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THE DIRECTED PATH PARTITION CONJECTURE

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Abstract

The Directed Path Partition Conjecture is the following: If D is a digraph that contains no path with more than λ vertices then, for every pair (a, b) of positive integers with $\lambda = a + b$, there exists a vertex partition (A, B) of D such that no path in $D\langle A\rangle$ has more than a vertices and no path in $D\langle B\rangle$ has more than b vertices.We develop methods for finding the desired partitions for various classes of digraphs.

Keywords: longest path, Path Partition Conjecture, vertex partition, digraph, prismatic colouring.

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1. Introduction

The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$ respectively, and the number of vertices is denoted by $n(D)$. For undefined concepts we refer the reader to [2].

A *directed* cycle (path, walk) in a digraph will simply be called a cycle (path, walk).

A digraph is called *traceable* if it contains a path that visits every vertex, and *walkable* if it contains a walk that visits every vertex.

A digraph D is called *strong* (or *strongly connected*) if every vertex of D is reachable from every other vertex. Thus a digraph D of order bigger than one is strong if and only if it contains a closed walk that visits every vertex.

A longest path in a digraph D is called a *detour* of D. The order of a detour of D is called the *detour order* of D and is denoted by $\lambda(D)$.

If (a, b) is a pair of positive integers, a partition (A, B) of the vertex set of a digraph D is called an (a, b)*-partition* if $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.

If a digraph D has an (a, b)-partition for every pair of positive integers (a, b) such that $a + b = \lambda(D)$, then D is called λ*-partitionable.* The Directed Path Partition Conjecture (DPPC) is the following:

DPPC: Every digraph is λ-partitionable.

The analogous conjecture for undirected graphs, known as the Path Partition Conjecture (PPC), was first stated in 1981 and is still an open problem.

(Cf. [4–10, 13, 14] for results supporting the PPC.) A digraph D is called *symmetric* if for every $xy \in E(D)$ the arc yx is also in $E(D)$. The PPC is, obviously, equivalent to the conjecture that every symmetric digraph is λ-partitionable.

An *oriented* graph is a digraph with no cycle of length 2. We can therefore obtain an *oriented* graph D from a graph G by assigning a direction to each edge of G. We call such a digraph D an *orientation* of G. We shall also consider the following conjecture, referred to as the Oriented Path Partition Conjecture (OPPC):

OPPC: Every oriented graph is λ-partitionable.

The DPPC implies both the PPC and the OPPC. We do not know the relationship between the OPPC and the PPC.

A directed version of the PPC that is perhaps stronger than the DPPC was stated by Bondy [1]. (His conjecture requires $\lambda(D\langle A\rangle) = a$ and $\lambda(D\langle B\rangle) = b$ while the DPPC only requires $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$.)

The DPPC is one of those conjectures that is easy to state but seems very difficult to settle. In this paper we present some techniques for finding (a, b)-partitions in certain classes of digraphs. We begin by considering the case $a = 1$.

2. The Case $a = 1$

Whereas the PPC has been proved for $a \leq 7$ (cf. [6] and [14]), the DPPC has not yet been proved for the case $a = 1$. In 1983 Laborde, Payan and Xuong formulated this case as a conjecture [13]. We shall call it the *Case* 1 *Conjecture* (C1). It can be stated as follows:

C1: If D is a digraph with detour order λ, then D has a $(1, \lambda - 1)$-partition.

The *detour source,* $S(D)$, of a digraph D is the set consisting of all initial vertices of detours of D. The following conjecture, which we shall call the *Strong Case* 1 *Conjecture* (SC1) is also stated in [13].

SC1: If D is a digraph with detour order λ, then D has a $(1, \lambda - 1)$-partition (A, B) with $A \subseteq S(D)$.

