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AN UPPER BOUND OF THE BASIS NUMBER OF THE STRONG PRODUCT OF GRAPHS

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Abstract

The basis number of a graph G is defined to be the least integer d such that there is a basis $\mathcal{B}$ of the cycle space of G such that each edge of G is contained in at most d members of $\mathcal{B}$. In this paper we give an upper bound of the basis number of the strong product of a graph with a bipartite graph and we show that this upper bound is the best possible.

Keywords: basis number; cycle space; strong product.

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1. Introduction

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [5].

The cycle space, $\mathcal{C}(G)$, of a graph G is the vector space over the two element field, Z_2, spanned by the cycles of G; the sum of two vectors is obtained by taking the symmetric difference of the corresponding sets of edges. It follows that the non-zero elements of $\mathcal{C}(G)$ are cycles and edge disjoint union of cycles. It is known that

$$\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1.$$

A basis of the cycle space of a graph G is called d-fold if each edge of G occurs in at most d of the cycles in the basis. The basis number of G,

$b(G)$, is the smallest non-negative integer d such that $\mathcal{C}(G)$ has a d-fold basis. Formally the basis number was introduced by Schmeichel [15], but it already appeared in MacLane [13] who classified graphs into planar and non planar with respect to their basis numbers. In fact, MacLane proved that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel proved that there are graphs with arbitrarily large basis numbers. And also, Schmeichel proved that $b(K_n) \leq 3$.

A required basis of $\mathcal{C}(G)$ is a basis that is $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\{\varphi(c)|c \in \mathcal{B}\}$ is called the corresponding (required) basis of $\mathcal{B}$ in H.

Our primary interest concerns studying the basis number of graph products. In particular, the strong product of graphs. Let G and H be two graphs. The G-layer G_v is the graph with the vertex set $V(G_v) = V(G) \times v$ and edge set $E(G_v) = \{(u_1, v)(u_2, v)|u_1u_2 \in E(G)\}$. Similarly, the H-layer H_u is the graph with the vertex set $V(H_u) = u \times V(H)$ and edge set $E(H_u) = \{(u, v_1)(u, v_2)|v_1v_2 \in E(H)\}$. The cartesian product $G \Box H$ is the graph with the vertex set $V(G \Box H) = V(G) \times V(H)$ and the edge set $E(G \Box H) = (\cup_{v \in H} E(G_v)) \cup (\cup_{u \in G} E(H_u))$. The direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$. The semi-strong product $G \bullet H$ is the graph with the vertex set $V(G \bullet H) = V(G) \times V(H)$ and the edge set $E(G \bullet H) = E(G \times H) \cup (\cup_{u \in G} E(H_u))$. The strong product $G \boxtimes H$ is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)\}$.

Finding an upper bound for the basis number of graph products has been studied by many authors. Schmeichel [15] and Ali [1] gave an upper bound for the semi-strong products of some special graphs. They proved the following results:

Theorem 1.1 (Schmeichel)**.** *For each* $n \geq 5, b(K_n \bullet P_2) \leq 1 + b(K_n)$.

Theorem 1.2 (Ali)**.** *For each* $n, m \geq 5, b(K_m \bullet K_n) \leq 3 + b(K_m) + b(K_n)$.

In fact, Schmeichel gave an upper bound for a more general case, when he proved that $b(K_{n,m}) \leq 4$. An upper bound of the basis number of the cartesian product was obtained by Ali and Marougi [3] who proved the following:

Theorem 1.3 (Ali and Marougi)**.** *For any two graphs G and H, we have $b(G\Box H) \leq \max\{b(G) + \Delta(T_H),\ b(H) + \Delta(T_G)\}$ where T_H and T_G will be defined latter.*

Recently, Alsardary [4] gave the following result:

Theorem 1.4 (Alsardary)**.** *For every $d \geq 1$ and $n \geq 2$, we have $b(K_n^d) \leq 2d+1$ where K_n^d is the d-fold cartesian product of K_n.*

The direct product was studied by Jaradat [9], who gave the following upper bounds of the direct product of some graphs. Moreover, Jaradat classified trees with respect to the basis number of their direct product with paths of order greater than or equal to 5.

Theorem 1.5 (Jaradat)**.** *For each bipartite graphs G and H, $b(G \times H) \leq 5 + b(G) + b(H)$.*

Theorem 1.6 (Jaradat)**.** *For each bipartite graph G and cycle C, $b(G \times C) \leq 3 + b(G)$.*

Ali and Marougi [2] investigated the basis number of the strong product of two paths, two cycles, a star and a cycle, a path and a cycle.

The results cited above triggers off the following question: Is there an upper bound of the basis number of the strong product of two graphs with respect to the factors?

The paper is organized as follows: In Section 2, we introduce the concept of degree vector set and we give an upper bound of the strong product of two trees. In Section 3, we give an optimal upper bound of the strong product of a graph with a bipartite graph.

After this manuscript was completed the author learned that some ideas used in this paper are closely related to some of the ideas employed in [8].

In the rest of this work $f_B(e)$ stands for the number of cycles in $B \subseteq \mathcal{C}(G)$ containing e and $E(B) = \cup_{b\in B} E(b)$. $\mathcal{B}_G$ stands for a required basis of G. $\lceil x \rceil$ stands for the least integer greater than or equal to x. $\lfloor x \rfloor$ stands for the greatest integer less than or equal to x.

