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ON THE p-DOMINATION NUMBER OF CACTUS GRAPHS

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Abstract

Let p be a positive integer and $G = (V, E)$ a graph. A subset S of V is a p-dominating set if every vertex of $V - S$ is dominated at least p times. The minimum cardinality of a p-dominating set a of G is the p-domination number $\gamma_p(G)$. It is proved for a cactus graph G that $\gamma_p(G) \leqslant (|V| + |L_p(G)| + c(G))/2$, for every positive integer $p \geqslant 2$, where $L_p(G)$ is the set of vertices of G of degree at most $p - 1$ and $c(G)$ is the number of odd cycles in G.

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1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is $n(G) = |V(G)|$ and the *degree* of a vertex v, denoted by $\deg_G(v)$, is the number of vertices adjacent to v. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex.* A vertex of V is called a *cut vertex* if removing it from G increases the number of components of G. A graph G is called a *cactus graph* if each edge of G is contained in at most one cycle. A *unicycle graph* is a graph with exactly one cycle. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex, is denoted by $S_{p,q}$.

For a positive integer p, a subset S of $V(G)$ is a *p-dominating set* if every vertex not in S is adjacent to at least p vertices of S. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of a p-dominating set of G. Note that every graph G has a p-dominating set, since $V(G)$ is such a set. Also the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. The concept of p-domination was introduced by Fink and Jacobson [2, 3]. For more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [4, 5].

We make a straightforward observation.

Observation 1. Every p-dominating set of a graph G contains any vertex of degree at most $p - 1$.

In this paper we present an upper bound for the p-domination number for cactus graphs in terms of the order, the number of odd cycles and the number of vertices of degrees at most $p - 1$.

The following result due to Blidia et al. [1] will be useful for the next. Let $L_p(G)$ denote the set $\{x \in V(G) : \deg_G(x) \leqslant p - 1\}$.

Theorem 2 (Blidia, Chellali and Volkmann [1])**.** *Let p be a positive integer. If G is a bipartite graph then*

$$\gamma_p(G) \leqslant (n + |L_p(G)|)/2\,.$$

2. Main Results

We begin by giving an upper bound for the p-domination number for connected unicycle graphs.

Theorem 3. *Let $p \geqslant 2$ be a positive integer. If G is a connected unicycle graph then*

$$\gamma_p(G) \leqslant (n + |L_p(G)| + 1)/2$$

and this bound is sharp.

Proof. Let G be a connected unicycle graph. If G is bipartite then the result is valid by Theorem 2. So assume that G contains an odd cycle denoted by C. If $G = C$, then $\gamma_p(G) = n$ if $p \geqslant 3$ and $\gamma_p(G) = (n+1)/2$ if $p = 2$, in both cases the result holds. Thus we assume that $G \neq C$, that is G contains at least one leaf.

Suppose that the result does not hold and let G be the smallest connected unicycle graph such that $\gamma_p(G) > (n+|L_p(G)|+1)/2$. We claim that every vertex on C has degree exactly p. Suppose to the contrary that there is a vertex $x \in C$ such that $\deg_G(x) \neq p$ and let y be one of its two neighbors on C. Let G' be the spanning graph of G obtained by removing the edge xy. Then G' is tree and so a bipartite graph. We also have $|L_p(G')| \leqslant |L_p(G)|+1$ and $n(G) = n(G')$. According to Theorem 2, we deduce that

$$\gamma_p(G) \leqslant \gamma_p(G') \leqslant (n(G') + \left|L_p(G')\right|)/2 \leqslant (n(G) + |L_p(G)| + 1)/2,$$

a contradiction with our assumption.

Similarly, we will show that every vertex not on C and different to a leaf has degree at least p. Assume to the contrary that there is a vertex $x \in V(G) - C$ different to a leaf with $\deg_G(x) \leqslant p - 1$ and let z be its neighbor in the unique path from x to C. Let G_1 be the connected unicycle subgraph of G containing x and obtained by removing all the edges incident to x excepted the edge xz, and let G_2 be the component containing x by removing the edge xz. Let D_1 and D_2 denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Clearly G_1 contains C and G_2 is a tree, $x \in D_1 \cap D_2$, $x \in L_p(G_1) \cap L_p(G_2)$, $|L_p(G_1)|+|L_p(G_2)| = |L_p(G)|+1$ and $n(G_1)+n(G_2) = n(G)+1$. Furthermore, $D_1 \cup D_2$ is a p-dominating set of G. In addition, G_1 and G_2 have order less than G and so satisfy the theorem, implying that

$$\begin{aligned}
\gamma_p(G) &\leqslant |D_1 \cup D_2| = \gamma_p(G_1) + \gamma_p(G_2) - 1 \\
&\leqslant (n(G_1) + |L_p(G_1)| + 1)/2 + (n(G_2) + |L_p(G_2)|)/2 - 1 \\
&\leqslant (n + |L_p(G)| + 1)/2,
\end{aligned}$$

contradicting the assumption.

