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TREES WITH α-LABELINGS AND DECOMPOSITIONS OF COMPLETE GRAPHS INTO NON-SYMMETRIC ISOMORPHIC SPANNING TREES

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Abstract

We examine constructions of non-symmetric trees with a flexible q-labeling or an α-like labeling, which allow factorization of K_{2n} into spanning trees, arising from the trees with α-labelings.

Keywords: graph decomposition and factorization, graph labeling, α-labeling, flexible q-labeling, α-like labeling.

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1. Introduction

Let G be a graph with at most n vertices. We say that the complete graph K_n has a *G-decomposition* if there are subgraphs $G_0, G_1, G_2, \ldots, G_s$, all isomorphic to G, such that each edge of K_n belongs to exactly one G_i. Then we say that G divides K_n, and write $G|K_n$. If G has exactly n vertices and none of them are isolated, then G is called a *factor* and the decomposition is called *G-factorization* of K_n. Many factorization methods are based on graph labelings, where a *labeling* of G with at most $2n+1$ vertices is an injection $\lambda : V(G) \to S$, $S \subseteq \{0, 1, \ldots, 2n\}$ and labels of vertices u, v (denote $\lambda(u), \lambda(v)$) induce uniquely the label or the *length* $\ell(e)$ of the edge $e = uv$. Let G have m edges and length of an edge $uv \in E(G)$ be $\ell(uv) = |\lambda(u) - \lambda(v)|$. If the set of all lengths of the m edges is equal to $\{1, 2, \ldots, m\}$

and $S \subseteq \{0, 1, \ldots, m\}$, then λ is a *graceful* or *β-labeling*. A graph with graceful labeling is called a *graceful graph*.

If there exists $\lambda_0 \in \{0, 1, ..., n-1\}$ in a graceful graph G such that for every edge $uv \in E(G), \lambda(u) < \lambda(v)$, it holds that $\lambda(u) \leq \lambda_0 < \lambda(v)$, then we say that G has an *α-labeling* with the *splitting value* λ_0.

A graceful labeling of a graph G with m edges yields decompositions of K_{2m+1} into $2m+1$ copies of G (see [5, 6]), whereas an α-labeling of a graph G with m edges can be used to yield decompositions of K_{2mk+1} into copies of G, for all positive integers k.

In this article we will usually identify a vertex v with its label $\lambda(v)$ and an edge uv with the pair $(\lambda(u), \lambda(v))$.

2. Definitions and Notations

In this section we introduce notation, definitions, and lemmas which are important for our further considerations.

Let G be a graph with $2n-1$ edges and at most $2n$ vertices and $\lambda : V(G) \to \{0, 1, 2, \ldots, 2n-1\}$ be an injection and length of an edge $uv \in E(G)$ be $\ell(uv) = \min\{|\lambda(u) - \lambda(v)|, 2n - |\lambda(u) - \lambda(v)|\}$. Then λ is called a *flexible q-labeling* if

(i) there is exactly one edge of length n,

(ii) for each m, $1 \leq m \leq n-1$, there are exactly two edges of length m, and

(iii) if $(r, r+m)$ with $1 \leq m \leq n-1$ is an edge of G, then the other edge of length m in G is $(r+2s+1, r+m+2s+1)$ for some $s, 0 \leq s \leq n-1$, where the labels are taken modulo $2n$.

If $(r, r+m)$ is an edge of length m, then the vertex r is called the *origin* and the vertex $r+m$ is called the *terminus*. We will refer to condition (iii) as to the *origin condition*.

Every flexible q-labeling λ in G, for which holds that

(iv) there exists $\lambda_0 \in \{0, 1, ..., 2n-1\}$ such that $\lambda(u) \leq \lambda_0 < \lambda(v)$ for each edge $uv \in E(G)$ is called an *α-like labeling* with the splitting value λ_0.

The existence of the previous two labelings, namely *flexible* and *α-like* in a tree T, implies the existence of a T-factorization of K_{2n} for n odd (see [2, 3]). Moreover, the existence of an α-like labeling of a tree T with $2n$

vertices, where n is odd, guarantees a factorization of K_{2nk} into isomorphic spanning trees with $2nk$ vertices for every positive odd number k.

Notice that there is only a limited number of classes of trees known to have an α-like or flexible q-labeling. It does not seem likely that a complete characterization of such trees would be found easily.

Peter Eldergill proved in [1] that a symmetric tree T allows cyclic T-factorization of K_{2n} if and only if T has a symmetric ρ-labeling, where a tree $T = T_1 \cup T_2 + uv$ is called *symmetric* if T_1, T_2 are trees, $u \in V(T_1), v \in V(T_2)$, and there exists isomorphism $f : V(T_1) \to V(T_2)$ such that $f(u) = v$.

