Discussiones Mathematicae
Graph Theory 25 (2005) 363–383

ON γ-LABELINGS OF TREES

Gary Chartrand

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008 USA

David Erwin

School of Mathematical Sciences
University of KwaZulu-Natal
Durban 4041, South Africa

Donald W. VanderJagt

Department of Mathematics
Grand Valley State University
Allendale, MI 49401 USA

and

Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008 USA

Abstract

Let G be a graph of order n and size m. A γ-labeling of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. The value of a γ-labeling f is $\text{val}(f) = \sum_{e \in E(G)} f'(e)$. The maximum value of a γ-labeling of G is defined as

$$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

while the minimum value of a γ-labeling of G is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

The values $\mathrm{val}_{\max}(S_{p,q})$ and $\mathrm{val}_{\min}(S_{p,q})$ are determined for double stars $S_{p,q}$. We present characterizations of connected graphs G of order n for which $\mathrm{val}_{\min}(G) = n$ or $\mathrm{val}_{\min}(G) = n + 1$.

Keywords: γ-labeling, value of a γ-labeling.

2000 Mathematics Subject Classification: 05C78, 05C05.

1. Introduction

For a graph G of order n and size m, a *γ-labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \to \{1, 2, \ldots, m\}$ of the edges of G defined by

$$f'(e) = |f(u) - f(v)| \text{ for each edge } e = uv \text{ of } G.$$

Therefore, a graph G of order n and size m has a γ-labeling if and only if $m \geq n - 1$. In particular, every connected graph has a γ-labeling. If the induced edge-labeling f' of a γ-labeling f is also one-to-one, then f is a *graceful labeling*, one of the most studied of graph labelings. An extensive survey of graph labelings as well as their applications has been given by Gallian [2].

Each γ-labeling f of a graph G of order n and size m is assigned a *value* denoted by $\mathrm{val}(f)$ and defined by

$$\mathrm{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since f is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge e in G and so

$$\mathrm{val}(f) \geq m. \tag{1}$$

Figure 1 shows nine γ-labelings $f_1, f_2, \ldots, f_9$ of the path P_5 of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each γ-labeling is shown in Figure 1 as well.

For a graph G of order n and size m, the *maximum value* of a γ-labeling of a graph G is defined as

$$\mathrm{val}_{\max}(G) = \max\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$

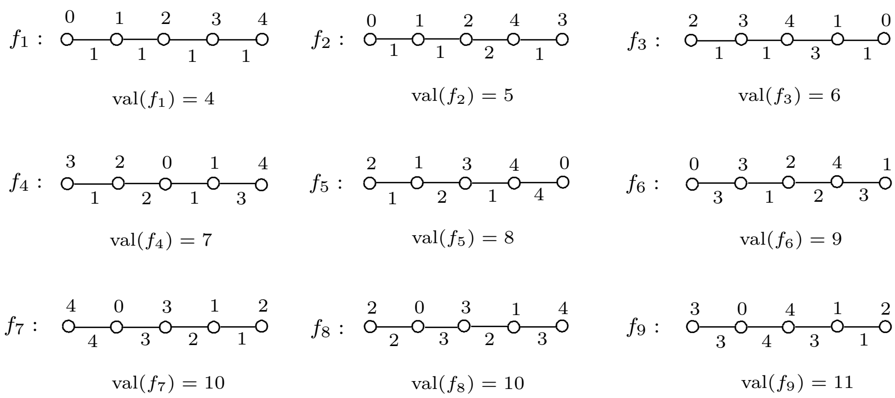

Figure 1: Some γ-labelings of P_5.

while the *minimum value* of a γ-labeling of G is

$$\mathrm{val}_{\min}(G) = \min\{\mathrm{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$

A γ-labeling g of G is a γ-max *labeling* if

$$\mathrm{val}(g) = \mathrm{val}_{\max}(G)$$

and a γ-labeling h is a γ-min *labeling* if

$$\mathrm{val}(h) = \mathrm{val}_{\min}(G).$$

Since $\mathrm{val}(f_1) = 4$ for the γ-labeling f_1 of P_5 shown in Figure 1 and the size of P_5 is 4, it follows that f_1 is a γ-min labeling of P_5. Although less clear, the γ-labeling f_9 shown in Figure 1 is a γ-max labeling. The concepts of a γ-labeling of a graph and the value of a γ-labeling were introduced in [1].

For a γ-labeling f of a graph G of size m, the *complementary labeling* $\overline{f} : V(G) \to \{0, 1, 2, \ldots, m\}$ of f is defined by

$$\overline{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\overline{f}$ a γ-labeling of G as well but $\mathrm{val}(\overline{f}) = \mathrm{val}(f)$. This gives us the following observation that appeared in [1].

Observation 1.1. Let f be a γ-labeling of a graph G. Then f is a γ-max labeling (γ-min labeling) of G if and only if $\overline{f}$ is a γ-max labeling (γ-min labeling).

A more general vertex labeling of a graph was introduced by Hegde in [3]. A vertex function f of a graph G is defined from $V(G)$ to the set of nonnegative integers that induces an edge function f' defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G. Such a function is called a *geodetic function* of G. A one-to-one geodetic function is a *geodetic labeling* of G if the induced edge function f' is also one-to-one. The following result was established by Hegde which provides an upper bound for $\text{val}_{\max}(G)$ (see [3]).

Theorem (Hegde). *For any geodetic γ-labeling f of a graph G of order n,*

$$\sum_{e \in E(G)} f'(e) \leq \sum_{i=0}^{n-1} (2i - n + 1) f(v_i).$$

The following results were obtained in [1] for the paths P_n and stars $K_{1,n-1}$ of order n.