Let $N^-(x)$ and $N^+(x)$ denote the in-neighbourhood and out-neighbourhood of a vertex x respectively. Then $N(x) = N^-(x) \cup N^+(x)$ is the neighbourhood of x. Let D be a digraph with detour source S. If $x, y \in S$ and $y \in N^+(x)$, i.e. y is an out-neighbour of x, then, obviously, x lies on every y-detour. (A y-detour is a detour of D with initial vertex y.) If the same is true for all the in-neighbours of x, then removal of x from D destroys all detours that originated in the neighbourhood of x. This leads to the following definition: A vertex $x \in S$ is a *local detour-destroyer* if, for every $y \in N^-(x)$, every y-detour contains x.

A class $\mathcal{P}$ of digraphs is called an *induced hereditary property* if it is closed under isomorphism and every induced subdigraph of every digraph in $\mathcal{P}$ is also in $\mathcal{P}$.

Proposition 2.1. *Suppose $\mathcal{P}$ is an induced hereditary property of digraphs such that every digraph in $\mathcal{P}$ contains a local detour-destroyer $x \in S(D)$. Then SC1 holds for the class $\mathcal{P}$.*

Proof. We prove the result by induction on the order of the digraph. Let D be a digraph in $\mathcal{P}$ with detour order λ and detour source S. Let $x \in S(D)$ be a local detour-destroyer of D. We consider two cases:

Case 1. $\lambda(D - x) < \lambda(D)$: Then, clearly, $(\{x\}, V - \{x\})$ is a $(1, \lambda - 1)$-partition of $V(D)$.

Case 2. $\lambda(D - x) = \lambda(D)$: Since $\mathcal{P}$ is induced hereditary, $D - x \in \mathcal{P}$, so, by the induction hypothesis, $D - x$ has a $(1, \lambda - 1)$-partition (A', B) with $A' \subseteq S(D - x)$. But, since x is a local detour-destroyer, $N(x) \cap S(D - x) = \emptyset$ and hence $A' \cup \{x\}$ is an independent set. Put $A = A' \cup \{x\}$. Then (A, B) is a $(1, \lambda - 1)$-partition of D, with $A \subseteq S(D)$. ■

The above result is used in [13] to show that SC1 holds for the class of digraphs with maximum degree at most 3. We shall prove that SC1 also holds for the class consisting of all digraphs with the property that no two cycles share an arc.

Corollary 2.2. *Let D be a digraph with the property that no two cycles share an arc. Then D has a $(1, \lambda - 1)$-partition, (A, B) such that $A \subseteq S(D)$.*

Proof. Let S be the detour source of D and suppose D has detour order λ. Since the property of having no two cycles sharing an arc is induced hereditary, it suffices to prove that D has a local detour-destroyer $x \in S$. Suppose no vertex in S is a local detour-destroyer and let x_1 be a vertex in S. Then there is a vertex $x_2 \in S \cap N^-(x_1)$ and an x_2-detour that does not contain x_1. Again, there is a vertex $x_3 \in S \cap N^-(x_2)$ and an x_3-detour that does not contain x_2. Continuing this procedure we construct a sequence $x_1, x_2, \ldots$ of elements of S such that $x_{i+1}x_i \in E(D)$. Since S is finite there is some $i < j$ such that $x_i = x_j$. Hence the subsequence $x_j x_{j-1} \ldots x_i$ is a closed walk and thus contains a cycle C. For a vertex v on C let v^+ and v^- be its out- and in-neighbour on C. Thus, for every vertex v on C, there is a v-detour not containing its successor v^+ on the cycle. Since v^- is contained in every v-detour, it follows that there exists a $v - v^-$-path P such that $v^+ \notin V(P)$. But then the arc v^-v is shared by the cycle C and the cycle v^-vPv^-. ■

An independent set of vertices K in a digraph D is a *kernel* of D if, for every vertex y in $D - K$ there is a vertex $x \in K$ such $xy \in E(D)$. It is shown in [13] that, if a digraph D with detour order λ has a kernel, then D has a $(1, \lambda - 1)$ partition.