Suppose now that $V(G) - C$ contains a support vertex. Let a be a support vertex of G of maximum distance from C. As seen above, a has degree at least p. Let $G' = G - (L_a \cup \{a\})$. Then $\gamma_p(G') + |L_a| = \gamma_p(G)$, $n(G') = n(G) - |L_a| - 1$ and $L_p(G) \geqslant L_p(G') + |L_a| - 1$. It follows that

$$\gamma_p(G') + |L_a| = \gamma_p(G) > (n(G) + |L_p(G)| + 1)/2$$

implying that

$$\gamma_p(G') > (n(G) + |L_p(G)| + 1 - 2\,|L_a|)/2$$

and so

$$\gamma_p(G') > (n(G') + \left|L_p(G')\right| + 1)/2$$

contradicting our assumption that G is the smallest graph that does not satisfy the theorem.

Consequently, every vertex of $V(G) - C$ must be a leaf and so every vertex on C is adjacent to exactly $p - 2$ leaves, which implies that

$$\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$$

a contradiction.

To see that this bound is sharp, consider the graph G formed by an odd cycle C where each vertex on C is adjacent to exactly $p - 2$ vertices. Then $\gamma_p(G) = n - (|V(C)| - 1)/2 = (n(G) + |L_p(G)| + 1)/2$. ■

Theorem 4. *Let $p \geqslant 2$ be a positive integer. If G is a connected cactus graph with $c(G)$ odd cycles then,*

$$\gamma_p(G) \leqslant (n + |L_p(G)| + c(G))/2,$$

and this bound is sharp.

Proof. If G is a bipartite graph, then by Theorem 2 the result holds. If G is a unicycle graph then by Theorem 3 the result is also valid. So consider a cactus graph G containing at least two cycles with one of odd length. Assume that the result does not hold and let G be the smallest cactus graph such that $\gamma_p(G) > (n(G) + |L_p(G)| + c(G))/2$. We also assume that among all such graphs, G is the one having the fewest edges.

First, let u be a vertex on an odd cycle C of G and assume that $\deg_G(u) \neq p$. Let G' be the spanning graph of G obtained by removing an edge of C incident with u. Then $|L_p(G')| \leqslant |L_p(G)|+1$ and $c(G') = c(G)-1$. Also G' satisfies the result and so

$$\begin{aligned}\gamma_p(G) &\leqslant \gamma_p(G') \leqslant (n(G') + \left|L_p(G')\right| + c(G'))/2 \\ &\leqslant (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction. Thus every vertex in an odd cycle has degree exactly p.

Now consider a vertex v different from a leaf and contained in no odd cycle. Then, either v is a cut vertex or v is on an even cycle and $\deg_G(v) = 2$. Suppose first that v is a cut vertex with $\deg_G(v) < p$. Let G_1 and G_2 be two connected cactus subgraphs of G with $V(G) = V(G_1) \cup V(G_2)$ having v as a unique common vertex. Then, $c(G) = c(G_1) + c(G_2), n(G) = n(G_1) + n(G_2) - 1, |L_p(G)| = |L_p(G_1)| + |L_p(G_2)| - 1$. Now let D_1 and D_2 denote a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Then $v \in D_1 \cup D_2$ and $D_1 \cup D_2$ is a p-dominating set of G. Since G_1 and G_2 satisfy the result,

$$\begin{aligned}\gamma_p(G) &\leqslant |D_1 \cup D_2| = |D_1| + |D_2| - 1 \\ &\leqslant (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 - 1 \\ &\leqslant (n(G) + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction. Consequently, every cut vertex contained in no odd cycle has degree at least p.

Now let v be a vertex on an even cycle with $\deg_G(v) = 2$. Since we have assumed in the beginning of the proof that G has at least two cycles, we have $p \geqslant 3$. We claim that each neighbor of v has degree exactly p. Indeed, let u be a neighbor of v and assume that $\deg_G(u) \neq p$. Then every $\gamma_p(G')$-set S is a p-dominating set of G where G' is obtained from G by removing the edge vu. So

$$\gamma_p(G) \leqslant |S| \leqslant (n(G') + \left|L_p(G')\right| + c(G'))/2 = (n(G) + |L_p(G)| + c(G))/2\,,$$

a contradiction. Thus $\deg_G(u) = p$.