Since we investigate non-cyclic T-factorizations of K_{2n}, where T is a non-symmetric tree, the basic question of this article is how to construct new non-symmetric trees with an α-like labeling or a flexible q-labeling from the trees with an α-labeling. Recall that for example all caterpillars, where a *caterpillar* is a tree with the property that removal of its endpoints leaves a path, allow an α-labeling.

Now we show that some symmetric trees allow α-like labeling.

Lemma 2.1. *If $T = T_1 \cup T_2 + e$ is a symmetric tree, where T_1, T_2 have α-labeling λ, $|T_1 \cup T_2| \equiv 2 \pmod 4$ and $e = uv, \lambda(u) = 0 = \lambda(v)$, then T allows an α-like labeling λ' with the splitting value $\lambda'_0 = |T_i| - 1$ for $i = 1, 2$.*

Proof. Suppose that T_1 and T_2 have an α-labeling λ with the splitting value λ_0 and (a, b) is an arbitrary edge belonging to T_1 or T_2, where $a, b \in \{0, 1, \ldots, |T_i| - 1\}, a \leq \lambda_0 < b$. And let $|T_1| = |T_2| = n = 2t + 1$. We form in T_1 and T_2 a new vertex labeling so that we transform every edge $(a, b) \in T_1$ to the edge (x, y), where $x = a, y = b + 2t + 1$. Every edge $(a, b) \in T_2$ we transform to the edge $(x + 2t + 1, y + 2t + 1)$. Thus the corresponding edges in T_1 and T_2 are $(a, b + 2t + 1) \in T_1$ and $(a + 2t + 1, b) \in T_2$. Since $2t + 1 + b - a > 2t + 1$, the origin of the edge $(a, b + 2t + 1)$ is $b + 2t + 1$ and its length is $4t + 2 - (2t + 1 + b - a) = 2t + 1 - (b - a)$, where $b - a = 1, 2, \ldots, 2t$. Since $a + 2t + 1 - b = 2t + 1 - (b - a) < 2t + 1$, the origin of the edge $(a + 2t + 1, b)$ is b and its length is $2t + 1 - (b - a)$.

Hence, every pair of corresponding edges in T_1 and T_2 have the same length $2t + 1 - (b - a)$, where $b - a = 1, 2, \ldots, 2t$, and their origins are of different parity. Further since $\lambda(u) = 0 = \lambda(v)$, the edge $(\lambda(u), \lambda(v))$ is transformed to the edge $(0, 2t + 1)$ of length $2t + 1$.

We see that T on $2n$ vertices contains the edge of length $2t + 1 = n$ exactly once, every edge of length ℓ, $\ell = 1, 2, \ldots, 2t = n - 1$ exactly twice and each pair of edges having the same length has origins of different parity. ■

A tree T on $4k$ vertices is called *α-like expandable* if it satisfies the conditions (i), (ii) and (iv) from the definitions of flexible q-labeling and α-like labeling, and in which both origins of edges having the same length are of the same parity, one origin less than or equal to λ_0 and the other one greater than λ_0. The following lemma shows that every α-like expandable tree is expandable to a tree with an α-like labeling.

Lemma 2.2. *For every α-like expandable tree T on $4k$ vertices there exists an α-like tree T' on $4k+2$ vertices.*

Proof. Let T be an α-like expandable tree on $4k$ vertices with the splitting value λ_0 and let $(r, r+l), (s, s+l)$, $r \leq \lambda_0, s > \lambda_0, r, s \in \{0, 1, \ldots, 4k-1\}$ be the edges having the same length l, $l \in \{1, 2, \ldots, 2k-1\}$, in T. We see that the origins of these edges are r, s and from the definition of an α-like expandable tree we know that they are of the same parity. We form in T a new vertex labeling λ' such that $\lambda'(x) = \lambda(x) + 1$ for $x \leq \lambda_0$ and $\lambda'(x) = \lambda(x) + 2$ for $x > \lambda_0$. Thus every pair of corresponding edges is transformed to a pair $(r+1, s+l+2), (s+2, s+l+1)$. Hence, if r and s were of the same parity then the new origins $r+1$ and $s+2$ are of different parity and the length of the corresponding edges is $l+1$. Further, the edge of length $2k$ is also transformed to an edge with length $2k+1$ in T. We see that in T the new labeling λ' has the splitting value $\lambda'_0 = \lambda_0 + 1$ and there are missing only two edges of length 1. We construct the tree T' so that we join 0 and $\lambda_0 + 2$, respectively, by extra edges to $4k+1$ and $\lambda_0 + 1$, respectively. We see that both new edges $(0, 4k+1)$ and $(\lambda_0 + 2, \lambda_0 + 1)$ have length 1 and $0 \leq \lambda_0 + 1 < 4k+1, \lambda_0 + 1 \leq \lambda_0 + 1 < \lambda_0 + 2$. Further we know that before we exchanged the labels in T the edges of length 1 to have origins $4k-1$ and λ_0 which were of the same parity. Thus λ_0 is odd. Therefore, the new corresponding edges of length 1 have origins of different parity. Hence, T' is an α-like tree on $4k+2$ vertices. ■