Not every digraph has a kernel, e.g. an odd cycle does not have a kernel. In some cases the subdigraph of D induced by its detour source might have a kernel, even if D itself does not have a kernel, in which case the following result is useful.

Theorem 2.3. *Let D be a digraph with detour order λ and detour source S. If the digraph $D\langle S\rangle$ has a kernel, then D has a $(1, \lambda - 1)$-partition.*

Proof. Let A be a kernel of $D\langle S\rangle$ and put $B = V(D) - A$. Suppose there is a path P of order λ in $D\langle B\rangle$. If y is the initial vertex of P, then $y \in S - A$ and hence there is a vertex x in A such that xy is an arc in D. But then xP is a path of order $\lambda + 1$ in D. This contradiction shows that $\lambda(B) \leq \lambda - 1$, and hence (A, B) is a $(1, \lambda - 1)$-partition of D. ■

The following result is due to Richardson [15]. Shorter proofs of this result can be found in [3] and [1].

Theorem 2.4 (Richardson)**.** *Every digraph without odd cycles has a kernel.*

Thus we have the following result:

Corollary 2.5. *Let D be a digraph with detour order λ and detour source S. If the digraph $D\langle S\rangle$ does not contain an odd cycle, then D has a $(1, \lambda - 1)$-partition.*

3. Partitions of Digraphs Without Odd Cycles

We have seen that a digraph D without odd cycles has a $(1, \lambda(D) - 1)$-partition. The following result, due to Harary, Norman and Cartwright [12], shows that strong digraphs without odd cycles are, in fact, $(1, 1)$-partitionable (and hence λ-partitionable).

Theorem 3.1 (Harary, Norman and Cartwright). *A strong digraph is bipartite if and only if it has no odd cycle.*

A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. In order to extend this result to digraphs with more than one strong component, we need some definitions.

First we define the function p by choosing, for each $v \in V(D)$, the value $p(v)$ as the maximum order of a path in D that ends at v.

The *strong component digraph*, $SC(D)$, of D is the digraph whose vertices $v_1, \ldots, v_k$ correspond to the strong components $D_1, \ldots, D_k$ of D and whose arc set is given by $E(SC(D)) = \{v_i v_j :$ there is a vertex in D_i that is adjacent to a vertex in $D_j\}$.

Let S be a strong component of D and let s be the vertex in $SC(D)$ corresponding to S. Note that since $s \in V(SC(D))$, $p(s)$ is now the maximum order of a path in $SC(D)$ that ends at s. We say that the strong component S lies on *level* $p(s)$ of D. We also say that every vertex of S lies on level $p(s)$ of D. The number of levels of a digraph D is called its *height* h and is therefore given by $h(D) = \lambda(SC(D))$.

Theorem 3.2. *If D is a digraph of height h without odd cycles, then D has an (h, h)-partition.*

Proof. By Theorem 3.1, each strong component S of D has a $(1, 1)$-partition (A_S, B_S). Let

$$A = \bigcup A_S \text{ and } B = \bigcup B_S,$$

where the unions are taken over all strong components S of D. Since there are no arcs from a strong component on level j to a strong component on level i if $i \le j$, it follows that every path in A contains at most one vertex from each level. Hence $\lambda(D\langle A\rangle) \le h$. Similarly, $\lambda(D\langle B\rangle) \le h$ and hence (A, B) is an (h, h)-partition of D. ∎

Corollary 3.3. *Let D be a digraph with no odd cycles. If D has height $h \leq 2$, then D is λ-partitionable.*

Proof. By Theorem 2.4 D has a kernel. Thus by Corollary 2.5 D is $(1, \lambda - 1)$-partitionable and the result therefore follows immediately from Theorem 3.2. ■

4. Gallai-Roy-Vitaver Partitions

The following result which was proved independently by Gallai [11], Roy [16] and Vitaver [17] relates the chromatic number $\chi(G)$ of a graph G to the detour orders of its orientations.