2. Strong Product of Two Trees

In this section, we focus our attention on obtaining an upper bound of the basis number of the strong product of two trees which is independent of their orders. We begin with a criteria for a strong product of two graphs to be non-planar which is a direct consequence of MacLane's theorem and a characterization of planar strong product given in [12] (see also, [7]).

Lemma 2.1. *Let G and H be two connected graphs such that one of them is of order ≥ 3 and the other is of order ≥ 4. Then $b(G \boxtimes H) \geq 3$. Moreover, the same result holds if one of them contains a cycle and the other is of order ≥ 2.*

In order to achieve an optimal upper bound for the basis number of the strong product of two graphs, we shall introduce the following concepts: Let G be a graph and $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of the edge set of G. For each e_i assign 1 to one of its two vertices and 2 to the other. Let u be a vertex which is incident with $e_{n_1}, e_{n_2}, \ldots, e_{n_r}$ where $n_1 < n_2 < \ldots < n_r$ and $r = d_G(u)$. Then u corresponds to a $(1,2)$-vector $(\xi_1, \xi_2, \ldots, \xi_r)$ where $\xi_i = 1$ if 1 is assigned to u in e_{n_i} and $\xi_i = 2$ if 2 is assigned to u in e_{n_i}. We call this vector a degree-vector of u and denote it by $DV_G(u)$. The set of all degree-vectors of G will be denoted by $DVS(G)$. Note that degree vector set of a graph G is not unique because the values of the components in each vector depend on the way we assign the 1's and 2's for the vertices of edges of G and on the way we label the edges of G.

Proposition 2.1. *For each tree T of order ≥ 2, there is a degree vector set $DVS(T)$ such that for each vertex $v \in T$ the number of coordinates of value 2 is less than or equal to $\lceil \frac{d_T(v)}{2} \rceil$ and at least one of the end vertices has the degree-vector (1).*

Proof. Let $e_1, e_2, \ldots, e_{|E(T)|}$ be a labeling of the edges of T in such away that $e_{|E(T)|} = vv^*$ where v^* is an end vertex. We now proceed using mathematical induction on $|V(T)|$. If $|V(T)| = 2$, then T is a path of order 2, and so $DVS(T) = \{(1), (2)\}$. Let T be a tree of order $n+1$. Then $T^{'} = T - v^*$ is a tree of order n. Thus, by the inductive step there is $DVS(T^{'})$ which satisfies the conditions that is stated in the theorem. Let $DV_{T'}(v) = (\xi_1, \xi_2, \ldots, \xi_{d_{T'}(v)})$ where ξ_i is 1 or 2. We now consider the following cases:

Case 1. $d_{T'}(v) = 1$ and $\xi_1 = 1$. Then assign 2 to v and 1 to v^* in $e_{|E(T)|} = vv^*$. Take $DV_T(u) = DV_{T'}(u)$ for each $u \in V(T) - \{v, v^*\}$, $DV_T(v) = (1,2)$ and $DV_T(v^*) = (1)$.

Case 2. $d_{T'}(v) > 1$; or $d_{T'}(v) = 1$ and $\xi_1 = 2$. Then assign 1 to v and 2 to v^* in $e_{|E(T)|} = vv^*$. Take $DV_T(u) = DV_{T'}(u)$ for each $u \in V(T) - \{v, v^*\}$, $DV_T(v) = (\xi_1, \xi_2, \dots, \xi_{d_{T'}(v)}, 1)$ and $DV_T(v^*) = (2)$. To this end, it is easy to see that $DVS(T)$ resulting from the above construction satisfies the conditions which are required in the proposition. ■

The dimension of K_4 is three and there are four possibilities for a basis consisting of triangles. Because we will use those bases frequently in the sequel in a specific manner we write them explicitly in the following lemma in detail.

Lemma 2.2. *Let* $P_2^{(i)} = a_ib_i$, *and* $Q_2^{(j)} = u_jv_j$ *be two edges of* G *and* H, *respectively. Let*

$$\mathcal{B}_{ij}^{(1)} = \{(a_i,u_j)(b_i,v_j)(a_i,v_j)(a_i,u_j), (a_i,u_j)(b_i,v_j)(b_i,u_j)(a_i,u_j),$$
$$(a_i,v_j)(b_i,u_j)(b_i,v_j)(a_i,v_j)\},$$
$$\mathcal{B}_{ij}^{(2)} = \{(a_i,u_j)(b_i,v_j)(a_i,v_j)(a_i,u_j), (a_i,u_j)(b_i,v_j)(b_i,u_j)(a_i,u_j),$$
$$(a_i,u_j)(b_i,u_j)(a_i,v_j)(a_i,u_j)\},$$
$$\mathcal{B}_{ij}^{(3)} = \{(a_i,u_j)(b_i,u_j)(a_i,v_j)(a_i,u_j), (b_i,u_j)(b_i,v_j)(a_i,v_j)(b_i,u_j),$$
$$(a_i,u_j)(b_i,u_j)(b_i,v_j)(a_i,u_j)\},$$
$$\mathcal{B}_{ij}^{(4)} = \{(a_i,u_j)(b_i,u_j)(a_i,v_j)(a_i,u_j), (b_i,u_j)(b_i,v_j)(a_i,v_j)(b_i,u_j),$$
$$(a_i,u_j)(b_i,v_j)(a_i,v_j)(a_i,u_j)\}.$$