In this section we introduced two labelings whose existence for a tree T_{2n} guarantees that T_{2n} factorizes K_{2n} for n odd. Further, we have shown that every α-like expandable tree T with $4k$ vertices, which has not to factorize K_{4k}, is easily expandable to a tree T' on $4k+2$ vertices that factorizes K_{4k+2}. Lemma 2.1 will be used in the proof of Theorem 3.3 in the following section.

3. rn-Free Almost α-Like Forests

First we introduce the definition of some forests with m components on n vertices each, in which there are missing just the edges of length rn, because these forests we can simply extend to the trees with a flexible or an α-like labeling.

Let F be a forest with m components on n vertices each, where m is even, that allows a labeling $\lambda : V(F) \to \{0, 1, \ldots, mn-1\}$ such that

(i) F contains every edge of length ℓ, where $\ell \in \{1, 2, \ldots, rn-1, rn+1, \ldots, mn/2-1\}$ and $r \in \{1, 2, \ldots, m/2-1\}$, exactly twice. The length $\ell(u,v)$ of an edge $uv \in E(F)$, $\lambda(u) < \lambda(v)$, is defined as $\ell(u,v) = min\{\lambda(v)-\lambda(u), mn-(\lambda(v)-\lambda(u))\}$,

(ii) F contains the edge of length $mn/2$ at most once,

(iii) there exists $\lambda_0 \in \{0, 1, \ldots, mn-1\}$ such that for every $uv \in E(F)$, $\lambda(u) < \lambda(v)$, it holds that $\lambda(u) \le \lambda_0 < \lambda(v)$, and

(iv) the origins of every pair of edges having the same length are either of the same parity for $mn \equiv 0 \pmod 4$ or of different parity for $mn \equiv 2 \pmod 4$.

Then F is called *rn-free almost α-like* or we say that F has an *rn-free almost α-like* labeling.

In the following example we show a simple construction of a $5r$-free almost α-like forest on 20 vertices.

Example 3.1. Let us have two trees T and T', respectively, of order 5 with α-labelings λ and λ', respectively, with the splitting values λ_0 and λ'_0, respectively.

Let $V(T) = \{0,1,2,3,4\}, E(T) = \{(0,4),(0,3),(0,2),(2,1)\}$ and $V(T') = \{0,1,2,3,4\}, E(T') = \{(0,4),(0,3),(0,2),(0,1)\}$.

We denote by T_1, T_4 two copies of T and by T_2, T_3 two copies of T'. Now we exchange the labels in T_1 and T_2 so that $E(T_1) = \{(0,19),(0,18),(0,17),(17,1)\}$ and $E(T_2) = \{(2,16),(2,15),(2,14),(2,13)\}$. Since $T_1 \cong T_4$ and $T_2 \cong T_3$ we can form new labels in T_3 and T_4, respectively, so that $E(T_3) = \{(i+10, j+10) | (i,j) \in T_2\}$ and $E(T_4) = \{(i'+10, j'+10) | (i',j') \in T_1\}$, respectively, where we denote by (i,j) two corresponding edges in T_2 and T_3, and we denote by (i',j') two corresponding edges in T_1 and T_4. Thus $E(T_4) = \{(10,9),(10,8),(10,7),(7,11)\}$ and $E(T_3) = \{(12,3),(12,4),12,5),(12,6)\}$.

If we determine the length of every edge $(r, s) \in E(F)$, where $r < s$, as $\ell(r, s) = \min\{s - r, 20 - (s - r)\}$ then we see that T_1, T_4 contain the edges of lengths $1, 2, 3, 4$ and T_2, T_3 the edges of lengths $6, 7, 8, 9$. More over for every $(r, s) \in E(F)$, $r < s$, it holds that $r \le 9 < s$ and the origins of the edges having the same length are of the same parity. Therefore forest $F = \bigcup_{k=1}^{4} T_k$ is $5r$-free almost α-like for $r = 1$. ■

In Theorem 3.2 we generalize our method from Example 3.1.