Theorem 4.1 (Gallai-Roy-Vitaver)**.** *If D is any orientation of a graph G, then*

$$\chi(G) \leq \lambda(D)$$

and equality holds for some orientation of G.

The proof, as presented in [18], relies on a vertex partition of a maximal acyclic spanning subdigraph M of D, which we shall call the *Gallai-Roy-Vitaver* (*GRV*) *partition* of M.

In general, if D is any acyclic digraph with detour order λ, we use the function p defined in the previous section to define the *GRV partition* of D as the partition $(V_1, \ldots, V_\lambda)$ of $V(D)$, where

$$V_i = \{v \in V(D) : p(v) = i\}, \; i = 1, \ldots, \lambda.$$

Lemma 4.2. *Let D be an acyclic digraph with detour order λ and GRV partition $(V_1, \ldots, V_\lambda)$. Then the following hold:*

1. *If $v_i v_j \in E(D)$, with $v_i \in V_i$ and $v_j \in V_j$, then $i < j$.*
2. *If P is a path in D, then*

$$|V(P) \cap V_i| \leq 1, \; \textit{for } i = 1, \ldots, \lambda.$$

Proof. 1. Since $v_i \in V_i$, there is a path P of order i ending at v_i. Since $v_i v_j \in E(D)$, we cannot have $v_j \in V(P)$, otherwise D would have a cycle. Hence Pv_j is a path of order $i+1$ ending at v_j, so $p(v_j) \geq i+1$. Since $p(v_j) = j$, it follows that $i < j$.

2. This follows immediately from (1). ■

Remark 4.3. Note that Lemma 4.2 implies that the function p strictly increases along each path in an acyclic digraph.

Theorem 4.4. *If D is an acyclic digraph, then D is λ-partitionable.*

Proof. Suppose $\lambda(D) = a + b$; $1 \leq a \leq b$. Form the GRV partition, $V_1, \ldots, V_{\lambda(D)}$ of D. Let

$$A = \bigcup_{i=1}^{a} V_i \text{ and } B = \bigcup_{i=a+1}^{\lambda(D)} V_i.$$

By Lemma 4.2, every path in D has at most one vertex in each V_i; hence $\lambda(D\langle A\rangle) \leq a$ and $\lambda(D\langle B\rangle) \leq b$. ■

Given any digraph D we can consider a maximal acyclic spanning subdigraph M of D and then form the GRV partition of M. If we let A be the union of any a of the sets of the partition and put $B = V(D) - A$, then (A, B) will be an $(a, \lambda(M) - a)$-partition of M. In some cases this will yield a suitable partition for D. The next two results, one for digraphs and the other for oriented graphs, illustrate this technique.

Theorem 4.5. *If D is unicyclic, then D is λ-partitionable.*

Proof. Suppose $\lambda(D) = a+b$, where $1 \leq a \leq b$. Let M be a maximal acyclic spanning subdigraph of D, i.e., $M = D - uv$, where uv is an arc on the cycle in D. Form the GRV partition $V_1, \ldots, V_{\lambda(M)}$ of M, where $\lambda(M) \leq \lambda(D)$. Let A be the union of any a of these sets, including the one containing u but excluding the one containing v, and put $B = V(D) - A$. Then, as in the proof of Theorem 4.4, $\lambda(M\langle A\rangle) \leq a$ and $\lambda(M\langle B\rangle) \leq b$. Since the removed arc uv goes from a vertex in A to a vertex in B, we have $M\langle A\rangle = D\langle A\rangle$ and $M\langle B\rangle = D\langle B\rangle$; hence (A, B) is an (a, b)-partition of D. ■

Theorem 4.6. *Suppose an oriented graph D contains a maximal acyclic spanning subdigraph M with detour order at most 3. Then D is λ-partitionable.*