Then $\mathcal{B}_{ij}^{(k)}$ *is a basis for* $\mathcal{C}(P_2^{(i)} \boxtimes Q_2^{(j)})$ *for each* $k = 1, 2, 3, 4$. *Moreover,*

(1) *each edge of* $A_1 = \{(a_i,u_j)(b_i,v_j),\ (b_i,u_j)(b_i,v_j),\ (b_i,v_j)(a_i,v_j)\}$ *is of* 2*-fold in* $\mathcal{B}_{ij}^{(1)}$ *and each edge of* $E(\mathcal{B}_{ij}^{(1)}) - A_1$ *is of* 1*-fold in* $\mathcal{B}_{ij}^{(1)}$.

(2) *Each edge of* $A_2 = \{(a_i,u_j)(b_i,v_j),\ (a_i,u_j)(a_i,v_j),\ (a_i,u_j)(b_i,u_j)\}$ *is of* 2*-fold in* $\mathcal{B}_{ij}^{(2)}$ *and each edge of* $E(\mathcal{B}_{ij}^{(2)}) - A_2$ *is of* 1*-fold in* $\mathcal{B}_{ij}^{(2)}$.

(3) *Each edge of* $A_3 = \{(a_i,v_j)(b_i,u_j),\ (a_i,u_j)(b_i,u_j),\ (b_i,u_j)(b_i,v_j)\}$ *is of* 2*-fold in* $\mathcal{B}_{ij}^{(3)}$ *and each edge of* $E(\mathcal{B}_{ij}^{(3)}) - A_3$ *is of 1-fold in* $\mathcal{B}_{ij}^{(3)}$.

(4) *Each edge of* $A_4 = \{(a_i,v_j)(b_i,u_j),\ (a_i,v_j)(b_i,v_j),\ (a_i,u_j)(a_i,v_j)\}$ *is of* 2*-fold in* $\mathcal{B}_{ij}^{(4)}$ *and each edge of* $E(\mathcal{B}_{ij}^{(4)}) - A_4$ *is of 1-fold in* $\mathcal{B}_{ij}^{(4)}$.

Throughout the rest of this paper we consider

$$(1) \qquad \mathcal{B}_{ij} = \begin{cases} \mathcal{B}_{ij}^{(1)}, & \text{if 1 is assigned to both of } a_i \text{ and } u_j, \\ \mathcal{B}_{ij}^{(2)}, & \text{if 1 is assigned to both of } b_i \text{ and } v_j, \\ \mathcal{B}_{ij}^{(3)}, & \text{if 1 is assigned to both of } a_i \text{ and } v_j, \\ \mathcal{B}_{ij}^{(4)}, & \text{if 1 is assigned to both of } b_i \text{ and } u_j. \end{cases}$$

Lemma 2.3. *Let $P_2^{(i)} = a_i b_i$ and H be a graph such that $E(H) = \{u_1v_1, u_2v_2, \ldots, u_{|E(H)|}v_{|E(H)|}\}$. Let $\mathcal{B}_i = \cup_{j=1}^{|E(H)|}\mathcal{B}_{ij}$. Then $\mathcal{B}_i$ is a linearly independent set of $\mathcal{C}(P_2^{(i)} \boxtimes H)$.*

Proof. We shall proceed by induction on $|E(H)|$. For $|E(H)| = 1$, the result is satisfied using Lemma 2.2. Clearly, $\mathcal{B}_i = \cup_{j=1}^{|E(H)|-1}\mathcal{B}_{ij} \cup \mathcal{B}_{i|E(H)|}$. Moreover, $E(\cup_{j=1}^{|E(H)|-1}\mathcal{B}_{ij}) \cap E(\mathcal{B}_{i|E(H)|}) \subseteq \{(a_i, u_{|E(H)|})(b_i, u_{|E(H)|}), (a_i, v_{|E(H)|})(b_i, v_{|E(H)|})\}$. Thus, any linear combination of cycles of $\mathcal{B}_{i|E(H)|}$ must contain at least one edge of the form $(a_i, u_{|E(H)|})(a_i, v_{|E(H)|})$, $(a_i, v_{|E(H)|})(b_i, u_{|E(H)|})$, $(a_i, u_{|E(H)|})$ $(b_i, v_{|E(H)|})$ and $(b_i, u_{|E(H)|})$ $(b_i, v_{|E(H)|})$ which is not in any cycle of $\cup_{i=1}^{|E(H)|-1}\mathcal{B}_{ij}$. Hence, $\mathcal{B}_i$ is linearly independent. ■

Knowing whether the graph is connected or not is very important in finding the dimension (a basis) of the cycle space of a graph, so we give the following result which goes back to [14].