Theorem A. *For each integer $n \geq 2$,*

$$\text{val}_{\min}(P_n) = n - 1 \text{ and } \text{val}_{\max}(P_n) = \left\lfloor \frac{n^2 - 2}{2} \right\rfloor.$$

Theorem B. *Let G be a connected graph of order n and size m. Then*

$$\text{val}_{\min}(G) = m \textit{ if and only if } G \cong P_n.$$

Theorem C. *For each integer $n \geq 3$,*

$$\text{val}_{\min}(K_{1,n-1}) = \binom{\left\lfloor \frac{n+1}{2} \right\rfloor}{2} + \binom{\left\lceil \frac{n+1}{2} \right\rceil}{2} \text{ and } \text{val}_{\max}(K_{1,n-1}) = \binom{n}{2}.$$

Theorem D. *For each integer $n \geq 3$,*

$$\text{val}_{\min}(C_n) = 2(n-1)$$

and

$$\mathrm{val}_{\max}(C_n) = \begin{cases} \dfrac{n(n+2)}{2} & \text{if } n \text{ is even,} \\ \dfrac{(n-1)(n+3)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

In this paper, we investigate γ-labelings of trees, beginning with double stars.

2. γ-Labelings of Double Stars

We now turn to the double star $S_{p,q}$ containing central vertices u and v such that $\deg u = p$ and $\deg v = q$ and determine $\mathrm{val}_{\min}(S_{p,q})$ and then $\mathrm{val}_{\max}(S_{p,q})$.

Proposition 2.1. *For integers $p, q \geq 2$,*

$$\mathrm{val}_{\min}(S_{p,q}) = \left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right)^2 + \left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right)^2 - \left(n_p \left\lfloor \frac{p+2}{2} \right\rfloor + n_q \left\lfloor \frac{q+2}{2} \right\rfloor + 1\right),$$

where

$$n_p = \begin{cases} 1 & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd} \end{cases} \quad \text{and} \quad n_q = \begin{cases} 1 & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Proof. Let $N(u) = \{v, u_1, u_2, \ldots, u_{p-1}\}$ and $N(v) = \{u, v_1, v_2, \ldots, v_{q-1}\}$. Since the proof is similar whether p and q are odd or even, we provide the proof in one of these four cases only, namely when p and q are odd. Let $p = 2s+1$ and $q = 2t+1$ for positive integers s and t. Define a γ-labeling f of $S_{p,q}$ by

$$f(x) = \begin{cases} s & \text{if } x = u, \\ 2s+t+1 & \text{if } x = v, \\ i-1 & \text{if } x = u_i,\ 1 \leq i \leq s, \\ i & \text{if } x = u_i,\ s+1 \leq i \leq 2s, \\ 2s+i & \text{if } x = v_i,\ 1 \leq i \leq t, \\ 2s+i+1 & \text{if } x = v_i,\ t+1 \leq i \leq 2t. \end{cases}$$

Thus exactly two edges in $\{uu_i:\ 1 \le i \le 2s\}$ are labeled a for each integer a with $1 \le a \le s$ and exactly two edges in $\{vv_i:\ 1 \le i \le 2t\}$ are labeled b for each integer b with $1 \le b \le t$. Furthermore, the edge uv is labeled $s+t+1$. Therefore,

$$\begin{aligned}\mathrm{val}(f) &= (s+t+1) + 2(1+2+\ldots+s) + 2(1+2+\ldots+t)\\ &= (s+t+1) + 2\binom{s+1}{2} + 2\binom{t+1}{2} = (s+1)^2 + (t+1)^2 - 1.\end{aligned}$$

Therefore,

$$\mathrm{val}_{\min}(S_{p,q}) \le (s+1)^2 + (t+1)^2 - 1.$$

Next, consider an arbitrary γ-labeling g of $S_{p,q}$. We may assume that $g(u) < g(v)$; otherwise, we could consider the complementary γ-labeling $\overline{g}$ of g. We show that

$$\mathrm{val}(g) \ge (s+1)^2 + (t+1)^2 - 1.$$

First, we make the following observations:

1. At most two edges in $\{uu_i : 1 \le i \le 2s\}$ can be labeled a for each integer a with $1 \le a \le s$ and this can occur only if the labels in $\{g(u) \pm a : 1 \le i \le s\}$ are available for the vertices u_i $(1 \le a \le 2s)$.
2. At most two edges in $\{vv_i : 1 \le i \le 2t\}$ can be labeled b for each integer b with $1 \le b \le t$ and this can occur only if the labels in $\{g(v) \pm b : 1 \le b \le t\}$ are available for the vertices v_i $(1 \le i \le 2t)$.

Therefore,

$$\sum_{e \in E(G)-\{uv\}} g'(e) \ge 2\binom{s+1}{2} + 2\binom{t+1}{2}.$$

Thus if $g'(uv) = g(v) - g(u) \ge s+t+1$, then

$$\mathrm{val}(g) \ge (s+t+1) + 2\binom{s+1}{2} + 2\binom{t+1}{2} = (s+1)^2 + (t+1)^2 - 1.$$

