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EXACT DOUBLE DOMINATION IN GRAPHS

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Abstract

In a graph a vertex is said to dominate itself and all its neighbours. A doubly dominating set of a graph G is a subset of vertices that dominates every vertex of G at least twice. A doubly dominating set is exact if every vertex of G is dominated exactly twice. We prove that the existence of an exact doubly dominating set is an NP-complete problem. We show that if an exact double dominating set exists then all such sets have the same size, and we establish bounds on this size. We give a constructive characterization of those trees that admit a doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

Keywords: double domination, exact double domination.

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1. Introduction

In a graph $G = (V, E)$, a subset $S \subseteq V$ is a *dominating set* of G if every vertex v of $V - S$ has a neighbour in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For a comprehensive treatment of domination in graphs and its variations, see [8, 9].

Harary and Haynes [7] defined and studied the concept of double domination, which generalizes domination in graphs. In a graph $G = (V, E)$, a subset S of V is a *doubly dominating set* of G if, for every vertex $v \in V$, either v is in S and has at least one neighbour in S or v is in $V - S$ and has at least two neighbours in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a doubly dominating set of G. Double domination was also studied in [2, 3, 4]. Analogously to exact (or perfect) domination introduced by Bange, Barkauskas and Slater [1], Harary and Haynes [7] defined an *efficient doubly dominating set* as a subset S of V such that each vertex of V is dominated by exactly two vertices of S. We will prefer here to use the phrase *exact doubly dominating set*.

Every graph $G = (V, E)$ with no isolated vertex has a doubly dominating set; for example V is such a set. In contrast, not all graphs with no isolated vertex admit an exact doubly dominating set; for example, the star $K_{1,p}$ $(p \geq 2)$ does not. In Section 2 we prove that the existence of an exact doubly dominating set is an NP-complete problem. We then show in Section 3 that if a graph G admits an exact doubly dominating set then all such sets have the same size, and we give some bounds on this number. Finally, we give in Section 4 a constructive characterization of those trees that admit an exact doubly dominating set, and we establish a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph.

Let us give some definitions and notation. In a graph $G = (V, E)$, the *open neighbourhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, the *closed neighbourhood* is the set $N[v] = N(v) \cup \{v\}$, and the *degree* of v is the size of its open neighbourhood, denoted by $\deg_G(v)$. We denote respectively by n, δ and Δ the *order* (number of vertices), *minimum degree* and *maximum degree* of a graph G.

2. NP-Completeness

In this section we consider the complexity of the problem of deciding whether a graph admits an exact doubly dominating set.

EXACT DOUBLY DOMINATING SET (X2D)

Instance: A graph G;

Question: Does G admit an exact doubly dominating set?

We show that this problem is NP-complete by reducing the following EXACT 3-COVER (X3C) problem to our problem.

EXACT 3-COVER (X3C)

Instance: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X;

Question: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

EXACT 3-COVER is a well-known NP-complete problem [6].

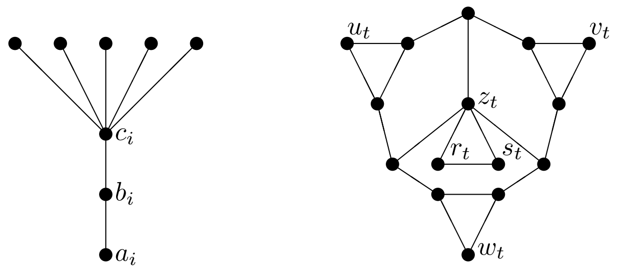
Theorem 1. *EXACT DOUBLY DOMINATING SET is NP-complete.*

Proof. Clearly, X2D is in NP. Let us now show how to transform any instance X, C of X3C into an instance G of X2D so that one of them has a solution if and only if the other has a solution.

For each $x_i \in X$, we build a "gadget" graph with vertices a_i, b_i, c_i and $d_i^1, \ldots, d_i^{k_i}$, where k_i is the number of elements of C that contain x_i, and with edges $a_i b_i, b_i c_i$ and $c_i d_i^j$ $(j = 1, \ldots, k_i)$. We view the d_i^j's as points of this gadget, each of them being associated with an element of C that contains x_i. See Figure 1.