Theorem 3.2. *Let $T_1, T_2, \ldots, T_{2k}$ be a collection of trees on n vertices each with α-labelings $\lambda_1, \lambda_2, \ldots, \lambda_{2k}$ and $(\lambda_1)_0, (\lambda_2)_0, \ldots, (\lambda_{2k})_0$ be their respective splitting values. Furthemore let for every $i = 1, 2, \ldots, k$ the trees T_i and T_{2k-i+1} be mutually isomorphic. Then the forest $F = T_1 \cup T_2 \cup \ldots \cup T_{2k}$ on mn vertices, where $m = 2k$, allows an rn-free almost α-like labeling with the splitting value $mn/2 - 1$.*

Proof. Suppose that $(u, v) \in E(T_i)$ for $i = 1, 2, \ldots, k$ and $u \le (\lambda_i)_0 < v$. We form in T_i for $i = 1, 2, \ldots, k$ a new vertex labeling so that we transform every edge (u, v) to the edge $(x, y) \in E(T_i)$, where $x = u + i - 1 + \sum_{j=1}^{i-1} (\lambda_j)_0$ and $y = v + (2k-1)n - (i-1)(n-1) + \sum_{j=1}^{i-1} (\lambda_j)_0$. Since T_i and T_{2k-i+1} are isomorphic the edge (u, v) exists also in T_{2k-i+1} and we transform this edge to the edge $(x + kn, y + kn)$, where the sums are taken modulo mn. Let for every edge (a, b) in T_i, $a < b$, the length $\ell(a, b)$ be defined as $\ell(a, b) = \min\{b - a, mn - (b - a)\}$. We see that $y > x$ and

$$y - x = (m - i)n + (v - u). \tag{1}$$

The expression (1) for $i = 1, 2, \ldots, k$ has minimal value for $i = k$ and thus $y - x > kn = mn/2$. From above it follows that the length of the edge (x, y) is $mn - (y - x) = mn - (m - i)n - (v - u) = in - (v - u)$ and the origin of the edge (x, y) is the vertex with the greater label, namely y. Further, from above it follows that T_1 contains the edges of lengths $n - 1, n - 2, \ldots, 1$, T_2 contains the edges of lengths $2n - 1, 2n - 2, \ldots, n + 1$ and at last T_k contains the edges of lengths $kn - 1, kn - 2, \ldots, (k-1)n + 1$. Now we show that $u + k - 1 + \sum_{j=1}^{k-1} (\lambda_j)_0 \le kn - 1$ and $kn - 1 < v + (2k-1)n - (k-1)(n-1) + \sum_{j=1}^{k-1} (\lambda_j)_0$. Since $(\lambda_j)_0 \le n - 1$ for every $j = 1, 2, \ldots, k - 1$ and $u \le n - 1$ it holds that

$$u + k - 1 + \sum_{j=1}^{k-1} (\lambda_j)_0 \le n - 1 + k - 1 + (k-1)(n-1). \tag{2}$$

Furthermore,

$$n-1+k-1+(k-1)(n-1)=(k-1)n+n-1=kn-1. \tag{3}$$

Thus from (2) and (3) it follows that

$$u+k-1+\sum_{j=1}^{k-1}(\lambda_j)_0 \leq kn-1. \tag{4}$$

Since $v \geq 0$ and $(\lambda_j)_0 \geq 0$ it holds that

$$v+(2k-1)n-(k-1)(n-1)+\sum_{j=1}^{k-1}(\lambda_j)_0 \geq (2k-1)n-(k-1)(n-1). \tag{5}$$

Moreover,

$$(2k-1)n-(k-1)(n-1)=2kn-n-kn+n+k-1=kn+k-1, k>0, \tag{6}$$

and from (5) and (6) we get

$$v+(2k-1)n-(k-1)(n-1)+\sum_{j=1}^{k-1}(\lambda_j)_0 > kn-1. \tag{7}$$

Since $0 \leq x \leq u+k-1+\sum_{j=1}^{k-1}(\lambda_j)_0 \leq kn-1$ and $kn-1 < v+(2k-1)n-(k-1)(n-1)+\sum_{j=1}^{k-1}(\lambda_j)_0 \leq y \leq mn-1$ it holds that $x \leq kn-1 < y$.

Let (x', y') be an arbitrary edge in T_{2k-i+1} such that $x' = x+kn$ and $y' = y+kn$. We say that the edges $(x', y') \in E(T_{2k-i+1})$ and $(x, y) \in T_i$ are corresponding. If $x \leq kn-1$ then $x' = x+kn > kn-1$ and if $y > kn-1$ then $y' = y+kn \leq kn-1$. Therefore the length of the edge $(x', y') \in T_{2k-i+1}$ is $x'-y' = x+kn-(y+kn) = -(y-x) = mn-(y-x) = in-(v-u)$. Thus the corresponding edges $(x, y) \in E(T_i)$ and $(x', y') \in E(T_{2k-i+1})$ have the same length l, where $l \in \{1, 2, \ldots, rn-1, rn+1, \ldots, kn-1\}$ for $r = 1, 2, \ldots, k-1$ and the origin of the edge (x, y) is the vertex y and the origin of the edge (x', y') is the vertex $y' = y+kn$.

