# THE DIRECTED PATH PARTITION CONJECTURE 

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#### Abstract

The Directed Path Partition Conjecture is the following: If $D$ is a digraph that contains no path with more than $\lambda$ vertices then, for every pair $(a, b)$ of positive integers with $\lambda=a+b$, there exists a vertex partition $(A, B)$ of $D$ such that no path in $D\langle A\rangle$ has more than $a$ vertices and no path in $D\langle B\rangle$ has more than $b$ vertices. We develop methods for finding the desired partitions for various classes of digraphs.


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## 1. Introduction

The vertex set and the arc set of a digraph $D$ are denoted by $V(D)$ and $E(D)$ respectively, and the number of vertices is denoted by $n(D)$. For undefined concepts we refer the reader to [2].

A directed cycle (path, walk) in a digraph will simply be called a cycle (path, walk).

A digraph is called traceable if it contains a path that visits every vertex, and walkable if it contains a walk that visits every vertex.

A digraph $D$ is called strong (or strongly connected) if every vertex of $D$ is reachable from every other vertex. Thus a digraph $D$ of order bigger than one is strong if and only if it contains a closed walk that visits every vertex.

A longest path in a digraph $D$ is called a detour of $D$. The order of a detour of $D$ is called the detour order of $D$ and is denoted by $\lambda(D)$.

If $(a, b)$ is a pair of positive integers, a partition $(A, B)$ of the vertex set of a digraph $D$ is called an $(a, b)$-partition if $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.

If a digraph $D$ has an $(a, b)$-partition for every pair of positive integers $(a, b)$ such that $a+b=\lambda(D)$, then $D$ is called $\lambda$-partitionable. The Directed Path Partition Conjecture (DPPC) is the following:

DPPC: Every digraph is $\lambda$-partitionable.
The analogous conjecture for undirected graphs, known as the Path Partition Conjecture (PPC), was first stated in 1981 and is still an open problem.
(Cf. [4-10, 13, 14] for results supporting the PPC.) A digraph $D$ is called symmetric if for every $x y \in E(D)$ the arc $y x$ is also in $E(D)$. The PPC is, obviously, equivalent to the conjecture that every symmetric digraph is $\lambda$-partitionable.

An oriented graph is a digraph with no cycle of length 2 . We can therefore obtain an oriented graph $D$ from a graph $G$ by assigning a direction to each edge of $G$. We call such a digraph $D$ an orientation of $G$. We shall also consider the following conjecture, referred to as the Oriented Path Partition Conjecture (OPPC):

OPPC: Every oriented graph is $\lambda$-partitionable.
The DPPC implies both the PPC and the OPPC. We do not know the relationship between the OPPC and the PPC.

A directed version of the PPC that is perhaps stronger than the DPPC was stated by Bondy [1]. (His conjecture requires $\lambda(D\langle A\rangle)=a$ and $\lambda(D\langle B\rangle)=b$ while the DPPC only requires $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.)

The DPPC is one of those conjectures that is easy to state but seems very difficult to settle. In this paper we present some techniques for finding ( $a, b$ )-partitions in certain classes of digraphs. We begin by considering the case $a=1$.

## 2. The Case $a=1$

Whereas the PPC has been proved for $a \leq 7$ (cf. [6] and [14]), the DPPC has not yet been proved for the case $a=1$. In 1983 Laborde, Payan and Xuong formulated this case as a conjecture [13]. We shall call it the Case 1 Conjecture (C1). It can be stated as follows:

C1: If $D$ is a digraph with detour order $\lambda$, then $D$ has a $(1, \lambda-1)$ partition.

The detour source, $S(D)$, of a digraph $D$ is the set consisting of all initial vertices of detours of $D$. The following conjecture, which we shall call the Strong Case 1 Conjecture (SC1) is also stated in [13].

SC1: If $D$ is a digraph with detour order $\lambda$, then $D$ has a $(1, \lambda-1)$ partition $(A, B)$ with $A \subseteq S(D)$.

