Discussiones Mathematicae Graph Theory 25 (2005) 325–329

DECOMPOSITIONS INTO TWO PATHS

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Abstract

It is proved that a connected multigraph G which is the union of two edge-disjoint paths has another decomposition into two paths with the same set, U, of endvertices provided that the multigraph is neither a path nor cycle. Moreover, then the number of such decompositions is proved to be even unless the number is three, which occurs exactly if G is a tree homeomorphic with graph of either symbol + or \perp . A multigraph on n vertices with exactly two traceable pairs is constructed for each $n \geq 3$. The Thomason result on hamiltonian pairs is used and is proved to be sharp.

Keywords: graph, multigraph, path decomposition, hamiltonian decomposition, traceable.

2000 Mathematics Subject Classification: 05C70, 05C35, 05C38, 05C45.

1. Introduction

Investigations presented in what follows have been inspired by discussion within GRAPHNET [1] in February 2001 on a two-path conjecture presented then and there by Ken W. Smith of Central Michigan University. The conjecture says that a connected graph G which is the edge-disjoint union of two paths of length n has at least one more subgraph which is a path of length n. In four days the discussion concluded with a note by Doug West in which he presented a proof (based on Thomason's paper [6]) of the result which we state below in terms of decompositions. For notation and terminology, see [7].

The above two-path conjecture of Smith brings to mind a result due to Sloane [4]. It answers in the affirmative Shen Lin's question of 1965 whether every 4-regular graph which is hamiltonian decomposable has another hamiltonian cycle. Lin's paper [2] originated related investigations into hamiltonian decompositions.

Recall that a decomposition of a graph G is a collection of edge-disjoint subgraphs whose union is equal to G. A decomposition is called *hamiltonian* (*traceable*) if all decomposition parts are hamiltonian cycles (hamiltonian paths), the decomposition is called a *hamiltonian pair* (*traceable pair*) if the number of parts in question is two. Let $h_2(G)$ and $t_2(G)$ be the *numbers* of respectively hamiltonian and traceable pairs of G, the numbers being 0 if $\Delta(G) > 4$. Similarly, let $p_2(G)$ stand for the *number of decompositions* of Ginto two nontrivial paths.

In what follows by a graph, G, we mean a multigraph (which is loopless), the phrase *simple graph* is used to emphasize that multiple (or parallel) edges do not appear. The degree of a vertex is the number of incident edges.

Theorem A (D. West). If G is connected and decomposable into two paths of length k (where k > 1), then G is decomposable into a different pair of paths. In particular, one of these two, different from the original pair, has length at least k.

The following theorem is a part of Thomason's related result.

Theorem B. Let G be a multigraph with three or more vertices that has a hamiltonian pair. Then the number $h_2(G)$ of hamiltonian decompositions of G is even and at least four. Moreover, for any two edges of G, the number of hamiltonian pairs in which the two edges are in the same part is also even.

Note that if G has a pair of parallel edges, a simple switch produces the pair of new decomposition parts. Nevertheless, there are large multigraphs with only few (i.e., two) traceable pairs. Our main result follows.

Theorem 1. Let G be a connected multigraph that is decomposable into two nontrivial paths whose set of endvertices is denoted by U. If G is neither a path nor a cycle, then G is the union of a different pair of edge-disjoint paths with the same set U of endvertices. In fact, the number $p_2(G)$ of such decompositions is then even unless G is homeomorphic with the graph of either symbol +, or \bot , and then $p_2 = 3$.

Corollary 2. The number of traceable pairs among multigraphs of order $n \geq 3$ is even.

The following result shows that Thomason's lower bound $h_2(G) \ge 4$ in his theorem above is sharp.

Proposition 3. For each $n \geq 3$ there are two n-vertex multigraphs M_n and M''_n which have exactly four hamiltonian pairs and two traceable ones, respectively.

2. Proofs and Examples

Proof of Theorem 1. Let P and Q denote the original paths whose union is G. Consider the only interesting case that the largest vertex valency, $\Delta(G)$, is 3 or 4 and G is not homeomorphic with + or \bot . Then both paths Pand Q are nontrivial and G has three or more edges. Moreover, $2 \leq |U| \leq 4$, the vertices in U have degree 3 or less, and U includes all vertices of G of odd degree (1 or 3). Furthermore, at least two vertices of G are of degree larger than one.