Notice that $2kn = mn \equiv 2 \pmod 4$ if and only if k and n are odd and $mn \equiv 0 \pmod 4$ if and only if k or n is even. Hence, if $mn \equiv 2 \pmod 4$ then the origins y, y' are of different parity and if $mn \equiv 0 \pmod 4$ then the origins y, y' are of the same parity.

We see that there exists an injection $\lambda : V(F = T_1 \cup T_2 \cup \ldots \cup T_{2k}) \to \{0, 1, \ldots, 2kn-1\}$ such that the forest $F = T_1 \cup T_2 \cup \ldots \cup T_{2k}$ contains every edge of length l, $l \in \{1, 2, \ldots, rn-1, rn+1, \ldots, kn-1\}$ for $r = 1, 2, \ldots, k-1$, exactly twice. Parity of the origins of the corresponding edges with the same length is different for $2kn \equiv 2 \pmod 4$ and the same for $2kn \equiv 0 \pmod 4$. And more over for every edge $(a, b) \in E(F), a < b$ it holds that $a \le kn - 1 = mn/2 - 1 < b$. The forest F is rn-free almost α-like. ■

In the following theorem we show that an rn-free almost α-like forest F from the previous theorem is expandable to an rn-free almost α-like forest F' with $m + 1$ components.

Theorem 3.3. *Let $F' = F \cup T$ be a forest on $(m+1)n$ vertices, where $n \equiv 2 \pmod 4$, F be the forest from Theorem 3.2, and T be a symmetric tree on n vertices from Lemma 2.1 with an α-like labeling λ' and the splitting value $\lambda'_0 = \frac{n}{2} - 1$. Then F' allows rn-free almost α-like labeling λ with the splitting value $\lambda_0 = \frac{(m+1)n}{2} - 1$.*

Proof. We form a vertex labeling of F similar to that in the proof of Theorem 3.2. Let $(u, v), u < v, u, v \in \{0, 1, \ldots, n-1\}$ be an arbitrary edge in T_i and in the copy of T_i, namely in T_{2k-i+1}, for $i = 1, 2, \ldots, k$ and let $n = 4t+2$. Then we construct a vertex labeling in F so that for $i = 1, 2, \ldots, k$ every T_i contains the edges (x, y), where $x = u + i - 1 + \sum_{j=1}^{i-1} (\lambda_j)_0$ and $y = v + mn - (i-1)(n-1) + \sum_{j=1}^{i-1} (\lambda_j)_0$. Similarly every T_{2k-i+1} contains the edges (x', y'), where $x' = x + (m+1)n/2$ and $y' = y + (m+1)n/2$.

Hence, $y - x = v + mn - (i-1)(n-1) + \sum_{j=1}^{i-1} (\lambda_j)_0 - [u + i - 1 + \sum_{j=1}^{i-1} (\lambda_j)_0] = (m-i+1)n + (v-u) \ge (m-k+1)n + (v-u) = (k+1)n + (v-u) > (2k+1)n/2 = (m+1)n/2$ and thus T_i contains the edges of lengths $(m+1)n-(y-x) = (m+1)n-(m-i+1)n-(v-u) = in-(v-u) < (m+1)n/2$ with the origin y. Further since $x' - y' = x - y = -(y-x) < (m+1)n/2$, the tree T_{2k-i+1} contains the edges of lengths $x' - y' = -(y-x) = (m+1)n-(y-x) = (m+1)n-(m-i+1)n-(v-u) = in-(v-u) < (m+1)n/2$ with the origin $y' = y + (2k+1)(2t+1)$. From above it follows that T_1, T_{2k} contain the edges of lengths $n-1, n-2, \ldots, 1$, T_2, T_{2k-1} contain the edges of lengths $2n-1, 2n-2, \ldots, n+1$,..., T_{k-1}, T_{k+2} contain the edges of lengths $(k-1)n-1, (k-1)n-2, \ldots, (k-2)n+1$, and T_k, T_{k+1} contain the edges of lengths $kn-1 = mn/2-1, kn-2 = mn/2-2, \ldots, (k-1)n+1$. Moreover, it holds that $x < (m+1)n/2 - 1 < y$, $y' \le (m+1)n/2 - 1 < x'$ and the origins of the corresponding edges (x,y) and (x',y'), which have the same length are of different parity.

