

ON A CONJECTURE OF QUINTAS AND
ARC-TRACEABILITY IN UPSET TOURNAMENTS

ARTHUR H. BUSCH

MICHAEL S. JACOBSON

Department of Mathematics
University of Colorado at Denver
Denver, CO 80217, USA

AND

K. BROOKS REID

Department of Mathematics
California State University San Marcos
San Marcos, CA 92096, USA

Abstract

A digraph $D = (V, A)$ is *arc-traceable* if for each arc xy in A , xy lies on a directed path containing all the vertices of V , i.e., hamiltonian path. We prove a conjecture of Quintas [7]: if D is arc-traceable, then the condensation of D is a directed path. We show that the converse of this conjecture is false by providing an example of an upset tournament which is not arc-traceable. We then give a characterization for upset tournaments in terms of their score sequences, characterize which arcs of an upset tournament lie on a hamiltonian path, and deduce a characterization of arc-traceable upset tournaments.

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1. Introduction

In this paper we will deal only with digraphs on a finite vertex set. A general reference on digraphs is the book by Bang-Jensen and Gutin [4]. For a survey of results on tournaments see the chapter by Reid in [9]. Given a digraph D we will use $V(D)$ to represent the vertex set and $A(D)$ to represent the arc set, or simply V and A when the meaning is clear. For vertices u and v , we will say u *dominates* v when $uv \in A$. Similarly, for subsets of vertices U and W , we will say U *dominates* W when U and W are connected by at least one arc and every arc between these sets originates in U and terminates in W . We denote the in-degree and out-degree of the vertex v by $d^-(v)$ and $d^+(v)$, respectively. Given two paths $P = u_0, a_1, u_1, \dots, u_{k-1}, a_k, u_k$ and $Q = v_0, b_1, v_1, \dots, v_{m-1}, a_m, v_m$ of a digraph D , if u_k dominates v_0 in D and $V(P) \cap V(Q) = \emptyset$, then we denote by PQ the directed path $u_0, a_1, u_1, \dots, u_{k-1}, a_k, u_k, a, v_0, a_1, v_1, \dots, v_{m-1}, a_m, v_m$, where a is the arc $u_k v_0$. If P (respectively, Q), is a path of length 0 (i.e., a single vertex) we will write $u_0 Q$ (respectively $P v_0$). A digraph D is *arc-traceable* if for each arc xy in A , xy lies on a directed path containing all the vertices of V , i.e., a hamiltonian path. If D is not arc-traceable, then we will call an arc xy *non-traceable* when the arc xy does not lie on a hamiltonian path. In [1], Balińska, Zwierzyński, Gargano and Quintas, investigate the analogous property in undirected graphs, and consider some very special cases of arc-traceable digraphs (directed graphs whose underlying graph is a path or a cycle). Additional results in the undirected case can be found in [2] and [3].

A *tournament* is a digraph obtained by orienting the complete undirected graph. In other words, for every pair of distinct vertices u and v , either $uv \in A$ or $vu \in A$ but not both. A tournament with n vertices is called an n -tournament. Note that every induced sub-digraph of a tournament is also a tournament. For a tournament T and $S \subset V(T)$, we denote the *sub-tournament* induced by S as $T[S]$, and the sub-tournament induced by $V \setminus S$ as $T - S$ or $T - v$ if $S = \{v\}$ for some vertex v . A tournament is *transitive* if the arcs induce a total ordering of the vertex set. It is easy to see that for such a tournament the vertices can be labeled $V = \{v_1, \dots, v_n\}$ so that for each i , $1 \leq i \leq n$, $d^+(v_i) = n - i$. An *upset tournament* on n vertices is a tournament T obtained from the transitive n -tournament by reversing some path P from the source (the vertex of out-degree $n - 1$) to the sink (the vertex of out-degree 0). We will call this reversed path U the *upset path* of the tournament T . The vertices of T can be labeled v_1, \dots, v_n

so that $v_i v_j \in A$ if and only if either $i < j$ or $v_i v_j$ is an arc of the upset path. Figure 1 shows the two upset tournaments on five vertices.