Case 1. G has exactly one vertex of degree bigger than 2 and |U| = 3. Then paths P and Q share one endvertex and intersect at another vertex which is of degree 3 or 4 in G. Hence $\delta(G) = 1$. It is easily seen that $p_2(G) = 2$.

Case 2. |U| = 2. Then both vertices in U are of degree two in G, G has a vertex of degree 4 and no vertex of odd degree. Let \hat{G} be obtained from G by joining vertices in U by two parallel edges, say e, f. Let G' be the 4-regular homeomorph of \hat{G} (obtained by contracting an edge incident to a degree-2 vertex, one after another until no such an edge remains). Then $p_2(G)$ is equal to the number of hamiltonian decompositions of G' in which a fixed length-2 path P_3 containing the edge e and its neighbor is in one decomposition part. Therefore $p_2(G)$ is even by Theorem B. Moreover, $p_2(G) = h_2(G')/2$.

Case 3. |U| = 3 and G has two or more vertices of degree bigger than 2. Hence U comprises a vertex, x, of degree 2 and two vertices, say y, z, of odd degree. Let $\hat{G} = G + \{xy, xz\}$ and let G' be the 4-regular homeomorph of \hat{G} . Then $p_2(G) = h_2(G') - h'_2$ where h'_2 counts the hamiltonian decompositions of G' such that one part has preimage in \hat{G} containing the path yxz. Hence, by Theorem B, $p_2(G)$ is even.

Case 4. |U| = 4. Then all vertices in U are of odd degree (1 or 3) and G has at least two vertices with degrees in the set $\{3, 4\}$. Add to G a new vertex, say w, together with four edges joining w to all vertices in U. Let \hat{G} and G' be the resulting multigraph and its 4-regular homeomorph, respectively. Then two-path decompositions of G (which must keep U fixed) are in one-one correspondence with hamiltonian pairs of G' whence $p_2(G)$ is even.

Proof of Proposition 3. Given the cycle C_n with $n \ge 5$ and a path $P_4 = stuv$ contained in C_n , let the multigraph M_n be obtained from the square C_n^2 of C_n by removing the two crossing chords su and tv and by doubling of edges st and uv. Thus M_n is a 4-regular multigraph with two pairs of parallel edges. Note that contracting any two parallel edges of M_n with $n \geq 6$ results in M_{n-1} . Assume that multigraphs M_4 and M_3 are obtained if this contracting is applied to M_5 and then to M_4 , respectively. Hence $M_3 = {}^2K_3$, the doubled triangle. Let M''_n be obtained from M_n by removing a pair of parallel edges. Hence $M_3'' = {}^2P_3$ and M_4'' is the join of the 2-cycle C_2 and $2K_1$. Assume that, for each $n \geq 4$, notation in M''_n is chosen so that degree-2 vertices are u and v (or u', v') and ^{2}st are the two parallel edges. Note that there exists a map $M''_n \mapsto M''_{n+1}$ for $n \ge 4$ in which the degree-2 vertex u with neighbors t and, say, v_1 ($v_1 = s$ if n = 4) is removed and replaced by two new vertices, say u', v', together with four edges $tu', u'v, v'v, v'v_1$. It is enough to show that $t_2(M''_n) = 2$. This equality is easily seen for n = 3, 4. Use the map $M''_n \mapsto M''_{n+1}$ to show by induction on $n \ge 4$ that in each traceable pair of M''_n the part containing the edge tuis either the v-u section of the cycle C_n or its switching at 2st .

3. Concluding Remarks

More examples of multigraphs M on n vertices (inclusive of the above examples M''_n) with the smallest possible nonzero number of traceable pairs $t_2(M) = 2$, and with |U| = 2, are given in author's paper [3] for each $n \ge 7$. Then $p_2(M) = 2$, with vertices in U being the only possible endvertices of

decomposition parts. Simple graphs G of each order $n \ge 5$ with $t_2(G) = 4$ and |U| = 3 are given in [3], too.

At the other extreme, $t_2({}^2P_n) = \frac{1}{2}h_2({}^2C_n) = 2^{n-2}$ and this is not the largest value of t_2 among *n*-vertex simple graphs. It is a challenging problem to find (good estimates of) the largest value of t_2 (and/or h_2) among simple graphs (or multigraphs) on *n* vertices.

Acknowledgements

The author thanks both referees for their helpful remarks.

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Received 8 March 2004 Revised 2 November 2004