Let $N^{-}(x)$ and $N^{+}(x)$ denote the in-neighbourhood and out-neighbourhood of a vertex $x$ respectively. Then $N(x)=N^{-}(x) \cup N^{+}(x)$ is the neighbourhood of $x$. Let $D$ be a digraph with detour source $S$. If $x, y \in S$ and $y \in N^{+}(x)$, i.e. $y$ is an out-neighbour of $x$, then, obviously, $x$ lies on every $y$-detour. (A $y$-detour is a detour of $D$ with initial vertex $y$.) If the same is true for all the in-neighbours of $x$, then removal of $x$ from $D$ destroys all detours that originated in the neighbourhood of $x$. This leads to the following definition: A vertex $x \in S$ is a local detour-destroyer if, for every $y \in N^{-}(x)$, every $y$-detour contains $x$.

A class $\mathcal{P}$ of digraphs is called an induced hereditary property if it is closed under isomorphism and every induced subdigraph of every digraph in $\mathcal{P}$ is also in $\mathcal{P}$.

Proposition 2.1. Suppose $\mathcal{P}$ is an induced hereditary property of digraphs such that every digraph in $\mathcal{P}$ contains a local detour-destroyer $x \in S(D)$. Then SC1 holds for the class $\mathcal{P}$.

Proof. We prove the result by induction on the order of the digraph. Let $D$ be a digraph in $\mathcal{P}$ with detour order $\lambda$ and detour source $S$. Let $x \in S(D)$ be a local detour-destroyer of $D$. We consider two cases:

Case 1. $\lambda(D-x)<\lambda(D)$ : Then, clearly, $(\{x\}, V-\{x\})$ is a $(1, \lambda-1)$ partition of $V(D)$.

Case 2. $\lambda(D-x)=\lambda(D)$ : Since $\mathcal{P}$ is induced hereditary, $D-x \in \mathcal{P}$, so, by the induction hypothesis, $D-x$ has a $(1, \lambda-1)$-partition $\left(A^{\prime}, B\right)$ with $A^{\prime} \subseteq S(D-x)$. But, since $x$ is a local detour-destroyer, $N(x) \cap S(D-x)=\emptyset$ and hence $A^{\prime} \cup\{x\}$ is an independent set. Put $A=A^{\prime} \cup\{x\}$. Then $(A, B)$ is a $(1, \lambda-1)$-partition of $D$, with $A \subseteq S(D)$.
The above result is used in [13] to show that SC1 holds for the class of digraphs with maximum degree at most 3 . We shall prove that SC 1 also holds for the class consisting of all digraphs with the property that no two cycles share an arc.

Corollary 2.2. Let $D$ be a digraph with the property that no two cycles share an arc. Then $D$ has a $(1, \lambda-1)$-partition, $(A, B)$ such that $A \subseteq S(D)$.

Proof. Let $S$ be the detour source of $D$ and suppose $D$ has detour order $\lambda$. Since the property of having no two cycles sharing an arc is induced
hereditary, it suffices to prove that $D$ has a local detour-destroyer $x \in S$. Suppose no vertex in $S$ is a local detour-destroyer and let $x_{1}$ be a vertex in $S$. Then there is a vertex $x_{2} \in S \cap N^{-}\left(x_{1}\right)$ and an $x_{2}$-detour that does not contain $x_{1}$. Again, there is a vertex $x_{3} \in S \cap N^{-}\left(x_{2}\right)$ and an $x_{3}$-detour that does not contain $x_{2}$. Continuing this procedure we construct a sequence $x_{1}, x_{2}, \ldots$ of elements of $S$ such that $x_{i+1} x_{i} \in E(D)$. Since $S$ is finite there is some $i<j$ such that $x_{i}=x_{j}$. Hence the subsequence $x_{j} x_{j-1} \ldots x_{i}$ is a closed walk and thus contains a cycle $C$. For a vertex $v$ on $C$ let $v^{+}$and $v^{-}$ be its out- and in-neighbour on $C$. Thus, for every vertex $v$ on $C$, there is a $v$-detour not containing its successor $v^{+}$on the cycle. Since $v^{-}$is contained in every $v$-detour, it follows that there exists a $v-v^{-}$-path $P$ such that $v^{+} \notin V(P)$. But then the arc $v^{-} v$ is shared by the cycle $C$ and the cycle $v^{-} v P v^{-}$.
An independent set of vertices $K$ in a digraph $D$ is a kernel of $D$ if, for every vertex $y$ in $D-K$ there is a vertex $x \in K$ such $x y \in E(D)$. It is shown in [13] that, if a digraph $D$ with detour order $\lambda$ has a kernel, then $D$ has a $(1, \lambda-1)$ partition.