Suppose then that $g'(uv) = s+t+1-k$ for some integer k with $1 \le k \le s+t$. Then there are $s+t-k$ vertices of $S_{p,q}$ that are labeled with integers between $g(u)$ and $g(v)$. Consequently, $s+t+k$ vertices of $S_{p,q}$ are assigned a label less than $g(u)$ or greater than $g(v)$, which implies that at least k vertices of $S_{p,q}$ are assigned a label less than $g(u)-s$ or greater than $g(v)+t$. For each vertex u_i, $1 \le i \le 2s$, assigned a label less than $g(u)-s$,

$$\sum_{i=1}^{2s} g'(uu_i) \text{ must exceed } 2\binom{s+1}{2}$$

by at least 1; while for each vertex v_i, $1 \le i \le 2s$, assigned a label greater than $g(v)+t$,

$$\sum_{i=1}^{2t} g'(vv_i) \text{ must exceed } 2\binom{t+1}{2}$$

by at least 1. Therefore,

$$\sum_{e \in E(G)-\{uv\}} g'(e) \ge 2\binom{s+1}{2} + 2\binom{t+1}{2} + k.$$

However then,

$$\begin{aligned} \text{val}(g) &= g'(uv) + \sum_{e \in E(G)-\{uv\}} g'(e) \\ &\ge (s+t+1-k) + \left[2\binom{s+1}{2} + 2\binom{t+1}{2} + k\right] \\ &= (s+1)^2 + (t+1)^2 - 1. \end{aligned}$$

In general, $\text{val}(g) \ge (s+1)^2 + (t+1)^2 - 1$. Therefore, $\text{val}_{\min}(S_{p,q}) = (s+1)^2 + (t+1)^2 - 1$. ■

Theorem 2.2 *For every pair p, q of positive integers,*

$$\text{val}_{\max}(S_{p,q}) = \frac{1}{2}\left[p^2 + q^2 + 4pq - 3p - 3q + 2\right].$$

Proof. Let u and v be the central vertices of $S_{p,q}$, where $\deg u = p$ and $\deg v = q$, and let f be the γ-labeling of $S_{p,q}$ in which we assign the label 0 to u, the label $p+q-1$ to v, the labels $1, 2, \ldots, q-1$ to the end-vertices adjacent to v, and the labels $q, q+1, \ldots, p+q-2$ to the end-vertices adjacent to u. The value of f is $(p^2+q^2+4pq-3p-3q+2)/2$, which is therefore a lower bound for $\text{val}_{\max}(S_{p,q})$.

We now show that $\text{val}_{\max}(S_{p,q}) \leq (p^2+q^2+4pq-3p-3q+2)/2$. First, we claim that $S_{p,q}$ has a γ-max labeling for which $\{f(u), f(v)\} = \{0, p+q-1\}$. We verify this claim by induction on $p+q$. The claim is clearly true for $p+q=2$. Assume that the claim is true for $p+q=k-1$, where $k \geq 3$. Let $T = S_{p,q}$, where $p+q=k$. Let f be a γ-max labeling of T. If $\{f(u), f(v)\} = \{0, p+q-1\}$, then the claim is true. Suppose that at least one $f(u)$ and $f(v)$ is neither 0 nor $p+q-1$. By Observation1.1, we may assume that $f(w) = p+q-1$ and $w \neq u, v$. The vertex w is therefore an end-vertex of T. Let $x \in \{u, v\}$ be the vertex of T that is adjacent to w. Then $T' = T - w$ is isomorphic to $S_{p',q'}$, where $p'+q' = k-1$. By the inductive hypothesis, T' has a γ-max labeling g for which $\{g(u), g(v)\} = \{0, p+q-2\}$. By Observation1.1, we may assume that $g(x) = 0$. Now

$$\text{(2)} \quad \text{val}(f) = (p+q-1-f(x)) + \sum_{e \in E(T')} f'(e) \leq p+q-1+\text{val}_{\max}(T').$$

We extend g to a γ-labeling h of T by defining $h(w) = p+q-1$. Then

$$\text{(3)} \qquad \text{val}(h) = p+q-1+\text{val}_{\max}(T').$$

By (2) and (3), $\text{val}(f) \leq \text{val}(h)$. Since f is a γ-max labeling of T, so too is h a γ-max labeling of T. Let $y \in \{u, v\}$ for which $h(y) = p+q-2$. Thus y is not adjacent to w. Next, let ϕ be the γ-labeling of T defined by

$$\phi(z) = \begin{cases} h(z) & \text{if } z \neq w, y, \\ p+q-1 & \text{if } z = y, \\ p+q-2 & \text{if } z = w. \end{cases}$$

Then $\text{val}(\phi) = \text{val}(h)$ if $\deg y \leq 2$; while $\text{val}(\phi) > \text{val}(h)$ if $\deg y \geq 3$. Since $\text{val}(\phi)$ cannot exceed $\text{val}(h)$, it follows that $\deg y \leq 2$, and ϕ has the desired property that verifies the claim. By the claim and Observation 1.1, there is a γ-max labeling f of $S_{p,q}$ with $f(u) = 0$ and $f(v) = p+q-1$.

If there is an end-vertex t_1 of $S_{p,q}$ adjacent to v with $f(t_1) = i > q-1$, then there is an end-vertex t_2 of $S_{p,q}$ adjacent to u with $f(t_2) = j$, where $1 \le j \le q-1$. Interchanging the labels of t_1 and t_2 produces a γ-labeling f_1 with $\mathrm{val}(f_1) > \mathrm{val}(f)$, which is impossible. Thus f is the γ-labeling described in the first paragraph of the proof and $\mathrm{val}(f) = (p^2+q^2+4pq-3p-3q+2)/2$. ■

3. Connected Graphs of Order n with Minimum Value n

We already mentioned (in Theorem B) that a connected graph G of order n has minimum value $n-1$ if and only if $G \cong P_n$. We now determine all those connected graphs G of order n for which $\mathrm{val}_{\min}(G) = n$. It is useful to present several lemmas first.