Lemma 2.4. *Let G and H be two graphs. Then $G \boxtimes H$ is connected.*

Theorem 2.1. *Let T_1 and T_2 be two trees. Then $b(T_1 \boxtimes T_2) \leq \max\{\lfloor \frac{3\Delta(T_1)+1}{2} \rfloor, \lfloor \frac{3\Delta(T_2)+1}{2} \rfloor\}$.*

Proof. Let $P_3^{(1)} = a_1b_1, P_3^{(2)} = a_2b_2, \ldots, P_3^{(|E(T_1|)} = a_{|E(T_1)|}b_{|E(T_1)|}$ be a labeling of the edges of T_1 in such away that $b_{|E(T_1)|}$ is an end vertex and let $u_1v_1, u_2v_2, \ldots, u_{|E(T_2)|}v_{|E(T_2)|}$ be a labeling of the edge set of T_2. Let $DVS(T_1)$ and $DVS(T_2)$ be the degree vector sets as in Proposition 2.1. Set $\mathcal{B}_i = \bigcup_{j=1}^{|E(T_2)|}\mathcal{B}_{ij}$. We now show that $\mathcal{B} = \bigcup_{i=1}^{|E(T_1)|}\mathcal{B}_i$ is linearly independent. By Lemma 2.3, $\mathcal{B}_1$ is linearly independent. Since $E(\mathcal{B}_{|E(T_1)|}) \cap E(\bigcup_{i=1}^{|E(T)|-1}\mathcal{B}_i) = E((T_2)_{a_{|E(T_1)|}})$ which is a tree, as a result any linear combination of cycles of $\mathcal{B}_{|E(T_1)|}$ contains either an edge of $(T_2)_{b_{|E(T_1)|}}$ or at least one edge of the form $(a_{|E(T_1)|}, u_j)(b_{|E(T_1)|}, v_j)$, $(a_{|E(T_1)|}, v_j)$ $(b_{|E(T_1)|}, u_j)$, $(a_{|E(T_1)|}, u_j)(b_{|E(T_1)|}, u_j)$ and $(a_{|E(T_1)|}, v_j)(b_{|E(T_1)|}, v_j)$ which is not in any cycle of $\bigcup_{i=1}^{|E(T_1)|-1} \mathcal{B}_i$. Thus, by inductive step, $\mathcal{B}$ is linearly independent. Counting the number of elements in the set $\mathcal{B}$, we have

$$\begin{aligned} |\mathcal{B}| &= \sum_{i=1}^{|E(T_1)|} \sum_{j=1}^{|E(T_2)|} |\mathcal{B}_{ij}| = \sum_{i=1}^{|E(T_1)|} \sum_{j=1}^{|E(T_2)|} 3 \\ &= 3|E(T_1)||E(T_2)| = \dim \mathcal{C}(T_1 \boxtimes T_2). \end{aligned}$$

Therefore, $\mathcal{B}$ is a basis for $\mathcal{C}(T_1 \boxtimes T_2)$. To complete the proof of the theorem, we should show that $\mathcal{B}$ satisfies the fold which is stated in the theorem. Let $e \in E(T_1 \boxtimes T_2)$, we have the following cases of e to consider:

Case 1. $e = (a_i, u_j)(a_i, v_j)$. Then

$$f_{\mathcal{B}}(e) = \sum_{j=1}^{|E(T_1)|} f_{\mathcal{B}_j}(e).$$

Let $P_2^{(i_1)}, P_2^{(i_2)}, \ldots, P_2^{(i_{d_{T_1}(a_i)})}$ be the edges of $E(T_1)$ which incident with a_i. Then, by Lemma 2.2, e appears in one or two cycles of $\mathcal{B}_{(i_k)j}$ for each $k = 1, 2, \ldots, d_{T_1}(a_i)$, and so of $\mathcal{B}_{i_k}$ for each $k = 1, 2, \ldots, d_{T_1}(a_i)$. Thus,

$$\begin{aligned} f_{\mathcal{B}}(e) &= \sum_{k=1}^{d_{T_1}(a_i)} f_{\mathcal{B}_{j_k}}(e) \\ &\leq \sum_{k=1 \textit{ and } 2 \textit{ is assigned to } a_i \textit{ in } P_2^{(i_k)}}^{d_{T_1}(a_i)} f_{\mathcal{B}_{j_k}}(e) \\ &+ \sum_{k=1 \textit{ and } 1 \textit{ is assigned to } a_i \textit{ in } P_2^{(i_k)}}^{d_{T_1}(a_i)} f_{\mathcal{B}_{j_k}}(e). \end{aligned}$$

By Proposition 2.1 and equation (1), we have

$$\begin{aligned} f_{\mathcal{B}}(e) &\leq 2\left\lceil \frac{d_{T_1}(a_i)}{2} \right\rceil + d_{T_1}(a_i) - \left\lceil \frac{d_{T_1}(a_i)}{2} \right\rceil \\ &\leq \frac{3d_{T_1}(a_i)+1}{2}. \end{aligned}$$

Thus,

$$f_{\mathcal{B}}(e) \leq \left\lfloor \frac{3d_{T_1}(a_i)+1}{2} \right\rfloor.$$

Similarly, if $e = (b_i, u_j)(b_i, v_j)$.

Case 2. $e = (a_i, u_j)(b_i, u_j)$ or $(a_i, v_j)(b_i, v_j)$. Because the strong product is commutative, as in Case 1, we have that

$$f_{\mathcal{B}}(e) \leq \left\lfloor \frac{3d_{T_2}(u_i)+1}{2} \right\rfloor.$$

Case 3. $e = (a_i, u_j)(b_i, v_j)$ or $e = (b_i, u_j)(a_i, v_j)$. Then $f_{\mathcal{B}_i}(e) \leq 2$. Since $e \notin E(\mathcal{B}_i) \cap E(\mathcal{B}_k)$ whenever $i \neq k$, as a result $f_{\mathcal{B}}(e) \leq 2$. ■

By the aid of Lemma 2.1 and specializing trees in Theorem 2.1 into paths, we have the following result.