Not every digraph has a kernel, e.g. an odd cycle does not have a kernel. In some cases the subdigraph of $D$ induced by its detour source might have a kernel, even if $D$ itself does not have a kernel, in which case the following result is useful.

Theorem 2.3. Let $D$ be a digraph with detour order $\lambda$ and detour source $S$. If the digraph $D\langle S\rangle$ has a kernel, then $D$ has a $(1, \lambda-1)$-partition.

Proof. Let $A$ be a kernel of $D\langle S\rangle$ and put $B=V(D)-A$. Suppose there is a path $P$ of order $\lambda$ in $D\langle B\rangle$. If $y$ is the initial vertex of $P$, then $y \in S-A$ and hence there is a vertex $x$ in $A$ such that $x y$ is an $\operatorname{arc}$ in $D$. But then $x P$ is a path of order $\lambda+1$ in $D$. This contradiction shows that $\lambda(B) \leq \lambda-1$, and hence $(A, B)$ is a $(1, \lambda-1)$-partition of $D$.
The following result is due to Richardson [15]. Shorter proofs of this result can be found in [3] and [1].
Theorem 2.4 (Richardson). Every digraph without odd cycles has a kernel.
Thus we have the following result:
Corollary 2.5. Let $D$ be a digraph with detour order $\lambda$ and detour source $S$. If the digraph $D\langle S\rangle$ does not contain an odd cycle, then $D$ has a $(1, \lambda-1)$ partition.

## 3. Partitions of Digraphs Without Odd Cycles

We have seen that a digraph $D$ without odd cycles has a $(1, \lambda(D)-1)$ partition. The following result, due to Harary, Norman and Cartwright [12], shows that strong digraphs without odd cycles are, in fact, $(1,1)$ partitionable (and hence $\lambda$-partitionable).

Theorem 3.1 (Harary, Norman and Cartwright). A strong digraph is bipartite if and only if it has no odd cycle.

A strong component of a digraph $D$ is a maximal induced subdigraph of $D$ which is strong. In order to extend this result to digraphs with more than one strong component, we need some definitions.

First we define the function $p$ by choosing, for each $v \in V(D)$, the value $p(v)$ as the maximum order of a path in $D$ that ends at $v$.

The strong component digraph, $S C(D)$, of $D$ is the digraph whose vertices $v_{1}, \ldots, v_{k}$ correspond to the strong components $D_{1}, \ldots, D_{k}$ of $D$ and whose arc set is given by $E(S C(D))=\left\{v_{i} v_{j}\right.$ : there is a vertex in $D_{i}$ that is adjacent to a vertex in $\left.D_{j}\right\}$.

Let $S$ be a strong component of $D$ and let $s$ be the vertex in $S C(D)$ corresponding to $S$. Note that since $s \in V(S C(D)), p(s)$ is now the maximum order of a path in $S C(D)$ that ends at $s$. We say that the strong component $S$ lies on level $p(s)$ of $D$. We also say that every vertex of $S$ lies on level $p(s)$ of $D$. The number of levels of a digraph $D$ is called its height $h$ and is therefore given by $h(D)=\lambda(S C(D))$.

Theorem 3.2. If $D$ is a digraph of height $h$ without odd cycles, then $D$ has an ( $h, h$ )-partition.

Proof. By Theorem 3.1, each strong component $S$ of $D$ has a (1,1)partition $\left(A_{S}, B_{S}\right)$. Let

$$
A=\bigcup A_{S} \text { and } B=\bigcup B_{S}
$$

where the unions are taken over all strong components $S$ of $D$. Since there are no arcs from a strong component on level $j$ to a strong component on level $i$ if $i \leq j$, it follows that every path in $A$ contains at most one vertex from each level. Hence $\lambda(D\langle A\rangle) \leq h$. Similarly, $\lambda(D\langle B\rangle) \leq h$ and hence $(A, B)$ is an $(h, h)$-partition of $D$.

Corollary 3.3. Let $D$ be a digraph with no odd cycles. If $D$ has height $h \leq 2$, then $D$ is $\lambda$-partitionable.

Proof. By Theorem $2.4 D$ has a kernel. Thus by Corollary $2.5 D$ is ( $1, \lambda-1$ )-partitionable and the result therefore follows immediately from Theorem 3.2.