Let $(a,b), a,b \in \{0,1,\dots,n-1\}, a \le (\lambda')_0 < b$ be an arbitrary edge in T. We also form in T a new vertex labeling so that we transform every edge (a,b) to the edge (e,f), where $e = a + k + \sum_{j=1}^{k}(\lambda_j)_0, f = b + mn + \sum_{j=1}^{k}(\lambda_j)_0 - k(n-1)$. The difference

$$\begin{aligned} f - e &= b + mn + \textstyle\sum_{j=1}^{k}(\lambda_j)_0 - k(n-1) - [a + k + \textstyle\sum_{j=1}^{k}(\lambda_j)_0] \\ &= mn - k(n-1) - k + b - a = (m-k)n + b - a = kn + b - a \end{aligned} \tag{8}$$

is less than $(2k+1)n/2 = (2k+1)(2t+1) = (m+1)n/2$ for $b - a = 1, 2, \dots, n/2 - 1$, because

$$kn + \frac{n}{2} - 1 = (2k+1)\frac{n}{2} - 1 < (2k+1)n/2 = (m+1)n/2, \tag{9}$$

and greater than $(m+1)n/2$ for $b - a = n/2 + 1, n/2 + 2, \dots, n - 1$, and equal to $(m+1)n/2$ for $b - a = n/2$.

Hence, T contains for $b - a = 1, 2, \dots, n/2 - 1$ the edges of lengths $kn+1 = mn/2+1, kn+2 = mn/2+2, \dots, (2k+1)n/2-1 = (m+1)n/2-1$, for $b - a = n/2$ the edge of length $(2k+1)n/2 = (m+1)n/2$ and for $b - a = n/2+1, n/2+2, \dots, n-1$ the edges of lengths $(2k+1)n/2 - 1 = (m+1)n/2 - 1, (2k+1)n/2 - 2 = (m+1)n/2 - 2, \dots, mn/2 + 1$. Then T contains each edge of length $l, l = (m+1)n/2-1, (m+1)n/2-2, \dots, mn/2+1$ exactly twice and the edge of length $(m+1)n/2$ exactly once. Moreover, since T contained an α-like labeling with the splitting value $n/2 - 1$, the edges in T which have the same length have origins of different parity and for each edge (e,f) it holds that $e \le (m+1)n/2 - 1 < f$.

From above it follows that the forest $F' = T \cup F$ allows a rn-free almost α-like labeling with the splitting value $(m+1)n/2 - 1$ for $r = 1, 2, \dots, \frac{m}{2}$. ■

Now we know that a forest F with $2k$ components on n vertices each, where the components are pairwise mutually isomorphic, allows an rn-free almost α-like labeling. Further, we know that if we add to the forest F a new component T, where T is a symmetric tree on n vertices from Lemma 2.1, then the forest $F' = F \cup T$ is also rn-free almost α-like. In the following section we will describe how we can "connect" the components of the forests F or F' to obtain a non-symmetric tree with a flexible q, α-like or α-like expandable labeling.

4. Grafting of rn-Free Almost α-Like Forests

Now we want to find a graph G that interconnects particular components of F or F', respectively, in such a way that the resulting tree on mn or $(m+1)n$, respectively, vertices is either flexible or α-like or α-like expandable. Therefore, we will introduce a commutative binary graph operation which describes the interconnection of components of F or F'.

Let us have two graphs G and F, respectively, with vertex labelings λ and λ', respectively. Furthermore let there exist $u_1, u_2, \ldots, u_k \in V(G)$, $v_1, v_2, \ldots, v_k \in V(F)$ such that for each $i = 1, 2, \ldots, k$ it holds that $\lambda(u_i) = \lambda'(v_i)$ and for every $i \neq j$, when $i, j \in \{1, 2, \ldots, k\}$, it holds that u_i, u_j or v_i, v_j are independent. If we construct a graph H from G and F so that we identify precisely the pairs of vertices $u_i \in V(G), v_i \in V(F)$, then we obtain graph H which is called the *grafting of G to F* or the *grafting of F to G*, denoted by $H = G \curvearrowright F$ or $H = F \curvearrowright G$. (Recall that the operation $\curvearrowright$ is commutative.)

Thus $V(G \curvearrowright F) = V(G) \cup V(F)$, where $V(G) \cap V(F) = \{u_1 = v_1, u_2 = v_2, \ldots, u_k = v_k\}$, and $E(G \curvearrowright F) = E(G) \cup E(F)$, where $E(G) \cap E(F) = \emptyset$.

Theorem 4.1. *For every forest $F = T_1 \cup T_2 \cup \ldots \cup T_{2k}, k \geq 5, n > 2$, with an rn-free almost α-like labeling from Theorem* 3.2 *there exists a graph G such that the grafting $G \curvearrowright F$ is a non-symmetric flexible q tree on mn vertices, where $mn \equiv 2 \pmod 4$.*

