. So there exists an $ug(H_{MP})$-mdp in D.

Case 2. There is no mdp in H whose terminal endpoint is u, and there exists a mdp from u to $C(H_{MP})$.
Clearly in this case we have a mdp from u to $g(H_{MP}) = C(H_{MP}) \cup D(H_{MP})$.

(4) The function $g' : \mathcal{K}^* \to \mathcal{K}$, where g' is the restriction of g to $\mathcal{K}^*$, is an injective function.

Let $N_{MP}, Q_{MP} \in \mathcal{K}^*$ such that $N_{MP} \neq Q_{MP}$. Let us suppose that $N_{MP} - Q_{MP} \neq \emptyset$ (the case $Q_{MP} - N_{MP} \neq \emptyset$ is completely analogous). Let $h \in (N_{MP} - Q_{MP})$, and u the terminal endpoint of h, so clearly $u \in g(N_{MP})$. Now we will prove $u \notin g(Q_{MP})$. Since Q_{MP} is absorbant by monochromatic paths and $h \notin Q_{MP}$, we have that there exists $k \in Q_{MP}$ and a hk-mdp in the inner m-colouration of $MP(D)$, let $v \in V(D)$ the terminal endpoint of k, it follows from Lemma 2.1 that $u \neq v$ and there exists a mdp say ℓ from the terminal endpoint of h to the initial endpoint of k whose colour is equal to that of k. Thus $\ell \cup k$ is a mdp (notice that D has no monochromatic directed cycles). Since $g(Q_{MP})$ is independent by monochromatic paths and $v \in g(Q_{MP})$ (recall that $k \in Q_{MP}$ and v is the terminal endpoint of k), we conclude that $u \notin g(Q_{MP})$, $u \in (g(N_{MP}) - g(Q_{MP}))$ and $g(N_{MP}) \neq g(Q_{MP})$. Finally notice that it follows from (2) and (4) that: $\operatorname{Card} \mathcal{K} \leq \operatorname{Card} \mathcal{K}^* \leq \operatorname{Card} \mathcal{K}$ and thus $\operatorname{Card} \mathcal{K} = \operatorname{Card} \mathcal{K}^*$. ∎

Remark 2.1. Theorem 2.1 generalizes Theorem 1.1.
Let D be a digraph, $|A(D)| = q$; consider the q edge-colouration which assigns to each arc of D one of q colours with and different arcs having different colours. Clearly a kmp of D is a kernel of D, $MP(D) \cong L(D)$ (the line digraph of D) and a kmp of the inner q-colouration of $L(D)$ is a kernel of $L(D)$ (as there are no mdp of length greather than 1 in $L(D)$).

Remark 2.2. Let D be an m-coloured digraph and $MP(D)$ its monochromatic path digraph; similarly as in the definition of inner colouration of $MP(D)$, we may define the outer m-colouration of $MP(D)$ as follows: If h is a mdp of D coloured c, then any arc of the form (h, x) in $MP(D)$ also is coloured c. However, Theorem 2.1 does not hold if we replace inner m-colour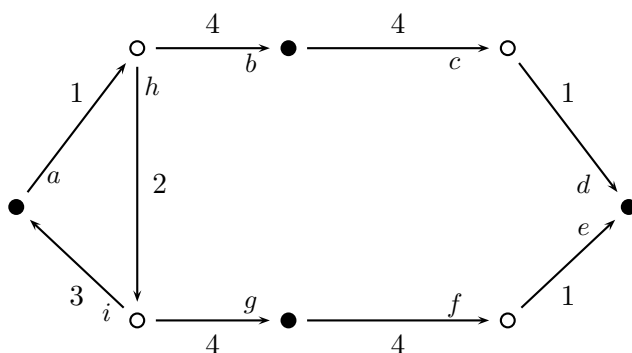ation of $MP(D)$ to outer m-colouration of $MP(D)$. In Figure 1 we show a digraph D without monochromatic directed cycles with one kmp

such that the outer m-colouration of $MP(D)$ (Figure 2) has no kmp. Figure 3 shows a digraph D without monochromatic directed cycles, with no kmp such that the outer m-colouration of $MP(D)$ (Figure 4) has a kmp.

Remark 2.3. Theorem 2.1 does not hold if we drop the hypothesis that D has no monochromatic directed cycles. In Figure 5 we show a digraph D with monochromatic directed cycles, which has two kmps, and the inner m-colouration of $MP(D)$ (Figure 6) has no kmp. And in Figure 7 we show a digraph with monochromatic directed cycles, without a kmp, and the inner m-colouration of $MP(D)$ has one kmp.