## 4. Gallai-Roy-Vitaver Partitions

The following result which was proved independently by Gallai [11], Roy [16] and Vitaver [17] relates the chromatic number $\chi(G)$ of a graph $G$ to the detour orders of its orientations.

Theorem 4.1 (Gallai-Roy-Vitaver). If $D$ is any orientation of a graph $G$, then

$$
\chi(G) \leq \lambda(D)
$$

and equality holds for some orientation of $G$.

The proof, as presented in [18], relies on a vertex partition of a maximal acyclic spanning subdigraph $M$ of $D$, which we shall call the Gallai-RoyVitaver (GRV) partition of $M$.

In general, if $D$ is any acyclic digraph with detour order $\lambda$, we use the function $p$ defined in the previous section to define the $G R V$ partition of $D$ as the partition $\left(V_{1}, \ldots, V_{\lambda}\right)$ of $V(D)$, where

$$
V_{i}=\{v \in V(D): p(v)=i\}, i=1, \ldots, \lambda .
$$

Lemma 4.2. Let $D$ be an acyclic digraph with detour order $\lambda$ and GRV partition $\left(V_{1}, \ldots, V_{\lambda}\right)$. Then the following hold:

1. If $v_{i} v_{j} \in E(D)$, with $v_{i} \in V_{i}$ and $v_{j} \in V_{j}$, then $i<j$.
2. If $P$ is a path in $D$, then

$$
\left|V(P) \cap V_{i}\right| \leq 1, \text { for } i=1, \ldots, \lambda .
$$

Proof. 1. Since $v_{i} \in V_{i}$, there is a path $P$ of order $i$ ending at $v_{i}$. Since $v_{i} v_{j} \in E(D)$, we cannot have $v_{j} \in V(P)$, otherwise $D$ would have a cycle. Hence $P v_{j}$ is a path of order $i+1$ ending at $v_{j}$, so $p\left(v_{j}\right) \geq i+1$. Since $p\left(v_{j}\right)=j$, it follows that $i<j$.
2. This follows immediately from (1).

Remark 4.3. Note that Lemma 4.2 implies that the function $p$ strictly increases along each path in an acyclic digraph.

Theorem 4.4. If $D$ is an acyclic digraph, then $D$ is $\lambda$-partitionable.
Proof. Suppose $\lambda(D)=a+b ; 1 \leq a \leq b$. Form the GRV partition, $V_{1}, \ldots, V_{\lambda(D)}$ of $D$. Let

$$
A=\bigcup_{i=1}^{a} V_{i} \text { and } B=\bigcup_{i=a+1}^{\lambda(D)} V_{i} .
$$

By Lemma 4.2, every path in $D$ has at most one vertex in each $V_{i}$; hence $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.
Given any digraph $D$ we can consider a maximal acyclic spanning subdigraph $M$ of $D$ and then form the GRV partition of $M$. If we let $A$ be the union of any $a$ of the sets of the partition and put $B=V(D)-A$, then $(A, B)$ will be an $(a, \lambda(M)-a)$-partition of $M$. In some cases this will yield a suitable partition for $D$. The next two results, one for digraphs and the other for oriented graphs, illustrate this technique.

Theorem 4.5. If $D$ is unicyclic, then $D$ is $\lambda$-partitionable.
Proof. Suppose $\lambda(D)=a+b$, where $1 \leq a \leq b$. Let $M$ be a maximal acyclic spanning subdigraph of $D$, i.e., $M=D-u v$, where $u v$ is an arc on the cycle in $D$. Form the GRV partition $V_{1}, \ldots, V_{\lambda(M)}$ of $M$, where $\lambda(M) \leq \lambda(D)$. Let $A$ be the union of any $a$ of these sets, including the one containing $u$ but excluding the one containing $v$, and put $B=V(D)-A$. Then, as in the proof of Theorem 4.4, $\lambda(M\langle A\rangle) \leq a$ and $\lambda(M\langle B\rangle) \leq b$. Since the removed arc $u v$ goes from a vertex in $A$ to a vertex in $B$, we have $M\langle A\rangle=D\langle A\rangle$ and $M\langle B\rangle=D\langle B\rangle$; hence $(A, B)$ is an $(a, b)$-partition of $D$.

Theorem 4.6. Suppose an oriented graph $D$ contains a maximal acyclic spanning subdigraph $M$ with detour order at most 3 . Then $D$ is $\lambda$-partitionable.