Proof. Since $2kn = mn \equiv 2 \pmod 4$, the numbers k and n have to be odd. Let us have a non-symmetric tree T' on $m = 2k$ vertices with flexible q-labeling $\lambda' : V(T') \to \{0, 1, \ldots, 2k-1 = m-1\}$. Thus T' contains each edge of length l, $l = 1, 2, \ldots, k-1$ exactly twice, the edge of length k exactly once and the origins of each pair of the edges having the same length are of different parity. We form in T' a new vertex labeling so that we multiply all original labels in T' by n. Recall that n is odd. Hence, the set of all lengths of the edges in T' is equal to $\{n, 2n, \ldots, (k-1)n, kn\}$ and the origins of the edges having the same length are of different parity again. Then T' contains precisely the edges of the lengths which are missing in F and more over we know that every vertex $rn, r = 1, 2, \ldots, 2k-1 = m-1$ belongs to exactly one component of F. Thus $F \curvearrowright T'$ has a flexible q-labeling. ∎

Notice that in [2, 3, 4, 7] are introduced the infinite classes of the trees on $4k+2$ vertices that allow a flexible q-labeling (which is equivalent to a blended ρ-labeling (see [3, 7])) and an α-like labeling for $k \geq 2$.

Theorem 4.2. *Let $F' = T \cup F$, where $F = T_1 \cup T_2 \cup \ldots \cup T_{2k}$, be an rn-free almost α-like forest from Theorem 3.3, $n > 2$, and suppose that for every T_i, $i = 1, 2, \ldots, k$, it holds that T_i is not isomorphic to a star on n vertices. Then there exists a graph G such that the grafting $F' \curvearrowright G$ is a non-symmetric flexible q tree for $(m+1)n \equiv 2 \pmod 4$.*

Proof. If $(2k+1)n = (m+1)n \equiv 2 \pmod 4$ then $n \equiv 2 \pmod 4$. We cannot use the method from the previous proof, because after multiplication of all labels by n, where n is even, the new labels will be all even. Thus G cannot satisfy the origin condition.

We know that every vertex rn, $r = 0, 1, 2, \ldots, 2k$ belongs to exactly one component of F'. We denote by $T_{i_0}, T_{i_1}, \ldots, T_{i_k}$ the components of F' which contain the vertices $0, n, 2n, \ldots, kn$ and by $T_{i_{k+1}}, T_{i_{k+2}}, \ldots, T_{i_{2k}}$ the components of F' which contain the vertices $(k+1)n, (k+2)n, \ldots, 2kn$. Thus T_{i_0} is T_1. Let A_{i_r} and B_{i_r}, respectively, be a partition of T_{i_r} that contains the vertices $a \leq mn/2 - 1$ and $b > mn/2 - 1$, respectively. Since T_{i_r} is not isomorphic to a star $K_{1,n-1}$, it follows that A_{i_r} contains at least two vertices. Hence, T_1 contains the vertices $0, 1$ and every $T_{i_l}, l = 1, 2, \ldots, k$ contains the vertex $ln - 1$ or $ln + 1$. More over we know that the original edge of length $n-1$ in T_1 is transformed to the edge $((m+1)n-1, 0)$ and thus T_1 contains also the vertex $(m+1)n - 1$. Now we construct the graph G so that $V(G)$ contains the vertices $0, 1, (m+1)n - 1, sn$ for $s = k+1, k+2, \ldots, 2k$ and $ln-1$ if $ln-1 \in T_{i_l}$ or $ln+1$ if $ln+1 \in T_{i_l}$. If $ln+1 \in T_{i_l}$ and $ln-1 \in T_{i_l}$ we choose exactly one of them. $E(G)$ contains $(0, sn)$ and $((m+1)n-1, ln-1)$ if $ln - 1 \in T_{i_l}$ or $(1, ln+1)$ if $ln+1 \in T_{i_l}$. From above it follows that G contains every edge of length $n, 2n, \ldots, kn$ exactly twice and their origins are of different parity. The method of previous construction of G guarantees that the grafting $G \curvearrowright F'$ is an acyclic connected graph with a flexible q-labeling, which is not symmetric. ∎

Theorem 4.3. *For every F there exists a graph G such that the grafting $G \curvearrowright F$ is either a non-symmetric α-like tree on mn vertices for $mn \equiv 2 \pmod 4$ or a non-symmetric α-like expandable tree on mn vertices for $mn \equiv 0 \pmod 4$, where $m = 2k, k \geq 5$.*

Proof. *Case* 1. Let k be odd.

Subcase 1.1. Let n be odd. Thus $mn \equiv 2 \pmod 4$. Let T be an arbitrary non-symmetric tree on 10 vertices with an α-like labeling and the splitting value 4.