Corollary 2.1 (Ali)**.** *For any paths P_n of order $n \geq 3$ and P_m of order $m \geq 4$, we have $b(P_n \boxtimes P_m) = 3$.*

A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ is called an n-special star if there is a vertex, say v, such that v is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \ldots, P^{(n)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v\}$ for each $i \neq j$ (see [9]).

Theorem 2.2. *For any path P_r of order $r \geq 3$ and any 3-special stars T and T^*, we have that $b(P_r \boxtimes T) = 3$ and $b(T \boxtimes T^*) = 3$.*

Proof. Let $V(T) = \{u_1, u_2, \ldots, u_{n_1}, \ldots, u_{n_2}, \ldots, u_n\}$, $V(T^*) = \{v_1, v_2, \ldots, v_{m_1}, \ldots, v_{m_2}, \ldots, u_m\}$ and $P_r = w_1 w_2 \ldots w_r$ where $d_T(u_1) = 3$, $d_{T^*}(v_1) = 3$ and u_{n_1}, u_{n_2}, u_n, v_{m_1}, v_{m_2} and u_m are the end points of T and T^*, respectively. Now, we may label the edges of T ,T^* and P_r in such a way that we can choose $DVS(T) = \{DV_T(u_1) = (1,1,1), DV_T(u_2) = (1,2), \ldots, DV_T(u_{n_1-1}) = (1,2), DV_T(u_{n_1}) = (2)$, $DV_T(u_{n_1+1}) = (1,2) \ldots$, $DV_T(u_{n_2-1}) = (1,2)$, $DV_T(u_{n_2}) = (2)$, $DV_T(u_{n_2+1}) = (1,2)$, $\ldots, DV_T(u_n) = (2)\}$, $DVS(T^*) = \{DV_{T^*}(v_1) = (1,1,1)$, $DV_{T^*}(v_2) = (1,2), \ldots$, $DV_{T^*}(v_{m_1-1}) = (1,2)$, $DV_{T^*}(v_{m_1}) = (2)$, $DV_{T^*}(v_{m_1+1}) = (1,2) \ldots$, $DV_{T^*}(v_{m_2-1}) = (1,2)$, $DV_{T^*}(v_{m_2}) = (2)$, $DV_{T^*}(v_{m_2+1}) = (1,2), \ldots$, $DV_{T^*}(v_m) = (2)\}$ and $DVS(P_r) = \{DV_{P_r}(w_1) = (2)$, $DV_{P_r}(w_2) = (1,2), \ldots$, $DV_{P_r}(w_{r_1}) = (1,2)$,

$DV_{P_r}(w_r) = (1)\}$. By applying the same arguments as in the proof of Theorem 2.1 on the above degree sequences, we have the result. ■

3. Strong Product of a Graph with a Bipartite Graph

In this section we give an optimal upper bound of the basis number of the strong product of a graph with a bipartite graph. In order to achieve our goal we give the following proposition.

Proposition 3.1. *For each graph G, there is a degree vector set $DVS(G)$ such that each degree-vector has at least one of its coordinates of value 1, except possibly one of the end vertices, if any, it may have degree-vector (2).*

Proof. Label the edges of G. Let T be a spanning tree of G. First, we assign values to the vertices of every edge of T. Choose an end vertex of T, say w^*, and let w^*w be an edge of T. Assign the value 2 to w^* and 1 to w in w^*w. Let $w^*w, w_1w, w_2w, \ldots, w_rw$ be the edges of T which are incident with w. For each $j = 1, 2, \ldots, r$ assign to w the value 2 and to w_j the value 1 in the edge w_jw. For each $j = 1, 2, \ldots, r$, let $w_jw, w_jw_{j_1}, \ldots, w_jw_{j_{r_j}}$ be the edges of T which are incident with w_j. For each $j = 1, 2, \ldots, r$, $i = 1, 2, \ldots, r_j$ assign to w_j the value 2 and to w_{j_i} the value 1 in the edge $w_jw_{j_i}$. By continuing this process, we have that every vertex in every edge of T is assigned a value (1 or 2) such that every non-end vertex of T has at least one of its values 1 and every end vertex of T has the value 1 except the vertex w^* which has value 2. Now, we assign values to the vertices of every edge of $E(G) - E(T)$. Let $e \in E(G) - E(T)$. Then we consider two cases.

Case 1. w^* is not a vertex of e (i.e., $e \neq w^*u$ for every $u \in V(G)$), then assign to one of the two end vertices of e the value 1 and to the other the value 2.

Case 2. w^* is a vertex of e (i.e., $e = w^*u$ for some $u \in V(G)$), then assign to w^* the value 1 and to the other vertex of e the value 2.

It is easy to see that $DVS(G)$ resulting from the above construction satisfies our proposition. ■

Note that we can prove Proposition 2.1 by using the same arguments as in the proof of Proposition 3.1.