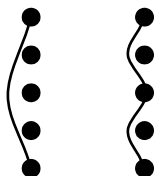


Figure 1. The two upset 5-tournaments. All arcs not drawn are oriented down. The path shown is oriented upwards.

In Section 2 we prove a necessary condition for digraphs to be arc-traceable, a condition first conjectured by Quintas. The converse of this result is easily seen to be false. We give an example of an upset tournament to show this, and in Section 3 we give a characterization of upset tournaments in terms of their score sequences. We then characterize which arcs of an upset tournament are non-traceable and deduce which upset tournaments are arc-traceable. We conclude by giving an example of an upset tournament with many non-traceable arcs and a proof that this example contains the maximum number of non-traceable arcs among all upset n -tournaments.

2. Proof of a Conjecture of Quintas

In this section we establish a necessary condition for arc-traceable digraphs, a condition first conjectured by Quintas ([7]). The *condensation* of a digraph D is a directed acyclic graph D^* whose vertices represent the strong components of D , and where $uw \in A(D^*)$ if and only if the strong component U (represented by u) dominates the strong component W (represented by w) in D .

Theorem 2.1. *If a digraph D is arc-traceable, then the condensation of D is a directed path.*

Proof. Let D be an arc-traceable digraph. Assume D has m strong components and let D^* be the condensation of D . If D is traceable, D^* must also be traceable. So, it suffices to show that when D is arc-traceable, then D^* is an oriented tree, as the only traceable oriented tree is a directed path. We establish this by showing that the underlying graph of D^* is acyclic.

Since D^* is acyclic, it contains some vertex of in-degree 0. Label such a vertex v_1^* . Now $D^* - \{v_1^*\}$ is also acyclic, and so this subgraph also has at least one vertex of in-degree 0. Choose such a vertex and label it v_2^* . Proceeding inductively, we can label all the vertices of D^* as $v_1^*, v_2^*, \dots, v_m^*$ such that v_p has in-degree 0 in $D^* - \{v_1, \dots, v_{p-1}\}$. Thus, for any $p > q$, $v_p^* v_q^* \notin A(D^*)$. This ordering is known as a topological sorting ([6]) or an acyclic ordering ([4]). We use this same ordering for the strong components of D : D_1, \dots, D_m so that v_i^* corresponds to component D_i .

Now, assume that C is a cycle of the underlying graph of D^* . Choose the smallest index i such that v_i^* is on C , and let v_j^* and v_k^* be the neighbors of v_i^* on this cycle. By the minimality of i we have $i < j$ and $i < k$, and without loss of generality we can assume that $j < k$. Since $v_i^* v_k^* \in E(C)$, the acyclic ordering requires that we must have $v_i^* v_k^* \in A(D^*)$. This in turn requires that there are distinct vertices x and y in D_i and D_k respectively such that $xy \in A(D)$. Now choose any vertex z in D_j .

Since D is arc-traceable, xy is on some hamiltonian path of D , and z must precede x or follow y on such a path, and so there must be either a path in D from z to x or a path in D from y to z . But clearly, any such path must contain an arc uv where $u \in D_p$ and $v \in D_q$ with $p > q$. This requires that $x_p^* x_q^* \in A(D^*)$, a contradiction. Hence, the underlying graph of D^* must be acyclic. ■

We note that the converse of Theorem 2.1 is not true. Many strong tournaments contain arcs that are not part of any hamiltonian path. For example, in the rightmost tournament in Figure 1, the arc from the top vertex to the bottom vertex is non-traceable. Generalizing this example, the upset tournament of order $n = 2k + 1$, where $k \geq 2$, obtained from the transitive n -tournament by reversing the 2-path from source x to sink y through the vertex with score k is not arc-traceable for any $k \geq 2$, since the arc xy is part of no hamiltonian path. The condensation of this tournament is a single vertex with no arcs and is consequently a directed path. In the next section we characterize the arcs of an upset tournament that are on no hamiltonian path, and as a corollary obtain a characterization of arc-traceable upset tournaments.