Lemma 3.1. *If G is a connected graph of size m and G' is a connected subgraph of G having size m', then*

$$\mathrm{val}_{\min}(G) \ge (m-m') + \mathrm{val}_{\min}(G').$$

Proof. Suppose that G has order n and G' has order n'. Let f be a γ-min labeling of G. Then the restriction h of f to G' is a one-to-one function. Suppose that the vertices of G' are labeled $a_1, a_2, \cdots, a_{n'}$ by h, where $0 \le a_1 < a_2 < \cdots < a_{n'} \le m$. Thus, for $1 \le i \ne j \le n'$, $|a_i - a_j| \ge |i-j|$. Consider the one-to-one function $g : \{a_1, a_2, \cdots, a_{n'}\} \to \{0, 1, 2, \cdots, m'\}$ defined by $g(a_i) = i-1$ for $1 \le i \le n'$. Then $\phi = g \circ h : V(G') \to \{0, 1, 2, \cdots, m'\}$ is a γ-labeling of G'. Furthermore,

$$\mathrm{val}_{\min}(G') \le \mathrm{val}(\phi) \le \sum_{e \in E(G')} h'(e) = \sum_{e \in E(G')} f'(e).$$

Since $f'(e) \ge 1$ for every edge e in G, it follows that

$$\begin{aligned}\mathrm{val}(f) &= \sum_{e \in E(G-G')} f'(e) + \sum_{e \in E(G')} f'(e) \\ &\ge (m-m') + \mathrm{val}_{\min}(G'),\end{aligned}$$

as desired. ■

Lemma 3.1 can be extended to obtain the following result.

Lemma 3.2. *If G is a connected graph of size m containing pairwise edge-disjoint connected subgraphs $G_1, G_2, \cdots, G_k$, where G_i has size m_i for $1 \leq i \leq k$, then*

$$\mathrm{val}_{\min}(G) \geq \left(m - \sum_{i=1}^{k} m_i\right) + \sum_{i=1}^{k} \mathrm{val}_{\min}(G_i).$$

Lemma 3.3. *Let G be a connected graph of order n with maximum degree Δ. Then*

$$\mathrm{val}_{\min}(G) \geq \begin{cases} (n-1) + k(k-1) & \text{if } \Delta = 2k, \\ (n-1) + k^2 & \text{if } \Delta = 2k+1. \end{cases}$$

Furthermore, this bound is sharp for stars.

Proof. Let $v \in V(G)$ with $\deg v = \Delta$ and let f be a γ-min labeling of G. Note that at most two edges incident with v can be labeled i for each i with $1 \leq i \leq \lfloor \Delta/2 \rfloor$. Thus, if $\Delta = 2k$, then

$$\mathrm{val}_{\min}(G) \geq (n - 1 - 2k) + 2(1 + 2 + \cdots + k) = (n-1) + k(k-1);$$

while if $\Delta = 2k + 1$, then

$$\mathrm{val}_{\min}(G) \geq [(n-1) - (2k+1)] + 2(1 + 2 + \cdots + k) + (k+1) = (n-1) + k^2.$$

That this bound is sharp for stars follows from Theorem C. ∎

The proof of the next lemma is straightforward and is therefore omitted.

Lemma 3.4. *Let f be a γ-labeling of a connected graph G. If P is a $u - v$ path in G, then*

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)|.$$

Lemma 3.5. *For the tree F of Figure* 2, $\mathrm{val}_{\min}(F) = 8$.

Proof. The γ-labeling f of F shown in Figure 2 has value 8 and so $\mathrm{val}_{\min}(F) \leq 8$. On the other hand, let g be γ-min labeling of F and

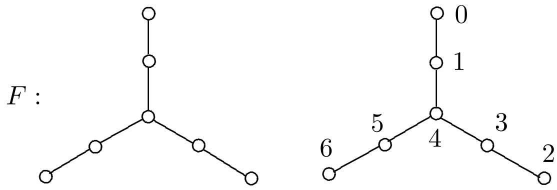

Figure 2: A tree F and a γ-labeling of F.

let $u, v \in V(F)$ such that $g(u) = 0$ and $g(v) = 6$. Suppose that P is a $u - v$ path in F. Then

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = 6$$

by Lemma 3.4. Since there are at least two edges of F not in P, it follows that $\mathrm{val}_{\min}(F) = \mathrm{val}(g) \geq 8$. ■

A *caterpillar* is a tree the removal of whose vertices results in a path. We are now able to characterize all connected graphs of order $n \geq 4$ whose minimum value is n.

Theorem 3.6. *Let G be a connected graph of order $n \geq 4$. Then $\mathrm{val}_{\min}(G) = n$ if and only if G is a caterpillar, $\Delta(G) = 3$, and G has a unique vertex of degree 3.*

Proof. Let T be the tree obtained from the path $v_1, v_2, \cdots, v_{n-1}$ by adding the vertex v_n and joining v_n to a vertex v_k, where $2 \leq k \leq n-2$. Thus v_k is the only vertex of degree 3 in T. Define a γ-labeling f of T by

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \leq i \leq k, \\ i & \text{if } k < i \leq n-1, \\ k & \text{if } i = n. \end{cases}$$

Since $\mathrm{val}(f) = n$, it follows that $\mathrm{val}_{\min}(T) \leq n$ and so $\mathrm{val}_{\min}(T) = n$ by Theorem B.

For the converse, let G be a connected graph of order $n \geq 4$ such that G is not a caterpillar with $\Delta(G) = 3$ containing a unique vertex of degree 3. We show that $\mathrm{val}_{\min}(G) \neq n$. This is certainly true if $G \cong P_n$ or if G is not a tree by Theorem B. Hence we may assume that G is a tree T with

$\Delta(T) \geq 3$. If $\Delta(T) \geq 4$, then $\mathrm{val}_{\min}(T) \geq (n-1)+2 = n+1$ by Lemma 3.3. Thus $\Delta(T) = 3$. We consider two cases.

Case 1. T contains two vertices u and v with degree 3.
If u and v are adjacent, then T contains the double star $S_{3,3}$ as a subgraph. By Theorem 2.2, $\mathrm{val}_{\min}(S_{3,3}) = 7$. Since the order of $S_{3,3}$ is 6, it then follows by Lemma 3.1 that $\mathrm{val}_{\min}(T) \geq (n-6)+7 = n+1$.