For each $C_t \in C$, we build a gadget graph with 15 vertices $y_t^0, \ldots, y_t^8$, z_t, r_t, s_t, u_t, v_t, w_t and edges $y_t^j y_t^{j+1}$ $(j = 0, \ldots, 8 \bmod 9)$ (so that the y_t^j's induce a C_9) and $z_t y_t^0, z_t y_t^3, z_t y_t^6, z_t r_t, z_t s_t, r_t s_t$ (so z_t, r_t, s_t induce a triangle), and $u_t y_t^1, u_t y_t^2$, $v_t y_t^4, v_t y_t^5$, $w_t y_t^7, w_t y_t^8$. We view u_t, v_t, w_t as the three points of this gadget, each of them being associated with an element of C_t. See Figure 1.

Now, for each C_t, if $C_t = \{x_i, x_j, x_k\}$ say, we identify the first, second and third point of the gadget of C_t with the corresponding point in the gadget of x_i, x_j, x_k respectively. We call G the resulting graph. Clearly the size of G is polynomial in the size of X and C.



Gadget for x_i (with $k_i = 5$) Gadget for C_t

Figure 1: Gadgets for an element x_i and a triple C_t.

1. Suppose that the instance X, C of X3C has a solution C'. We build a set S of vertices of G as follows: for each $C_t \in C'$, we put in S the vertices $u_t, y_t^1, v_t, y_t^4, w_t, y_t^7, z_t, r_t$; for each $C_t \in C - C'$, we put in S the vertices $y_t^1, y_t^2, y_t^4, y_t^5, y_t^7, y_t^8, r_t, s_t$; for each $x_i \in X$, we put in S the vertices a_i, b_i (note that exactly one of the d_i^j's has been put in S). It is a routine matter to check that S is an exact doubly dominating set in G.

2. Conversely, suppose that G has an exact doubly dominating set S. Note the gadget of a given C_t is in exactly one of the following two possible states:

(a) $z_t \in S$, and so exactly one of r_t, s_t is in S, none of y_t^0, y_t^3, y_t^6 is in S, the other six y_t^j's are in S, and none of u_t, v_t, w_t is in S; or

(b) $z_t \notin S$, both r_t, s_t are in S, none of y_t^0, y_t^3, y_t^6 is in S, exactly one of $\{y_t^1, y_t^4, y_t^7\}$, $\{y_t^2, y_t^5, y_t^8\}$ is in S and the other is in $V - S$, and each of u_t, v_t, w_t is in S.

Clearly, for each $x_i \in X$, we have $a_i, b_i \in S$ (else a_i would not be doubly dominated), then $c_i \notin S$ (else b_i would be dominated three times), and it follows that exactly one of the d_i^j's is in S. For each $i = 1, \ldots, 3q$, let $t(i)$ be the integer such that this special d_i^j is equal to one point of $C_{t(i)} \in C$, and let us say that $C_{t(i)}$ is selected by x_i. Thus the gadget of $C_{t(i)}$ is in state (b), which means that $C_{t(i)}$ is selected by each of its 3 elements. Therefore, the collection C' of all selected elements of C (i.e., those whose three points are in S) is an exact 3-cover. ■

3. Exact Doubly Dominating Sets

We begin by the following observation which is a straightforward property of exact doubly dominating sets in graphs. A *matching* in a graph G is a set of pairwise non-incident edges of E.

Observation 2. The vertex set of every exact doubly dominating set induces a matching.

Next, we show that all exact doubly dominating sets (if any) have the same size.