3. Non-Traceable Arcs in Upset Tournaments

We now turn our attention to upset tournaments. We choose to study arc-

traceability in upset tournaments for two reasons. Firstly, it is well known that all tournaments are traceable [8] which is clearly a necessary condition for a digraph to be arc-traceable. Secondly, upset tournaments are strong, and thus satisfy the necessary condition described in the previous section but as indicated at the end of Section 2, upset tournaments yield examples that show that this condition is not sufficient.

We begin by observing that such tournaments are completely characterized by their score sequence. This was first shown using tournament matrices by Brualdi and Li [5]; we present a short induction proof here.

Theorem 3.1 (Brualdi and Li [5]). *Let $n \geq 4$. An n -tournament T is an upset tournament if and only if the score sequence of T is $(1, 1, 2, 3, \dots, n-3, n-2, n-2)$.*

Proof. Necessity follows by noting that the reversal of the arcs of a path of the transitive tournament from source to sink leaves the scores of the vertices on the interior of the path unchanged, decreases the score of the source by 1 and increases the score of the sink by 1. This produces a tournament with the desired score sequence.

For sufficiency, we use induction. For $n = 4$, the tournament with score sequence $(1, 1, 2, 2)$ is unique and it is easily seen that this is an upset tournament. For $n > 4$, let T be a n -tournament with the given score sequence, let u and v be the vertices of T with score 1, and assume that $uv \in A$. Consider the tournament $T' = T - u$. Clearly, the score of every vertex x in T' is one less than the score of x in T , apart from v , whose score is unchanged. Thus, the score sequence of the $(n-1)$ -tournament T' is $(1, 1, 2, 3, \dots, n-4, n-3, n-3)$. By the induction hypothesis, T' is an upset tournament. Let U be the upset path of T' . The first vertex of U has out-degree 1 in T' , and we may assume that this vertex is v . It is clear that reversing the path U in the original tournament T would result in the transitive n -tournament, and hence U is also an upset path of T and thus T is an upset tournament. ■

We now address the question of arc-traceability in upset tournaments. We begin by characterizing which arcs of an upset tournament lie on a hamiltonian path.

Theorem 3.2. *Let T denote an upset tournament on $n \geq 6$ vertices with upset path U . If V is labeled $V = \{v_1, \dots, v_n\}$, so that $v_i v_j \in A$ if and only*

if either $i < j$ or $v_i v_j$ is an arc of U , then the arc $v_r v_s$ of T is non-traceable

if and only if all of the following hold:

- Both $v_r \in V(U)$ and $v_s \in V(U)$.
- $v_r v_s$ is not an arc of the upset path.
- For each vertex $v_k \in V(U)$ with $r < k < s$, neither v_{k-1} nor v_{k+1} are vertices of the upset path.

Proof. Let $v_r v_s$ be an arc of T that is on no hamiltonian path. First, we show that both v_r and v_s are vertices of the upset path. This follows from the observation that $T - v_i$ is an upset tournament for any v_i not on the upset path. Since upset tournaments are strong, this tournament has a hamiltonian cycle and subsequently, a hamiltonian path beginning or ending at any specified vertex. So if v_r is not on the upset path, we can choose H , a hamiltonian path of $T - v_r$ that begins at the vertex v_s . But then $v_r H$ is a hamiltonian path of T containing the arc $v_r v_s$, a contradiction.

Similarly, if v_s is not on the upset path, we choose H , a hamiltonian path of $T - v_s$ that ends at the vertex v_r and $H v_s$ is a hamiltonian path containing $v_r v_s$. As no such path exists, $v_s \in V(U)$.

Next, we show that $v_r v_s$ is not an arc of the upset path. This follows from the observation that $T - V(U)$ is a transitive tournament whose source is dominated by v_1 in T . Then if H is the hamiltonian path of $T - V(U)$, UH is a hamiltonian path of T containing the arc $v_r v_s$. As no such path exists, $v_r v_s$ can not be an arc of the upset path. Thus $r < s$, and for at least one $r < k < s$, $v_k \in V(U)$.