Lemma 3.1. *For every tree T and graph G, $b(T \boxtimes H) \leq \max\{b(H) + 1, 2\Delta(H) - 1, \lfloor \frac{3\Delta(T)+1}{2} \rfloor\}$.*

Proof. Let $a_1b_1, a_2b_2, \ldots, a_{|E(T)|}b_{|E(T)|}$ be a labeling of the edge set of T in such away that $b_{|E(T)|}$ is an end vertex and u_1v_1, $u_2v_2, \ldots, u_{(|E(H)|)}v_{(|E(H)|)}$ be a labeling of the edge set of H. Let $DVS(T)$ and $DVS(H)$ be the degree vector sets as in Proposition 2.1 and Proposition 3.1, respectively. Set $\mathcal{B}_j = \bigcup_{i=1}^{|E(T)|} \mathcal{B}_{ij}$ and $\mathcal{B}^* = \bigcup_{j=1}^{|E(H)|} \mathcal{B}_j$. We use mathematical induction on $|E(H)|$ to prove that $\mathcal{B}^*$ is linearly independent. The first step of the induction holds by Lemma 2.2. Assume $|E(H)| \geq 2$ and it is true for less than $|E(H)|$. Since $u_{(|E(H)|)}$ and $v_{(|E(H)|)}$ are the end vertices of $u_{(|E(H)|)}v_{(|E(H)|)}$,

$$\begin{aligned} &E(\mathcal{B}_{|E(H)|}) \cap E(\cup_{j=1}^{|E(H)|-1}\mathcal{B}_j) \\ &\subseteq \{(a_i, u_{|E(H)|})(b_i, u_{|E(H)|}), (a_i, v_{|E(H)|})(b_i, v_{|E(H)|}) | i = 1, 2, \ldots |E(T)|\} \end{aligned}$$

which forms an edge set of a forest. Thus, no linear combination of cycles of $\mathcal{B}_{|E(H)|}$ can be written as a linear combination of cycles of $\cup_{j=1}^{|E(H)|-1}\mathcal{B}_j$. Therefore, by the inductive step, $\mathcal{B}^*$ is linearly independent. Let a^* be an end vertex of T such that a^* has the degree-vector (1), let $\mathcal{B}^{**}$ be the corresponding required basis of $\mathcal{B}_H$ in G_{a^*}. We now prove that $\mathcal{B}^* \cup \mathcal{B}^{**}$ is linearly independent. Let C be a linear combination of cycles of $\mathcal{B}^{**}$ and $e \in E(C)$, say $e = (a^*, u_j)(a^*, v_j)$. Then e belongs only to cycles of $\mathcal{B}_j$. Moreover, there are no two edges of C occurs in some cycles of one $\mathcal{B}_l$ for some $1 \leq l \leq |E(G)|$. Assume that C can be written as a linear combination of cycles of $\mathcal{B}^*$. Then

$$C = D_j + \sum_{k=1}^{\delta} D_{n_k} \pmod 2$$

where D_{n_k} and D_j are linear combinations of $\mathcal{B}_{n_k}$ and $\mathcal{B}_j$, respectively. Then

$$D_j = C + \sum_{k=1}^{\delta} D_{n_k} \pmod 2.$$

Hence,

$$E(D_j) = E(C \oplus D_{n_1} \oplus \cdots \oplus D_{n_\delta})$$

which is an edge set of a cycle or an edge set of an edge disjoint union of cycles where $\oplus$ is the ring sum. On the other hand,

$$E(\mathcal{B}_j) \cap \left(E\left(\cup_{k=1 \ and \ k\neq j}^{|E(H)|}\mathcal{B}_k\right) \cup E(\mathcal{B}^{**}) \cup E(C)\right)$$

is an edge set of a forest and

$$\begin{aligned} E(D_j) &= E(C \oplus D_{n_1} \oplus \cdots \oplus D_{n_\delta}) \\ &\subseteq E(\mathcal{B}_j) \cap \left(E\left(\cup_{k=1 \ and \ k\neq j}^{|E(H)|}\mathcal{B}_k\right) \cup E(\mathcal{B}^{**}) \cup E(C)\right). \end{aligned}$$

This is a contradiction. Now,

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}^*| + |\mathcal{B}^{**}| = \sum_{i=1}^{|E(T)|}\sum_{j=1}^{|E(H)|}|\mathcal{B}_{ij}| + \dim \mathcal{C}(G) \\ &= 2|E(T)||E(H)| + |E(H)||V(T)| + |E(T)||V(H)| - |V(H)||V(T)| + 1 \\ &= \dim \mathcal{C}(T \boxtimes H). \end{aligned}$$

To this end, we have shown that $\mathcal{B}$ is a basis for $E(T\boxtimes H)$. Thus, to complete the proof of the theorem we show that $\mathcal{B}$ satisfies the fold which is stated in the lemma. Let $e \in E(T \boxtimes H)$.

(1) If $e = (a_i, u_j)(a_i, v_j)$ or $(b_i, u_j)(b_i, v_j)$ where neither $a_i = a^*$ nor $b_i = a^*$, then $f_{\mathcal{B}}(e) \leq \lfloor \frac{3\Delta(T)+1}{2} \rfloor$.

(2) If $e = (a^*, u_j)(a^*, v_j)$, then $f_{\mathcal{B}}(e) \leq b(H) + 1$.

(3) If $e = (a_i, w^*)(b_i, w^*)$, then $f_{\mathcal{B}}(e) \leq 2$ where w^* is the end vertex as in Proposition 3.1.