Proof. If $\lambda(M) \leq 2$ then $D$ is acyclic (since $D$ contains no 2-cycles) and then the result follows from Theorem 4.4. Now assume $\lambda(M)=3$ and let $V_{1}, V_{2}, V_{3}$ be the GRV partition of $M$. Suppose $u v \in E(D)-E(M)$. By the maximality of $M$, the arc $u v$ must lie on a cycle in $M+u v$ and hence there is a $v u$-path $P$ in $M$. Since $D$ contains no 2 -cycles, $P$ must have at least three vertices. Therefore, from Lemma $4.2, v \in V_{1}$ and $u \in V_{3}$. Now put $A=V_{1}$ and $B=V_{2} \cup V_{3}$. Then $(A, B)$ is a ( 1,2 )-partition of $M$ and, since every arc in $D$ that is not an arc in $M$ goes from $B$ to $A$, it follows that $(A, B)$ is also a $(1,2)$-partition of $D$. This serves as an $(a, b)$-partition of $D$ for every pair of positive integers $(a, b)$ such that $\lambda(D)=a+b$.
In the next section we generalize the idea of a GRV partition for digraphs that are not necessarily acyclic. This will allow us to construct $(a, b)$ partitions for various digraphs.

## 5. Prismatic Colourings

The proof of Theorem 4.4 relies on the fact that we obtained a partition of $V(D)$ into $\lambda(D)$ sets, such that no path in $D$ contains more than one vertex from the same set. This leads to the following definition:

A prismatic colouring of a digraph $D$ is a vertex colouring of $D$ such that no two vertices lying on the same path in $D$ have the same colour. Thus a prismatic $k$-colouring of $D$ corresponds to a partition of $V(D)$ into $k$ subsets $V_{1}, \ldots, V_{k}$ such that, for every path $P$ in $D$,

$$
\left|V(P) \cap V_{i}\right| \leq 1, \text { for } i=1, \ldots, k
$$

The prismatic number, $\varphi(D)$ of $D$ is the minimum $k$ such that $D$ has a prismatic $k$-colouring.

Proposition 5.1. If $D$ is a digraph such that $\varphi(D)=\lambda(D)$, then $D$ is $\lambda$-partitionable.

Proof. Suppose $\lambda(D)=a+b$. Consider a prismatic colouring of $D$ with colour classes $V_{1}, \ldots, V_{\varphi(D)}$. Let $A$ be the union of any $a$ of these colour classes and let $B$ be the union of the remaining $b$ colour classes. Since every path in $D$ has at most one vertex in each colour class, $(A, B)$ is clearly an $(a, b)$-partition of $D$.

The chromatic number $\chi(D)$ of a digraph $D$ is defined to be the chromatic number of the underlying graph of $D$.

Proposition 5.2. For any digraph $D$ we have

$$
\chi(D) \leq \lambda(D) \leq \varphi(D)
$$

Proof. The first inequality is given by Theorem 4.1 and the second follows from the fact that in any prismatic colouring of $D$ a detour requires $\lambda(D)$ different colours.


An example of a digraph $D$ with $\chi(D)=3, \lambda(D)=5$ and $\varphi(D)=7$.

If $D$ is an acyclic digraph, then the GRV partition is a prismatic colouring in $\lambda(D)$ colours, so $\varphi(D)=\lambda(D)$.

We call a digraph $D$ detour-perfect if $\chi\left(D^{\prime}\right)=\lambda\left(D^{\prime}\right)$ for every induced subdigraph $D^{\prime}$ of $D$. We shall show that for a detour-perfect graph all three the parameters of Proposition 5.2 are equal. First we present a characterization of detour-perfect digraphs.

Proposition 5.3. A digraph $D$ is detour-perfect if and only if for every path $P$ in $D$ the subdigraph induced by $V(P)$ is semicomplete (i.e., there is at least one arc between every pair of vertices in $P$ ).

Proof. Suppose $D$ is detour-perfect and $P$ is a path of order $n$ in $D$. Then $\chi(\langle P\rangle)=n$, so the underlying graph $\langle V(P)\rangle$ is isomorphic to $K_{n}$.