Thus we may assume that u and v are not adjacent. Let $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Then $v \notin N(u)$ and $u \notin N(v)$. For any γ labeling g of T, $g'(e) \geq 2$ for at least one edge e in $\{uu_i : 1 \leq i \leq 3\}$ and at least one edge e in $\{vv_i : 1 \leq i \leq 3\}$. Therefore, at least two edges in T are labeled 2 or more by g and so $\mathrm{val}_{\min}(T) \geq \mathrm{val}(g) \geq n+1$.

Case 2. T has exactly one vertex with degree 3.
Thus T contains the graph F in Lemma 3.5 as a subgraph. Since $\mathrm{val}_{\min}(F) = 8$ by Lemma 3.5 and the order of F is 7, it then following by Lemma 3.1 that $\mathrm{val}_{\min}(T) \geq (n-7)+8 = n+1$. ∎

4. Some Results on the Minimum Value of a Tree in Terms of Its Order and Other Parameters

In Theorem 3.6, we considered caterpillars T having maximum degree 3 and a unique vertex of degree 3. We now compute the minimum value of all such trees that are not necessarily caterpillars.

Theorem 4.1. *Let T be a tree of order $n \geq 4$ such that $\Delta(T) = 3$ and T has a unique vertex v of degree 3. If d is the distance between v and a nearest end-vertex, then*

$$\mathrm{val}_{\min}(T) = n + d - 1.$$

Proof. Let x, y, and z be the three end-vertices of T, where $d(v,x) = d$, $d(v,y) = d'$, and $d(v,z) = d''$, where $d \leq d' \leq d''$. Let $P : v = v_0, v_1, \cdots, v_d = x$, $P' : v = u_0, u_1, \cdots, u_{d'} = y$, and $P'' : v = w_0, w_1, \cdots, w_{d''} = z$ denote the $v-x$ path, $v-y$ path, and $v-z$ path in T. Let $f : V(T) \to \{0, 1, 2, \cdots, n-1\}$ be the γ-labeling of T for which $f(w_i) = d'' - i$ for $0 \leq i \leq d''$, $f(v_i) = d'' + i$ for $1 \leq i \leq d$, and $f(u_i) = i - d' + n - 1$ for $1 \leq i \leq d'$. Since $\mathrm{val}(f) = n + d - 1$, it follows that $\mathrm{val}_{\min}(T) \leq n + d - 1$.

It remains therefore to show that $\text{val}_{\min}(T) \geq n + d - 1$. Let $g : V(T) \to \{0, 1, 2, \cdots, n-1\}$ be an arbitrary γ-labeling of T, and suppose that $g(v) = i$. Let

$$S = \{u \in V(T) : d(u, v) \leq d\}.$$

Thus $|S| = 3d + 1$. Let a denote the smallest label assigned by g to a vertex of S and let b denote the largest such label. We now consider two cases.

Case 1. The vertices in S labeled a and b belong to two of the three paths P, P', and P'', say P and P', respectively. Then

$$\sum_{e \in E(P)} g'(e) \geq i - a \text{ and } \sum_{e \in E(P')} g'(e) \geq b - i.$$

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (i - a) + (b - i) + d = b - a + d \geq 3d + d = 4d.$$

Since there are $(n - 1) - 3d$ edges of T not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 4d + (n - 1 - 3d) = n + d - 1.$$

Case 2. The vertices in S labeled a and b belong to one of the three paths P, P', and P'', say P. Then

$$\sum_{e \in E(P)} g'(e) \geq b - a.$$

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (b - a) + 2d \geq 3d + 2d = 5d.$$

Since there are $(n - 1) - 3d$ edges of T not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 5d + (n - 1 - 3d) = n + 2d - 1.$$

In general, $\sum_{e \in E(T)} g'(e) \geq n + d - 1$ and so $\text{val}_{\min}(T) \geq n + d - 1$. ■

Next, we generalize Theorem 3.6 to caterpillars T with $\Delta(T) = 3$ having an arbitrary number of vertices of degree 3.

Theorem 4.2. *If T is a caterpillar of order $n \geq 4$ such that $\Delta(T) = 3$ and T has exactly k vertices of degree 3, then*

$$\mathrm{val}_{\min}(T) = n + k - 1.$$

Proof. Let T be a caterpillar of order $n \geq 4$ with $\Delta(T) = 3$ such that T contains k vertices of degree 3. Then $\mathrm{diam}(T) = n - k - 1$. Let $P : v_0, v_1, v_2, \cdots, v_{n-k-1}$ be a path of length $n - k - 1$ in T. Let $i_1, i_2, \cdots, i_k$ be integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n - k - 2$ and $\deg v_{i_j} = 3$ for $1 \leq j \leq k$. Let u_j be the vertex not on P that is adjacent to v_{i_j}, where $1 \leq j \leq k$. Furthermore, let $f : V(T) \to \{0, 1, \cdots, n-1\}$ be the γ-labeling of T defined by

$$f(v_t) = \begin{cases} d(v_t, v_0) & \text{if } t \leq i_1, \\ d(v_t, v_0) + \max\{j : i_j < t\} & \text{otherwise} \end{cases}$$

and

$$f(u_j) = 1 + f(v_{i_j}).$$

Since $\mathrm{val}(f) = n + k - 1$, it follows that $\mathrm{val}_{\min}(T) \leq n + k - 1$.