(4) If $e = (a_i, u_j)(b_i, u_j)$ or $(a_i, v_j)(b_i, v_j)$ where neither u_j nor v_j is w^*, then $f_{\mathcal{B}}(e) \leq 2\Delta(H) - 1$ because every vertex of H has at least one of its components of value 1 except possibly w^*. ■

In the rest of this paper T_G denotes a spanning tree of G with maximal degree as small as possible and $\Delta(T_G)$ denotes the maximal degree of T_G (see Ali and Marougi [3]). The following proposition (see [16, 12, 9]) with the aid of Lemma 3.1 will play a key role in proving the main result in this section.

Proposition 3.2. *Let G be a bipartite graph and P_2 be a path of order 2. Then $G \times P_2$ consists of two components G_1 and G_2 each of which is isomorphic to G.*

Theorem 3.1. *If G is a bipartite graph and H is a graph, then*

$$b(G\boxtimes H) \leq \max \left\{ b(H)+1, 2\Delta(H)+b(G)-1, \left\lfloor \frac{3\Delta(T_G)+1}{2} \right\rfloor, b(G)+2 \right\}.$$

Proof. Let $\{a_1, a_2, \ldots, a_{|V(G)|}\}$ be the vertex set of H. Let $DVS(T_G)$ and $DVS(H)$ be the degree vector sets of T_G and H as in Proposition 2.1 and Proposition 3.1, respectively. Assume $\mathcal{B}^*$ be the basis of $\mathcal{C}(T_G \boxtimes H)$ as in Lemma 3.1. Set $\mathcal{B}^{**} = \bigcup_{i=1}^{|V(H)|} \mathcal{B}_{a_i}$ where $\mathcal{B}_{a_i}$ is the corresponding required basis of $\mathcal{B}_G$ in G_{a_i}. Clearly, for each $i \neq k$, $E(G_{a_i}) \cap E(G_{a_k}) = \phi$, thus $\mathcal{B}^{**}$ is linearly independent. Moreover, each cycle of G_{a_i} contains at least one edge of $E(G_{a_i}) - E((T_G)_{a_i})$ which is not in any cycle of $\mathcal{B}^*$. Therefore, $\mathcal{B}^* \cup \mathcal{B}^{**}$ is a linearly independent set. To this end, by Proposition 3.2, for each $e \in E(H), G \times e$ consists of two components, each of which is isomorphic to G. Thus, we set $\mathcal{B}^{***} = \bigcup_{e\in E(H)} \mathcal{B}^{(e)}$ where $\mathcal{B}^{(e)} = \mathcal{B}_1^{(e)} \cup \mathcal{B}_2^{(e)}$, such that $\mathcal{B}_1^{(e)}$ and $\mathcal{B}_2^{(e)}$ are the corresponding required bases of $\mathcal{B}_G$ in the two copies of G in $G\times e$. Note that, $E(\mathcal{B}_1^{(e)}) \cap E(\mathcal{B}_2^{(e)}) = \phi$ for each e and $E(\mathcal{B}^{(e')}) \cap E(\mathcal{B}^{(e)}) = \phi$ for each $e' \neq e$. Thus, $\mathcal{B}^{***}$ is linearly independent. Since $E(\mathcal{B}^* \cup \mathcal{B}^{**}) = E(T_G \boxtimes H) \cup E(\cup_{i=1}^{|V(H)|} G_{a_k})$ and $(E(T_G \boxtimes H) \cup E(\cup_{i=1}^{|V(H)|} G_{a_k})) \cap E((G - T_G) \times H) = \phi$ and since each linear combination of cycles of $\mathcal{B}^{***}$ contains at least one edge of $E((G - T_G) \times H)$, we have that $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}^{**} \cup \mathcal{B}^{***}$ is linearly independent. Since,

$$\begin{aligned}
|\mathcal{B}| &= |\mathcal{B}^*| + |\mathcal{B}^{**}| + |\mathcal{B}^{***}| \\
|\mathcal{B}| &= 2|E(T_G)||E(H)| + |E(H)||V(T_G)| + |E(T_G)||V(H)| - |V(H)||V(T_G)| \\
&\quad +1 + |V(H)|\dim \mathcal{C}(G) + 2|E(H)|\dim \mathcal{C}(G) \\
&= 2|E(G)||E(H)| + |E(G)||V(H)| + |V(G)||E(H)| - |V(G)||V(H)| + 1 \\
&= \dim \mathcal{C}(G \boxtimes H),
\end{aligned}$$

$\mathcal{B}$ is a basis for $\mathcal{C}(G\boxtimes H)$ which can be seen easily that it satisfies the required fold. ■

Theorem 3.2. *If H and G are two bipartite graphs, then*

$$\begin{aligned}
&b(G \boxtimes H) \\
&\leq \max \left\{ b(H)+b(G)+2, \left\lfloor \frac{3\Delta(T_G)+1}{2} \right\rfloor + b(H), \left\lfloor \frac{3\Delta(T_H)+1}{2} \right\rfloor + b(G) \right\}.
\end{aligned}$$