Conversely, suppose every path in $D$ induces a semicomplete digraph. Consider a subdigraph $D^{\prime}$ of $D$. Let $P$ be a path of order $\lambda\left(D^{\prime}\right)$ in $D^{\prime}$. Then the underlying graph induced by $V(P)$ is isomorphic to $K_{\lambda\left(D^{\prime}\right)}$ and hence $\chi\left(D^{\prime}\right) \geq \lambda\left(D^{\prime}\right)$. Thus, by Theorem 4.1, $\chi\left(D^{\prime}\right)=\lambda\left(D^{\prime}\right)$ and hence $D$ is detour-perfect.

Corollary 5.4. If $D$ is detour-perfect, then $\chi(D)=\lambda(D)=\varphi(D)$.
Proof. Consider any colouring $c$ of $D$ in $\chi(D)$ colours. Let $P$ be any path in $D$. Then it follows from Proposition 5.3 that no two vertices of $P$ lie in the same colour class, and hence $c$ is a prismatic colouring. This proves that $\varphi(D) \leq \chi(D)$ and hence, by Proposition 5.2, $\chi(D)=\lambda(D)=\varphi(D)$.
By Corollary 5.4 and Proposition 5.3 every detour-perfect digraph is $\lambda$ partitionable and every transitive digraph is detour-perfect. Thus we have

Corollary 5.5. Every transitive digraph is $\lambda$-partitionable.
As a matter of interest, we note that neither an odd cycle of order at least 5 nor its complement has a detour-perfect orientation. By the Perfect Graph Theorem, recently proved by Chudnovsky, Robertson, Seymour and Thomas (see [5]), only a perfect graph can have a detour-perfect orientation. However, not every perfect graph has a detour-perfect orientation, e.g. the net (a $K_{3}$ with exactly one pendant vertex attached to each of its three vertices) has no detour-perfect orientation.

The converse of Corollary 5.4 is not true. For example, let $D$ be the digraph with vertices $\{a, b, c, d, e\}$ and $\operatorname{arcs}\{a b, c b, d c, d e, a e\}$. Then $\chi(D)=\lambda(D)=\varphi(D)=3$, but $D$ is not detour-perfect, because $d c b$ is an induced path.

A walkable component $W$ of a digraph is a maximal induced subdigraph which is walkable. Clearly, if $D$ is a walkable digraph, then $\varphi(D)=n(D)$ where $n(D)$ is the order of $D$. In general we have

Theorem 5.6. If $D$ is a digraph, then

$$
\varphi(D)=\max \{n(W): W \text { is a walkable component of } D\} .
$$

Proof. First we define $D_{i}$ to be the subdigraph of $D$ induced by all the vertices on the first $i$ levels of $D$. Note that if $D$ has $h$ levels then $D_{h}=D$. Let $w\left(D_{i}\right)=\max \left\{n(W): W\right.$ is a walkable component of $\left.D_{i}\right\}$. The inequality $\varphi(D) \geq w(D)$ is obvious. To prove the reverse inequality we construct a level by level prismatic colouring $c$ of the vertices of a digraph $D$ which uses exactly $w(D)$ colours.

Let $S$ be any strong component of $D_{1}$ and colour its vertices with colours $1, \ldots, n(S)$. Do so with all the strong components of $D_{1}$. The number of
colours used to colour the vertices of $D_{1}$ is therefore given by $\max \{n(S): S$ is a strong component of $\left.D_{1}\right\}=w\left(D_{1}\right)$.
Apply the next iterative steps for each level of the digraph $D$ consecutively: Suppose $D_{i-1}$ has already been coloured. Let $S$ be any strong component on level $i$. First consider all the walkable components in $D_{i-1}$ that have a neighbour in $S$. Let $r$ be the largest number used in colouring these components. Colour the vertices of $S$ with colours $r+1, \ldots, r+n(S)$. Do the same for all strong components on level $i$ of $D$. Thus if $D_{i}$ has a vertex coloured $k$, then that vertex lies in a walkable component of $D_{i}$ of order at least $k$. Hence $D_{i}$ is coloured with $w\left(D_{i}\right)$ colours. This holds for $i=1, \ldots, h$ so $D$ is coloured with $w(D)$ colours. Since no colour is repeated in any walkable component and since two vertices can only lie on the same path if they are on the same walkable component of $D$, the colouring $c$ is clearly a prismatic colouring of the vertices of $D$.

Corollary 5.7. If $D$ is a digraph such that at least one of its walkable components of largest order is traceable, then $D$ is $\lambda$-partitionable.

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