Next, we show that $\mathrm{val}_{\min}(T) \geq n + k - 1$. Let

$$f : V(T) \to \{0, 1, 2, \cdots, n-1\}$$

be an arbitrary γ-labeling of T and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let P be a $u - v$ path in T. The length of P is at most $\mathrm{diam}(T) = n - k - 1$. Also, by Lemma 3.3

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = n - 1.$$

Since there are at least k edges of T not on P, it follows that

$$\mathrm{val}(f) = \sum_{e \in E(T)} f'(e) \geq (n-1) + k,$$

and so $\mathrm{val}_{\min}(T) \geq n + k - 1$. ■

We now present a lower bound for the minimum value of a tree in terms of its order, maximum degree, and diameter.

Theorem 4.3. *If T is a tree of order $n \geq 4$, maximum degree Δ, and diameter d, then*

$$\mathrm{val}_{\min}(T) \geq \frac{8n + \Delta^2 - 6\Delta - 4d + \delta_\Delta}{4},$$

where

$$\delta_\Delta = \begin{cases} 0 & \text{if } \Delta \text{ is even,} \\ 1 & \text{if } \Delta \text{ is odd.} \end{cases}$$

Furthermore, this bound is sharp for paths and stars.

Proof. Let f be a γ-labeling of T and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let P be a $u - v$ path in T. Let x be a vertex of T with $\deg x = \Delta$. We consider two cases.

Case 1. $\Delta = 2k$ *for some integer* $k \geq 1$. Since (1) at most two edges of T incident with x can be labeled by i for each i with $1 \leq i \leq (k-1)$ and (2) the length of P is at most d, it follows that

$$\begin{aligned} \mathrm{val}(f) &\geq (n-1) + 2[1 + 2 + \cdots + (k-1)] + [(n-1-d) - (2k-2)] \\ &= 2n + k^2 - 3k - d = 2n + \frac{\Delta^2}{4} - \frac{3\Delta}{2} - d \\ &= \frac{8n + \Delta^2 - 6\Delta - 4d}{4}. \end{aligned}$$

Case 2. $\Delta = 2k + 1$ *for some integer* $k \geq 1$. By the same reasoning used in Case 1,

$$\begin{aligned} \mathrm{val}(f) &\geq (n-1) + 2[1 + 2 + \cdots + (k-1)] + k + [(n-1-d) - (2k-1)] \\ &= 2n - 1 + k^2 - 2k - d = 2n + \frac{(\Delta-1)^2}{4} - \Delta - d \\ &= \frac{8n + \Delta^2 - 6\Delta - 4d + 1}{4}. \end{aligned}$$

That this bound is sharp for paths and stars follows by Theorems B and C. ∎

5. Connected Graphs of Order n with Minimum Value $n+1$

In Theorem 3.6, all connected graphs of order $n \geq 4$ having minimum value n are characterized. In particular, if T is a caterpillar of order $n \geq 4$ whose only vertex of degree exceeding 2 has degree 3, then $\mathrm{val}_{\min}(T) = n$. In this section, we characterize those connected graphs of order $n \geq 5$ having minimum value $n+1$. First, we show that every caterpillar of order $n \geq 5$ whose unique vertex of degree exceeding 2 has degree 4 must have minimum value $n+1$.

Lemma 5.1. *Let T be a caterpillar of order $n \geq 5$. If T has a unique vertex v with degree greater than 2 and $\deg v = 4$, then*

$$\mathrm{val}_{\min}(T) = n+1.$$

Proof. By Lemma 3.3, $\mathrm{val}_{\min}(T) \geq n+1$. It remains to show that $\mathrm{val}_{\min}(T) \leq n+1$. Suppose that T is obtained from path $v_1, v_2, \cdots, v_{n-2}$ by adding the vertices v_{n-1} and v_n and joining each of v_{n-1} and v_n to a vertex v_k, where $2 \leq k \leq n-3$. Thus v_k is the only vertex of degree greater than 2 in T and $\deg v_k = 4$. Define a γ-labeling f of T by

$$f(v_i) = \begin{cases} i-1 & \text{if } 1 \leq i \leq k-1, \\ i & \text{if } i = k, \\ i+1 & \text{if } k+1 \leq i \leq n-2, \\ k-1 & \text{if } i = n-1, \\ k+1 & \text{if } i = n. \end{cases}$$

Since $\mathrm{val}(f) = n+1$, it follows that $\mathrm{val}_{\min}(T) \leq n+1$. ■

For a fixed integer n, let $\mathcal{T}_1$ be the set of caterpillars T of order $n \geq 5$ such that T has a unique vertex v with degree greater than 2 and $\deg v = 4$ (as described in Lemma 5.1), let $\mathcal{T}_2$ be the set of trees T of order n such that T is a caterpillar of order $n \geq 6$ with $\Delta(T) = 3$ and T has exactly two vertices of degree 3, and let $\mathcal{T}_3$ be the set of trees T of order $n \geq 7$ such that T has a unique vertex v of degree greater than 2 and $\deg v = 3$, where the distance between v and a nearest end-vertex of T is 2. By Lemma 5.1 and Theorems 4.1 and 4.2, we have the following.

Corollary 5.2. *Let T be a tree of order n. If $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then* $\text{val}_{\min}(T) = n + 1$.

Lemma 5.3. *Each of the threes F_1, F_2, and F_3 in Figure 3 of order $n = 9, 8, 8$, respectively, has minimum value $n + 2$, that is,*

$$\text{val}_{\min}(F_1) = 11 \textit{ and } \text{val}_{\min}(F_2) = \text{val}_{\min}(F_3) = 10.$$

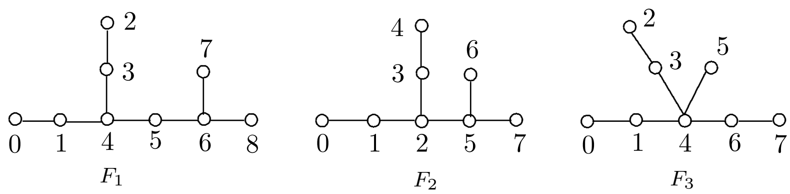

Figure 3: The graphs F_1, F_2, and F_3.