For example let $V(T) = \{0,1,2,\ldots,9\}$ and $E(T) = \{(0,9),(0,8),(0,7),$ $(7,2),\ (2,6),\ (2,5),(5,1),(5,3),(5,4)\}$. We know (see [4]) that for each α-like tree on $4t+2$ vertices with the splitting value $2t$ there exists an α-like tree on $4(t+1)+2$ vertices with the splitting value $2t+2$. Hence, the tree T is expandable step by step to an α-like tree T' on $2k, k \geq 5$, vertices with the splitting value $k-1$. Let T' have the desired α-like labeling with the splitting value $k-1$. Now we form in T' a new labeling so that we multiply all labels by n. Thus T' contains the edges of lengths $n, 2n, \ldots, (k-1)n$ exactly twice and the edge of length kn exactly once, the origins of the edges having the same length are of different parity, and the splitting value is $n(k-1) = kn - n = mn/2 - n = (mn/2 - 1) - (n-1)$. Hence, T' is a non-symmetric tree such that $T' \curvearrowright F$ is a non-symmetric α-like tree on mn vertices with the splitting value $mn/2 - 1$.

Subcase 1.2. Let n be even. Thus $mn \equiv 0 \pmod 4$. Let T be an arbitrary non-symmetric tree with an α-labeling on 10 vertices with the splitting value 4 in which after the redefining of a length of an edge $(u,v) \in E(T), u \leq 4 < v$ so that $\ell(u,v) = \min\{v-u, mn-(v-u)\}$, it holds that one origin of the pair of the edges having the same length is less than or equal to 4 and the other one is greater than 4. Recall that the original definition of the length of an edge $uv \in E(T)$ was $\ell(u,v) = v-u$. And such a definition does not allow us to determine the origin of an edge $uv \in E(T)$. Let T have $V(T) = \{0,1,\ldots,9\}$ and $E(T) = \{(0,9),\ (9,1),\ (1,8),\ (8,2),\ (2,7),$ $(2,6),\ (6,3),\ (3,5),\ (5,4)\}$. Again as in *Subcase* 1.1 T is expandable to tree G on $2k, k \geq 5$, vertices with an α labeling and the splitting value $k-1$. If we multiply all labels in G by n then we have in G the vertex labeling $\lambda : V(G) \to \{0, n, 2n, \ldots, (2k-1)n\}$ with the splitting value $(k-1)n = kn - n = mn/2 - 1 - (n-1)$ and the set of all lengths of the edges in G is equal to $\{n, 2n, \ldots, kn = mn/2\}$. Further we know that the origins of the edges having the same length in G and F are of the same parity and one of them is less than or equal to $mn/2 - 1$ and the other one is greater than $mn/2 - 1$. Therefore the grafting $G \curvearrowright F$ is a non-symmetric α-like expandable tree on mn vertices with the splitting value $mn/2 - 1$.

Case 2. Let k be even.

Subcase 2.1. Let n be odd. Let T be a non-symmetric α-like expandable tree on 12 vertices with the splitting value 5. Let T have $V(T) =$

$\{0, 1, \ldots, 11\}$ and $E(T) = \{(0, 11), (0, 10), (0, 9), (10, 2), (1, 8), (1, 7), (7, 2), (2, 6), (6, 3), (6, 4), (6, 5)\}$.

Recall that the origins of the edges having the same length are of the same parity in T. Again T is expandable step by step to an α-like expandable tree G on $2k, k \geq 6$ so that we add 1 to all labels that are less than or equal to 5 and 3 to the labels that are greater than 5. Further we label four new vertices by 15,0,7,8 and we join by two extra edges the pair of the vertices 0,15 and the pair of the vertices 7,8. After we join the pair of adjacent vertices 0,15 by the edge $(15, 1)$ and the pair of adjacent vertices 7,8 by the edge $(7, 6)$ to the tree T, we see that we obtain an α-like expandable tree on 16 vertices with the splitting value 7. If we continue in this procedure step by step we receive an infinite class of the desired trees. Again if we multiply all labels in G by n and construct the grafting $G \curvearrowright F$ then we obtain a non-symmetric α-like expandable tree on mn vertices with the splitting value $mn/2 - 1$.

Subcase 2.2. Let n be even. This proof is essentialy similar and therefore can be omitted.

We define the tree T the same way as in *Subcase* 1.2, but on 12 vertices. Notice that tree T in *Subcase* 1.2 is expandable to tree G such that $G \curvearrowright F$ is a non-symmetric α-like expandable tree for k and n even.
Now the proof is complete. ■

From above it follows that if we take a non-symmetric tree on m vertices, where m is even, with flexible q, α-like or α-like expandable labeling and "glue" to every vertex of such tree always one component of the forest F, then we receive a tree that either factorizes K_{mn} if $mn \equiv 2$ modulo 4, or is easily expandable to a tree on $mn + 2$ vertices that factorizes K_{mn+2} if $mn \equiv 0$ modulo 4. Further, we know a method how to "connect" the components of the forest F' to obtain a tree on $(m + 1)n$ vertices that factorizes $K_{(m+1)n}$ for $(m + 1)n \equiv 2$ modulo 4.

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