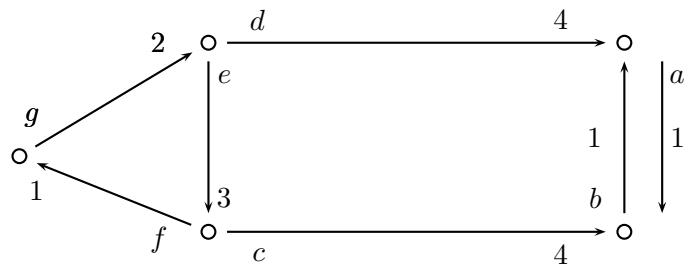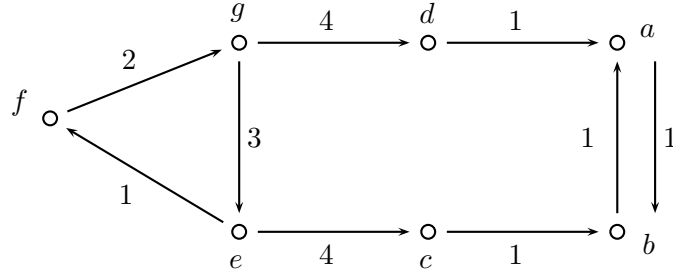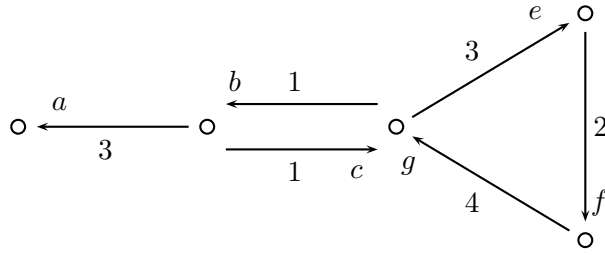Figure 1. D without monochromatic directed cycles with one kmp.
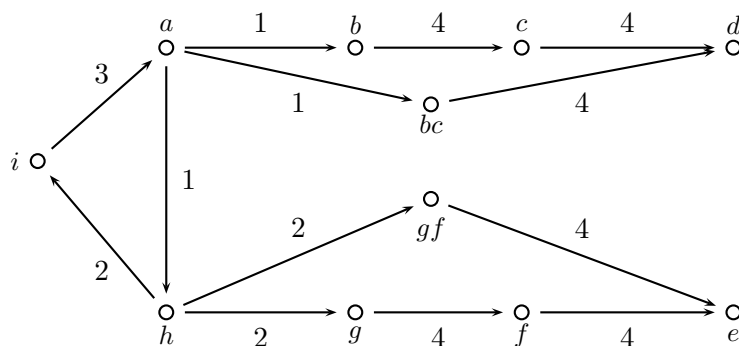
Figure 2. The outer m-colouration of $MP(D)$ has no kmp.
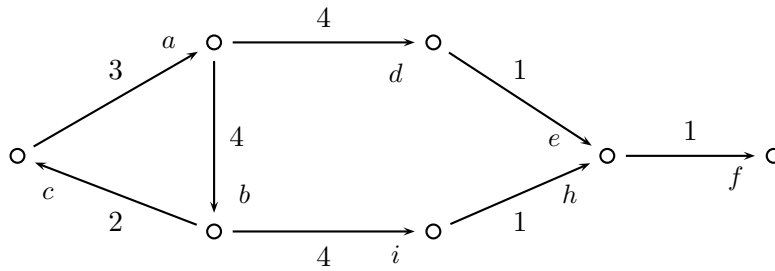
Figure 3. D without monochromatic directed cycles with no kmp.
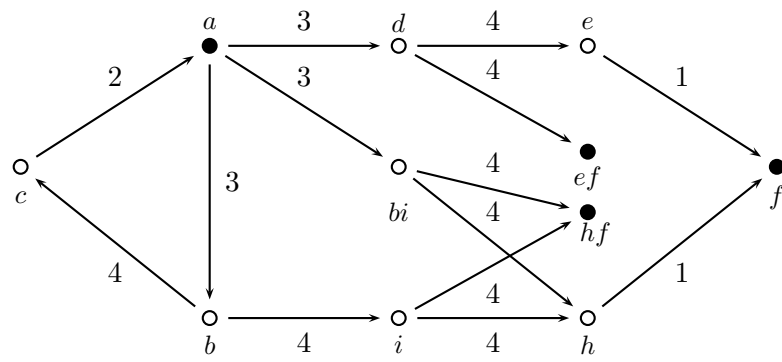
Figure 4. The outer m-colouration of $MP(D)$

Figure 5. D with monochromatic directed cycles and with two kmp.

Figure 6. $MP(D)$ The inner m-colouration of $MP(D)$, without a kmp.

Figure 7. D with monochromatic directed cycles and without a kmp.
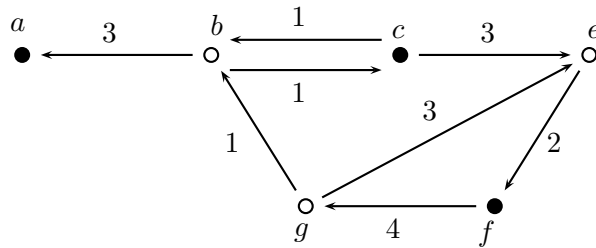
Figure 8. $MP(D)$ The inner m-colouration of $MP(D)$ has one kmp.

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