Now, we show that for each vertex $v_k \in V(U)$ where $r < k < s$, neither v_{k-1} nor v_{k+1} is on the upset path. Suppose that for some such k , v_{k-1} is on the upset path. So, $v_k v_{k-1}$ is an arc of U . Then, both $T_1 = T[\{v_1, \dots, v_{k-1}\}]$ and $T_2 = T[\{v_k, \dots, v_n\}]$ are upset tournaments and hence strong. We can then choose a hamiltonian path H_1 of T_1 that ends at v_r and a hamiltonian path H_2 of T_2 that begins at the vertex v_s . But then $H_1 H_2$ is then a hamiltonian path of T containing $v_r v_s$, a contradiction. Similarly, if v_{k+1} is on the upset path then by the same argument with $T_1 = T[\{v_1, \dots, v_k\}]$ and $T_2 = T[\{v_{k+1}, \dots, v_n\}]$, $v_r v_s$ is on a hamiltonian path. So, neither v_{k-1} nor v_{k+1} are on the upset path.

For the converse, suppose that $v_r v_s$ is an arc of T , where both $v_r \in V(U)$ and $v_s \in V(U)$, $v_r v_s$ is not an arc of the upset path, and that for each v_k on the upset path between v_r and v_s , neither v_{k-1} nor v_{k+1} are on the upset path. Let Q be a longest path of T that contains the arc $v_r v_s$, and let $U[v_s, v_r]$ be the sub-path of U beginning at v_s and ending at v_r .

First, note that Q can not contain $U[v_s, v_r]$ as a sub-path, since v_s follows v_r on this path but v_r follows v_s on Q . So we can choose an upset arc $v_q v_p$ of $U[v_s, v_r]$ that is not part of the path Q . Note that by assumption $p+1 \neq q$, and every path from v_{p+1} to v_r must contain the arc $v_q v_p$. Since Q does not contain this arc, v_{p+1} does not precede v_r on Q . Similarly, every path from v_s to v_{p+1} must include the arc $v_q v_p$, and hence v_{p+1} does not follow v_s on Q . But v_r and v_s are consecutive on Q , and so v_{p+1} is not on the path Q . Thus Q is not a hamiltonian path, and since the length of Q is maximal, no hamiltonian path containing $v_r v_s$ exists. ■

Corollary 3.1. *An upset tournament T on $n \geq 6$ vertices is arc-traceable if and only if for every vertex v_k on the interior of the upset path, either v_{k-1} or v_{k+1} is also on the upset path.*

Proof. If T is a tournament satisfying the given condition, then Theorem 3.2 implies that there is no arc that is non-traceable, i.e., T is arc-traceable. For the converse, assume there is some v_k on the upset path with neither v_{k-1} nor v_{k+1} on the upset path. Let v_i and v_j be the vertices immediately preceding and succeeding v_k on U , respectively. Then, by Theorem 3.2 $v_j v_i$ is on no hamiltonian path. ■

Remark. The above results fail for $n = 5$ vertices, as the upset tournament obtained from reversing the unique hamiltonian path of the transitive tournament on five vertices is not arc-traceable, despite satisfying the conditions stated above. This is a consequence of the fact that this tournament is isomorphic to the tournament on the right in Figure 1, the tournament obtained from the transitive tournament by reversing the 2-path containing the vertex of score 2, and this tournament does not meet the criteria indicated in Corollary 3.1. In fact, Figure 1 shows the two non-isomorphic upset tournaments on five vertices, and the tournament on the left is easily seen to be arc-traceable.

Next, we give an example of an upset tournament with many non-traceable arcs, and prove that this example is maximal.