Proof. If $\lambda(M) \leq 2$ then D is acyclic (since D contains no 2-cycles) and then the result follows from Theorem 4.4. Now assume $\lambda(M) = 3$ and let V_1, V_2, V_3 be the GRV partition of M. Suppose $uv \in E(D) - E(M)$. By the maximality of M, the arc uv must lie on a cycle in $M + uv$ and hence there is a vu-path P in M. Since D contains no 2-cycles, P must have at least three vertices. Therefore, from Lemma 4.2, $v \in V_1$ and $u \in V_3$. Now put $A = V_1$ and $B = V_2 \cup V_3$. Then (A, B) is a $(1, 2)$-partition of M and, since every arc in D that is not an arc in M goes from B to A, it follows that (A, B) is also a $(1, 2)$-partition of D. This serves as an (a, b)-partition of D for every pair of positive integers (a, b) such that $\lambda(D) = a + b$. ∎

In the next section we generalize the idea of a GRV partition for digraphs that are not necessarily acyclic. This will allow us to construct (a, b)-partitions for various digraphs.

5. Prismatic Colourings

The proof of Theorem 4.4 relies on the fact that we obtained a partition of $V(D)$ into $\lambda(D)$ sets, such that no path in D contains more than one vertex from the same set. This leads to the following definition:

A *prismatic colouring* of a digraph D is a vertex colouring of D such that no two vertices lying on the same path in D have the same colour. Thus a prismatic k-colouring of D corresponds to a partition of $V(D)$ into k subsets $V_1, \ldots, V_k$ such that, for every path P in D,

$$|V(P) \cap V_i| \leq 1, \text{ for } i = 1, \ldots, k.$$

The *prismatic number,* $\varphi(D)$ of D is the minimum k such that D has a prismatic k-colouring.

Proposition 5.1. *If D is a digraph such that $\varphi(D) = \lambda(D)$, then D is λ-partitionable.*

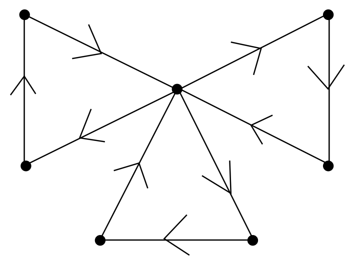
Proof. Suppose $\lambda(D) = a + b$. Consider a prismatic colouring of D with colour classes $V_1, \ldots, V_{\varphi(D)}$. Let A be the union of any a of these colour classes and let B be the union of the remaining b colour classes. Since every path in D has at most one vertex in each colour class, (A, B) is clearly an (a, b)-partition of D. ∎

The *chromatic number* $\chi(D)$ of a digraph D is defined to be the chromatic number of the underlying graph of D.

Proposition 5.2. *For any digraph D we have*

$$\chi(D) \leq \lambda(D) \leq \varphi(D).$$

Proof. The first inequality is given by Theorem 4.1 and the second follows from the fact that in any prismatic colouring of D a detour requires $\lambda(D)$ different colours. ∎



An example of a digraph D with $\chi(D) = 3$, $\lambda(D) = 5$ and $\varphi(D) = 7$.

If D is an acyclic digraph, then the GRV partition is a prismatic colouring in $\lambda(D)$ colours, so $\varphi(D) = \lambda(D)$.

We call a digraph D *detour-perfect* if $\chi(D') = \lambda(D')$ for every induced subdigraph D' of D. We shall show that for a detour-perfect graph all three the parameters of Proposition 5.2 are equal. First we present a characterization of detour-perfect digraphs.

Proposition 5.3. *A digraph D is detour-perfect if and only if for every path P in D the subdigraph induced by $V(P)$ is semicomplete* (*i.e., there is at least one arc between every pair of vertices in P*).

Proof. Suppose D is detour-perfect and P is a path of order n in D. Then $\chi(\langle P \rangle) = n$, so the underlying graph $\langle V(P) \rangle$ is isomorphic to K_n.