Now let C denote an odd cycle of length at least 5 and let w be a vertex on C, a and b its neighbors on C. Delete the edges wa, wb. The remaining graph has two components for otherwise wa or wb would be contained in two cycles. Let G_1 be the component containing w and G_2 the other component where a new edge is added joining a and b. Then both G_1 and G_2 verify the theorem. Also $\deg_{G_2}(a) = \deg_{G_2}(b) = p$, $|L_p(G_1)| + |L_p(G_2)| \leqslant |L_p(G)| + 1$

and $c(G_1) + c(G_2) = c(G) - 1$. Let D_1 and D_2 be a $\gamma_p(G_1)$-set and a $\gamma_p(G_2)$-set, respectively. Then D_1 contains w since $\deg_{G_1}(w) = p - 2$. It can be checked that $D_1 \cup D_2$ is a p-dominating set of G. It follows that

$$\begin{aligned}\gamma_p(G) &\leqslant |D_1 \cup D_2| \\ &\leqslant (n(G_1) + |L_p(G_1)| + c(G_1))/2 + (n(G_2) + |L_p(G_2)| + c(G_2))/2 \\ &\leqslant (n(G) + |L_p(G)| + 1 + c(G) - 1)/2 = (n(G) + |L_p(G)| + c(G))/2\end{aligned}$$

contradicting our assumption. Thus it remains to investigate the case that each odd cycle is a triangle.

Let $C = uvw$ be a triangle of G. If $p = 2$ then as claimed before $G = C_3$ and the theorem is valid. So assume that $p \geqslant 3$. Let G_u, G_v and G_w be the three components of G containing u, v, w, respectively, by removing the edges uv, uw and vw. Suppose that each component contains at most one vertex of degree at least p and let j the number of vertices of degree at least p in the three components. Then $j \leqslant 3$ and $|L_p(G)| = n - 3 - j$. In this case, G_u is either a star of center vertex u with $p-2$ leaves, or star of order at least 4 where u is a leaf if $p = 3$, or a double star $S_{p-3,p-1}$ with u as a support vertex if $p \geqslant 4$, or a graph formed by a cycle C_4 where $u \in V(C_4)$ and is adjacent to $p - 4$ leaves (if $p \geqslant 4$), its neighbors on the cycle have degree 2 and the remaining vertex of the cycle is adjacent to $p - 2$ leaves. Likewise G_v and G_w. If each component is a tree then G is a unicycle and the result follows by Theorem 3. So we assume that G_u is a component containing the cycle C_4. Now it is a routine matter to check that

$$\gamma_p(G) = n - (j + 1) \leqslant (n(G) + |L_p(G)| + c(G))/2 = n - 1 - j/2\,,$$

a contradiction.

Thus we may assume, without loss of generality, that G_u contains at least two vertices of degree at least p. Let G' be the component containing v, w by removing the edges uv, uw. Let G_0 be the graph constructed from G' by attaching v and w to the support vertices say a, b of a double star $S_{p-2,p-2}$ (so v, w, a, b induce a cycle C_4) and let D_u and D_0 a $\gamma_p(G_u)$-set and a $\gamma_p(G_0)$-set, respectively. Then, without loss of generality, D_0 contains v, w, a all the leaves adjacent to a and b. Also D_u contains u since it has degree at most $p-2$. Obviously $D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))$ is a p-dominating set of G. It is easy to check that G_u contains at least $2p - 1$ vertices. Thus G_0 has order less than G since we have added $2p - 2$ vertices and so both

G_u, G_0 verify the result. On the other hand, $n(G) = n(G_u) + n(G_0) - 2p + 2$, $L_p(G) = L_p(G_u) - 1 + L_p(G_0) - 2p + 4$, $c(G) = c(G_u) + c(G_0) + 1$. Consequently

$$\begin{aligned}\gamma_p(G) &\leqslant |D_u \cup (D_0 - (\{a\} \cup L_a \cup L_b))| = \gamma_p(G_u) + \gamma_p(G_0) - 2p + 3 \\ &\leqslant (n(G_u) + |L_p(G_u)| + c(G_u))/2 \\ &\quad + (n(G_0) + |L_p(G_0)| + c(G_0))/2 - 2p + 3 \\ &\leqslant (n(G) + |L_p(G)| + c(G))/2\,,\end{aligned}$$

a contradiction with our assumption.

That this bound is sharp may be seen by considering the graph G_k formed by $k \geqslant 1$ triangles where each vertex of the triangle is attached to $p-2$ leaves, and identifying a vertex of every triangle with a vertex of a path P_k. Then $n(G_k) = (3p-3)k$, $|L_p(G_k)| = 3(p-2)k$, $c(G_k) = k$ and $\gamma_p(G) = (n(G_k) + |L_p(G_k)| + c(G_k))/2 = (3p-4)k$. ■

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