Corollary 3.2. *If T is an upset tournament with $n \geq 5$ vertices, n odd, and the upset path of T is $v_n v_{n-2} v_{n-4} \dots v_3 v_1$, then exactly $\frac{n^2-4n+3}{8} = \frac{1}{4} \cdot \binom{n}{2} - \frac{3(n-1)}{8} = \frac{(n-1)(n-3)}{8}$ arcs of T are not on a hamiltonian path.*

Proof. By Theorem 3.2, the arcs of T on no hamiltonian path are of the form $v_i v_{i+(2k+2)}$ for i odd, $1 \leq i \leq n-4$ and $1 \leq k \leq \frac{n-1}{2} - 1$. Thus,

for a fixed $i = 2j + 1$, there are exactly $\frac{n-i}{2} - 1 = \frac{n-(2j+1)}{2} - 1 = \frac{n-3}{2} - j$ non-traceable arcs starting at vertex v_i . Summing all possible values of j , we obtain

$$\sum_{j=0}^{\frac{n-5}{2}} \left(\frac{n-3}{2} - j \right) = \left(\frac{n-3}{2} \right) \cdot \left(\frac{n-3}{2} \right) - \frac{\frac{n-5}{2} \frac{n-3}{2}}{2} = \frac{n^2 - 4n + 3}{8}$$

non-traceable arcs in T . ■

Next, we show that this family of examples has the maximal number of non-traceable arcs among all upset n -tournaments.

Theorem 3.3. *An upset n -tournament T , $n \geq 5$, has at most*

$$\frac{n^2 - 4n + 3}{8} = \frac{1}{4} \cdot \binom{n}{2} - \frac{3(n-1)}{8} = \frac{(n-1)(n-3)}{8}$$

non-traceable arcs.

Proof. We prove the result by induction on n . For $n = 5$, it is easy to verify that the unique non-traceable upset tournament has $\frac{5^2 - 4 \cdot 5 + 3}{8} = 1$ arc that is not on any hamiltonian path.

Next, assume the result for upset tournaments with fewer than $n > 5$ vertices. Let i be the vertex that immediately precedes the vertex v_1 on the upset path. As observed earlier, $T_1 = T[v_i, \dots, v_n]$ is an upset tournament, and so by the induction hypothesis, there are at most

$$\frac{(n-i+1)^2 - 4(n-i+1) + 3}{8} = \frac{n^2 - 4n + 3}{8} - (i-1) \frac{2n-4-(i-1)}{8}$$

non-traceable arcs in T_1 . By applying Theorem 3.2 twice (once for necessity in T_1 and again for sufficiency in T), we note that each of these arcs is also non-traceable in T .

All that remains is to count the non-traceable arcs of T that are not arcs of T_1 . Clearly, no arc incident with the vertices v_2, \dots, v_{i-1} is non-traceable so any non-traceable arc of T that is not an arc of T_1 is incident with v_1 . Additionally, if $i = 2$, then by Theorem 3.2, every arc incident with v_1 is also on a hamiltonian path. So, for $i = 2$, the Theorem follows by observing that $(i-1) \frac{2n-4-(i-1)}{8} = \frac{2n-5}{8} > 0$ as $n > 5$.

So we may assume that $i \geq 3$. In this case, as $n > 5$ it follows that

$$\frac{(n-i+1)^2 - 4(n-i+1) + 3}{8} \leq \frac{(n-2)^2 - 4(n-2) + 3}{8} = \frac{n^2 - 8n + 15}{8}.$$

Thus we must show that at most $\frac{4n-12}{8} = \frac{n-3}{2}$ arcs incident with v_1 are non-traceable. Let j be the largest index such that the arc v_1v_j is non-traceable. As a consequence of Theorem 3.2, at most $\frac{j-1}{2}$ of the vertices v_1, \dots, v_j are on the upset path of T and v_i is one of these vertices. Since v_iv_1 is part of the upset path (and thus is part of a hamiltonian path), this means that at most $\frac{j-1}{2} - 1 \leq \frac{n-3}{2}$ arcs incident with the vertex v_1 are not a part of any hamiltonian path, and the result follows. ■

4. Concluding Remarks

In this paper, in addition to settling the question posed by Quintas, we offer a characterization of arc-traceable upset tournaments. A natural question would be to develop a characterization of arc-traceable tournaments in general as well as to study the general extremal question suggested by Theorem 3.3.

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