Proof. For each integer i with $1 \leq i \leq 3$, a γ-labeling f_i of F_i is shown in Figure 3. Since $\text{val}(f_1) = 11$ and $\text{val}(f_2) = \text{val}(f_3) = 10$, it follows that $\text{val}_{\min}(F_1) \leq 11$, $\text{val}_{\min}(F_2) \leq 10$, and $\text{val}_{\min}(F_3) \leq 10$.

Next, we show that $\text{val}_{\min}(F_1) \geq 11$. Let g be γ-min labeling of F_1 and let $u, v \in V(F_1)$ such that $g(u) = 0$ and $g(v) = 8$. Suppose that P is a $u - v$ path in F_1. Then $\sum_{e \in E(P)} f'(e) \geq 8$ by Lemma 3.4. Since there are at least three edges of F_1 not in P, it follows that $\text{val}_{\min}(F_1) = \text{val}(g) \geq 8 + 3 = 11$. A similar argument shows that $\text{val}_{\min}(F_2) \geq 10$, and $\text{val}_{\min}(F_3) \geq 10$. ■

We now characterize all trees of order $n \geq 5$ whose minimum value is $n + 1$.

Theorem 5.4. *Let T be a tree of order $n \geq 5$. Then* $\text{val}_{\min}(T) = n + 1$ *if and only if $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$.*

Proof. By Corollary 5.2, if $T \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, then $\text{val}_{\min}(T) = n + 1$. It therefore remains to verify the converse. We begin by establishing the following three claims.

Claims. Let T be a tree of order $n \geq 7$ such that $\text{val}_{\min}(T) = n + 1$ and $T \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. Then:

(1) $3 \leq \Delta(T) \leq 4$.

(2) T has at most two vertices of degree greater than 2.

(3) If v is a vertex of T with $\deg v \geq 3$, then the distance between v and a nearest end-vertex in T is at most 2.

Proof of Claims. Since $\mathrm{val}_{\min}(T) = n + 1$, it follows that T is not a path by Theorem B and so $\Delta(T) \geq 3$. If $\Delta(T) \geq 5$, then $\mathrm{val}_{\min}(T) \geq (n-1) + 2^2 = n + 3$ by Lemma 3.3, a contradiction. Thus $3 \leq \Delta(T) \leq 4$ and so Claim (1) holds.

Next we verify Claim (2). Suppose that T has $k \geq 3$ vertices of degree greater than 2. Then T contains a caterpillar T' of order n' as a subgraph with $\Delta(T') = 3$ such that T' has exactly three vertices of degree 3. By Theorem 4.2, $\mathrm{val}_{\min}(T') = n' + 2$. It then follows from Lemma 3.1 that

$$\mathrm{val}_{\min}(T) \geq [(n-1) - (n'-1)] + \mathrm{val}_{\min}(T') \geq (n - n') + (n' + 2) = n + 2,$$

a contradiction. Thus Claim (2) holds.

We now verify Claim (3). Let v be a vertex of T with $\deg v \geq 3$. If the distance between v and a nearest end-vertex in T is greater than 2, then T contains a subtree T'' of order n'' such that (a) $\Delta(T'') = 3$ and T'' has a unique vertex v of degree 3 and (b) the distance d between v and a nearest end-vertex in T'' is greater than 2. By Theorem 4.1,

$$\mathrm{val}_{\min}(T'') = n' + d - 1 \geq n' + 2.$$

Again, by Lemma 3.1,

$$\mathrm{val}_{\min}(T) \geq [(n-1) - (n'-1)] + \mathrm{val}_{\min}(T') \geq (n - n') + (n' + 2) = n + 2,$$

a contradiction. Thus Claim (3) holds. This completes the proof of the three claims.

We continue with the proof of the theorem. Assume, to the contrary, that there is a tree T of order $n \geq 7$ with $\mathrm{val}_{\min}(T) = n + 1$ such that $T \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. By Claim (1), $3 \leq \Delta(T) \leq 4$. We consider two cases, according to whether $\Delta(T) = 3$ or $\Delta(T) = 4$.

Case 1. $\Delta(T) = 3$. If T is a caterpillar, then T contains exactly two vertices of degree 3 by Theorem 4.2. However then, $T \in \mathcal{T}_2$, a contradiction. Thus T is not a caterpillar. If T has exactly one vertex x of degree 3, then the distance between x and a nearest end-vertex of T is 2 by Theorem 4.1. However then, $T \in \mathcal{T}_3$, again a contradiction. Thus T is not a caterpillar and T contains exactly two vertices u and v of degree 3 by Claim (2). Furthermore, we may assume that the distance d from u to a nearest end-vertex of T is 2 by Claim (3). We consider three subcases.

Subcase 1.1. $d(u,v) \geq 3$. Then T contains two edge-disjoint subgraphs H_1 and H_2 such that H_1 is isomorphic to the graph F in Lemma 3.5 and H_2 is isomorphic to $K_{1,3}$. Let f be a γ-min labeling of T. Since $\text{val}_{\min}(H_1) = 8$ by Lemma 3.5 and $\text{val}_{\min}(H_2) = 4$ by Theorem C, it follows by Lemma 3.2 that

$$\text{val}_{\min}(T) \geq [(n-1) - 6 - 3] + (8 + 4) = n + 2,$$

a contradiction.

Subcase 1.2. $d(u,v) = 2$. Then T contains the graph F_1 of Lemma 5.3 as a subgraph. Since the size of F_1 is 8 and $\text{val}_{\min}(F_1) = 11$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n-1)-8]+11 = n+2$, which produces a contradiction.