Proof. Let $V(G) = \{a_1, a_2, \ldots, a_{|V(G)|}\}$ and $V(H) = \{u_1, u_2, \ldots, u_{|V(H)|}\}$. Let $\mathcal{B}^*$ be the basis of $T_G \boxtimes T_H$ as in Theorem 2.1. Set

$$\mathcal{B}^{**} = \bigcup_{i=1}^{|V(G)|} \mathcal{B}_{a_i} \text{ and } \mathcal{B}^{***} = \bigcup_{j=1}^{|V(H)|} \mathcal{B}_{u_j}$$

where $\mathcal{B}_{a_i}$ and $\mathcal{B}_{u_j}$ are the corresponding required bases of $\mathcal{B}_H$ and $\mathcal{B}_G$ in H_{a_i} and G_{u_j}, respectively. Also, set

$$\mathcal{B}^{'} = \bigcup_{e \in E(T_G)} \left(\mathcal{B}_1^{(e)} \cup \mathcal{B}_2^{(e)}\right) \quad \text{and} \quad \mathcal{B}^{''} = \bigcup_{e' \in E(H)} \left(\mathcal{B}_1^{(e')} \cup \mathcal{B}_2^{(e')}\right)$$

where $\mathcal{B}_1^{(e)}$ and $\mathcal{B}_2^{(e)}$ are the corresponding required bases of $\mathcal{B}_H$ in the two copies of H in $e \times H$, also $\mathcal{B}_1^{(e')}$ and $\mathcal{B}_2^{(e')}$ are the corresponding required basis of $\mathcal{B}_G$ in the two copies of G in $G \times e^{'}$. By using the same arguments as in Theorem 3.1, we can prove that

$$\mathcal{B} = \mathcal{B}^* \bigcup \mathcal{B}^{**} \bigcup \mathcal{B}^{***} \bigcup \mathcal{B}^{'} \bigcup \mathcal{B}^{''}$$

is linearly independent. Now,

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}^*| + |\mathcal{B}^{**}| + |\mathcal{B}^{***}| + |\mathcal{B}^{'}| + |\mathcal{B}^{''}| \\ &= 2|E(T_G)||E(T_H)| + |E(T_G)||E(T_H)| + |V(G)|\dim \mathcal{C}(H) \\ &\quad +|V(H)|\dim \mathcal{C}(G) + 2|E(H)|\dim \mathcal{C}(G) + 2|E(T_G)|\dim \mathcal{C}(H). \end{aligned}$$

But,

$$2|E(H)|\dim \mathcal{C}(G) = 2\dim \mathcal{C}(G)|E(T_H)| + 2\dim \mathcal{C}(H)\dim \mathcal{C}(H).$$

Also,

$$\begin{aligned} |E(T_H)||E(T_G)| &= (|V(H)| - 1)|E(T_G)| \\ &= |V(H)||E(T_G)| + |V(G)||E(T_H)| - |V(G)||V(H)| + 1. \end{aligned}$$

Thus,

$$\begin{aligned}|\mathcal{B}| &= 2|E(T_G)||E(T_H)| + |V(H)||E(T_G)| + |V(G)||E(T_H)| - |V(G)||V(H)| \\ &\quad +1 + |V(G)|\text{dim } \mathcal{C}(H) + |V(H)|\text{dim } \mathcal{C}(G) + 2\text{dim } \mathcal{C}(G)|E(T_H)| \\ &\quad +2\text{dim } \mathcal{C}(H)\text{dim } \mathcal{C}(H) + 2|E(T_G)|\text{dim } \mathcal{C}(H) \\ &= 2|E(H)||E(G)| + |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 1 \\ &= \text{dim } \mathcal{C}(G \boxtimes H).\end{aligned}$$

We conclude that $\mathcal{B}$ is a basis for $\mathcal{C}(G \boxtimes H)$. It is an easy task to see that $\mathcal{B}$ satisfies the required fold. ■

The following corollary is a straightforward consequence of Theorem 3.1 and Theorem 3.2.

Corollary 3.1. *For every two bipartite graphs G and H, we have:*

$$\begin{aligned}&b(G \boxtimes H) \leq \\ &\min\Bigg\{\max\Bigg\{\left\lfloor\frac{3\Delta(T_G)+1}{2}\right\rfloor + b(H), \left\lfloor\frac{3\Delta(T_H)+1}{2}\right\rfloor + b(G), b(G)+b(H)+2\Bigg\}, \\ &\max\Bigg\{\left\lfloor\frac{3\Delta(T_H)+1}{2}\right\rfloor, 2\Delta(G)-1+b(H), b(G)+1, b(H)+2\Bigg\}, \\ &\max\Bigg\{\left\lfloor\frac{3\Delta(T_G)+1}{2}\right\rfloor, 2\Delta(H)-1+b(G), b(H)+1, b(G)+2\Bigg\}\Bigg\}.\end{aligned}$$

The following corollary with the help of Lemma 2.1 shows that the upper bound in Theorem 3.1 is the best possible.

Corollary 3.2 (Ali)**.** *For every path P of order ≥ 2 and cycle C, we have $b(P \boxtimes C) = 3$.*

We remark that by specializing the graph and the bipartite graph in Theorem 3.1 into a cycle and an even cycle, respectively, and by the aid of Lemma 2.1 we have the following result.

Corollary 3.3 (Ali). *For any cycle C and any even order cycle C^**, $3 \leq b(C \boxtimes C^*) \leq 4$.

The proof of the following result follows by taking $DVS(T)$ as in Theorem 2.2 and employing the same argument as in the proof of Lemma 3.1.

Corollary 3.4. *For any cycle C and any 3-special star T, we have* $b(C \boxtimes T) = 3$.

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