Conversely, suppose every path in D induces a semicomplete digraph. Consider a subdigraph D' of D. Let P be a path of order $\lambda(D')$ in D'. Then the underlying graph induced by $V(P)$ is isomorphic to $K_{\lambda(D')}$ and hence $\chi(D') \geq \lambda(D')$. Thus, by Theorem 4.1, $\chi(D') = \lambda(D')$ and hence D is detour-perfect. ∎

Corollary 5.4. *If D is detour-perfect, then $\chi(D) = \lambda(D) = \varphi(D)$.*

Proof. Consider any colouring c of D in $\chi(D)$ colours. Let P be any path in D. Then it follows from Proposition 5.3 that no two vertices of P lie in the same colour class, and hence c is a prismatic colouring. This proves that $\varphi(D) \leq \chi(D)$ and hence, by Proposition 5.2, $\chi(D) = \lambda(D) = \varphi(D)$. ∎

By Corollary 5.4 and Proposition 5.3 every detour-perfect digraph is λ-partitionable and every transitive digraph is detour-perfect. Thus we have

Corollary 5.5. *Every transitive digraph is λ-partitionable.*

As a matter of interest, we note that neither an odd cycle of order at least 5 nor its complement has a detour-perfect orientation. By the Perfect Graph Theorem, recently proved by Chudnovsky, Robertson, Seymour and Thomas (see [5]), only a perfect graph can have a detour-perfect orientation. However, not every perfect graph has a detour-perfect orientation, e.g. the net (a K_3 with exactly one pendant vertex attached to each of its three vertices) has no detour-perfect orientation.

The converse of Corollary 5.4 is not true. For example, let D be the digraph with vertices $\{a, b, c, d, e\}$ and arcs $\{ab, cb, dc, de, ae\}$. Then $\chi(D) = \lambda(D) = \varphi(D) = 3$, but D is not detour-perfect, because dcb is an induced path.

A *walkable component* W of a digraph is a maximal induced subdigraph which is walkable. Clearly, if D is a walkable digraph, then $\varphi(D) = n(D)$ where $n(D)$ is the order of D. In general we have

Theorem 5.6. *If D is a digraph, then*

$$\varphi(D) = \max\{n(W) : W \textit{ is a walkable component of } D\}.$$

Proof. First we define D_i to be the subdigraph of D induced by all the vertices on the first i levels of D. Note that if D has h levels then $D_h = D$. Let $w(D_i) = \max\{n(W) : W \text{ is a walkable component of } D_i\}$. The inequality $\varphi(D) \geq w(D)$ is obvious. To prove the reverse inequality we construct a level by level prismatic colouring c of the vertices of a digraph D which uses exactly $w(D)$ colours.

Let S be any strong component of D_1 and colour its vertices with colours $1, \ldots, n(S)$. Do so with all the strong components of D_1. The number of

colours used to colour the vertices of D_1 is therefore given by $\max\{n(S) : S$ is a strong component of $D_1\} = w(D_1)$.
Apply the next iterative steps for each level of the digraph D consecutively: Suppose D_{i-1} has already been coloured. Let S be any strong component on level i. First consider all the walkable components in D_{i-1} that have a neighbour in S. Let r be the largest number used in colouring these components. Colour the vertices of S with colours $r+1, \ldots, r+n(S)$. Do the same for all strong components on level i of D. Thus if D_i has a vertex coloured k, then that vertex lies in a walkable component of D_i of order at least k. Hence D_i is coloured with $w(D_i)$ colours. This holds for $i = 1, \ldots, h$ so D is coloured with $w(D)$ colours. Since no colour is repeated in any walkable component and since two vertices can only lie on the same path if they are on the same walkable component of D, the colouring c is clearly a prismatic colouring of the vertices of D. ■

Corollary 5.7. *If D is a digraph such that at least one of its walkable components of largest order is traceable, then D is λ-partitionable.*

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