Subcase 1.3. $d(u,v) = 1$. Then T contains the graph F_2 of Lemma 5.3 as a subgraph. Since the size of F_2 is 7 and $\text{val}_{\min}(F_2) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n-1) - 7] + 10 = n + 2$, a contradiction.

Case 2. $\Delta(T) = 4$. There are two subcases.

Subcase 2.1. *T has a unique vertex v of degree exceeding* 2. Then $\deg v = 4$. If T is a caterpillar, then $T \in \mathcal{T}_1$, a contradiction. Thus T is not a caterpillar. However then, T contains the graph F_3 of Lemma 5.3 as a subgraph. Since the size of F_3 is 7 and $\text{val}_{\min}(F_3) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n-1) - 7] + 10 = n + 2$, a contradiction.

Subcase 2.2. *T has two vertices u and v of degree exceeding* 2. If T is not a caterpillar, then $\text{val}_{\min}(T) \geq n + 2$ by the proofs of Subcases 1.1, 1.2, and 1.3 in Case 1, which is a contradiction. Thus we may assume that T is a caterpillar and $\deg u = 4$. There are two subcases.

Subcase 2.2.1. $d(u,v) \geq 2$. Then T contains two edge-disjoint subgraphs isomorphic to $K_{1,4}$ and $K_{1,3}$, respectively. Let f be a γ-min labeling of T.

Since $\text{val}_{min}(K_{1,4}) = 6$ and $\text{val}_{min}(K_{1,3}) = 4$ by Theorem C, it follows from Lemma 3.2 that $\text{val}_{\text{min}}(T) \geq [(n-1)-4-3]+6+4 = n+2$, a contradiction.

Subcase 2.2.2. $d(u, v) = 1$. Then T contains the double star $S_{4,3}$ as a subgraph. Since the size of $S_{4,3}$ is 6 and $\text{val}_{\text{min}}(S_{4,3}) = 9$ by Proposition 2.1, it follows by Lemma 3.1 that $\text{val}_{\text{min}}(T) \geq [(n-1)-6]+9 = n+2$, a contradiction. ■

We next characterize all connected graphs G of order n for which $\text{val}_{\text{min}}(G) = n+1$. First, we present two lemmas. Since the proofs are straightforward, we omit them.

Lemma 5.5. *For the graph H of Figure* 4, $\text{val}_{\text{min}}(H) = 9$.

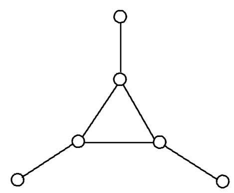

Figure 4: The graph H of Lemma 5.5.

Let $\mathcal{F}$ be the set of all graphs of order $n \geq 3$ obtained from the path $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+2} for some i with $1 \leq i \leq n-2$.

Lemma 5.6. *If $F \in \mathcal{F}$, then* $\text{val}_{\text{min}}(F) = n+1$.

Theorem 5.7. *Let G be a connected graph of order n. Then* $\text{val}_{\text{min}}(G) = n+1$ *if and only if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}$.*

Proof. We have seen in Theorem 5.4 and Lemma 5.6 that if $G \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{F}$, then $\text{val}_{\text{min}}(G) = n+1$. For the converse, let G be a connected graph for which $\text{val}_{\text{min}}(G) = n+1$ such that $G \notin \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$. It then follows from Theorem 5.4 that G is not a tree. Hence G contains cycles. By Theorem B, G contains exactly one cycle C and so G has size n. Suppose that C is a k-cycle, where $k \geq 3$. Since $\text{val}_{\text{min}}(G) = 2k-2$ by Theorem D, it follows by Lemma 3.1 that

$$\text{val}_{\text{min}}(G) \geq (n-k) + (2k-2) = n+k-2.$$

Since $\text{val}_{\min}(G) = n + 1$, the cycle C is a triangle. If G contains the graph H of Figure 4 as a subgraph, then by Lemmas 5.5 and 3.1,

$$\text{val}_{\min}(G) \geq (n-6) + \text{val}_{\min}(H) = (n-6) + 9 = n+3,$$

which is impossible. Therefore, at least one vertex of C has degree 2 in G. Furthermore, G contains no vertex of degree 4 or more; for otherwise, G contains $K_{1,4}$ as a subgraph and by Lemma 3.1 and Theorem C,

$$\text{val}_{\min}(G) \geq (n-4) + \text{val}_{\min}(K_{1,4}) = (n-4) + 6 = n+2,$$

a contradiction. Also, observe that there cannot be a vertex of degree 3 that does not belong to C; for otherwise, G contains edge-disjoint subgraphs K_3 and $K_{1,3}$ and by Lemma 3.2, Theorems C and D,

$$\begin{aligned} \text{val}_{\min}(G) &\geq (n-3-3) + \text{val}_{\min}(K_3) + \text{val}_{\min}(K_{1,3}) \\ &= (n-6) + 4 + 4 = n+2, \end{aligned}$$

which is impossible. This implies that $G \in \mathcal{F}$. ■

Acknowledgments

We are grateful to the referees whose valuable suggestions resulted in an improved paper.

References

[1] G. Chartrand, D. Erwin, D.W. VanderJagt and P. Zhang, *γ-Labelings of graphs*, Bull. Inst. Combin. Appl. **44** (2005) 51–68.

[2] J.A. Gallian, *A dynamic survey of graph labeling*, Electron. J. Combin. #DS6 (Oct. 2003 Version).

[3] S.M. Hegde, *On (k,d)-graceful graphs*, J. Combin. Inform. System Sci. **25** (2000) 255–265.

Received 16 April 2004
Revised 6 November 2004