Proposition 3. *If G has an exact doubly dominating set then all such sets have the same size.*

Proof. Let D_1, D_2 be two exact doubly dominating sets of G. Let us write $I = D_1 \cap D_2$, and let X_0 and X_1 be the subsets of $D_1 - I$ such that every vertex of X_0 has zero neighbours in I and every vertex of X_1 has one neighbour in I. Clearly $D_1 - I = X_0 \cup X_1$. We define similarly subsets Y_0 and Y_1 of $D_2 - I$. We claim that $|X_1| = |Y_1|$. Indeed, let x be any vertex of X_1, adjacent to a vertex $z \in I$. Since D_2 is an exact doubly dominating set, z has a unique neighbour y in D_2. We have $y \in D_2 - I$, for otherwise z has two neighbours x, y in D_2, a contradiction. Thus $y \in Y_1$. The symmetric argument holds for every vertex of Y_1, and so $|X_1| = |Y_1|$. Since D_2 is an exact doubly dominating set, every vertex of X_1 has exactly one neighbour in $Y_0 \cup Y_1$ and every vertex of X_0 has exactly two neighbours in $Y_0 \cup Y_1$. The same holds about the vertices of Y_1 and Y_0. This implies $|X_0| = |Y_0|$, and thus $|D_1| = |D_2|$. ■

The next result relates the size of an exact doubly dominating set with the order and minimum degree δ of a graph G.

Proposition 4. *If S is an exact doubly dominating set of a graph G, then $|S| \leq 2n/(\delta + 1)$.*

Proof. Let S be an exact doubly dominating set of a graph G and let t denote the number of edges joining the vertices of S to the vertices of $V - S$. Then $t = 2|V - S|$ since S is an exact doubly dominating set. By Observation 2, S induces a matching of G, and so every vertex v of S has exactly $\deg_G(v) - 1$ neighbours in $V - S$. Thus $t = \sum_{v \in S}(\deg_G(v) - 1)$. So $|S|(\delta - 1) \leq t = 2|V - S|$. Hence $|S| \leq 2n/(\delta + 1)$. ■

In [7], Harary and Haynes gave a lower bound for the doubly domination number:

Theorem 5 ([7]). *If G has no isolated vertices, then $\gamma_{\times 2}(G) \geq 2n/(\Delta+1)$.*

From Proposition 4 and Theorem 5, we have:

Corollary 6. *If S is an exact doubly dominating set of a regular graph G, then $|S| = 2n/(\Delta+1)$.*

Next, we establish a bound on the double domination number based on the neighbourhood packing number for any graph with no isolated vertices. Recall that a set $R \subseteq V(G)$ is a *neighbourhood packing set* of G if $N[x] \cap N[y] = \emptyset$ holds for any two distinct vertices $x, y \in R$. The *neighbourhood packing number* $\rho(G)$ is the maximum cardinality of a neighbourhood packing in G. It is easy to see (see [8]) that every graph G satisfies $\rho(G) \leq \gamma(G)$.

Theorem 7. *If G is a graph without isolated vertices, then $\gamma_{\times 2}(G) \geq 2\rho(G)$.*

Proof. Let R be a maximum neighbourhood packing set of G. Then for every $v \in R$, every doubly dominating set of G contains at least 2 vertices of $N[v]$ to doubly dominate v. Since $N[v] \cap N[u] = \emptyset$ holds for each pair of vertices v, u of R, we have $\gamma_{\times 2}(G) \geq 2|S|$. ■

Corollary 8. *If S is an exact doubly dominating set of G, then $|S| \geq 2\rho(G)$.*

Farber [5] proved that the domination number and neighbourhood packing number are equal for any strongly chordal graph. Thus we have the following corollary to Theorem 7 which extends the result of Blidia *et al.* [3] for trees.

Corollary 9. *If G is a strongly chordal graph without isolated vertices, then $\gamma_{\times 2}(G) \geq 2\gamma(G)$.*

4. Graphs with Exact Doubly Dominating Sets

We first consider paths and cycles. The double domination number for cycles C_n and nontrivial paths P_n were given in [7] and [3] respectively:

[7] $\gamma_{\times 2}(C_n) = \lceil \frac{2n}{3} \rceil$.

[3] $\gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil + 1$ if $n \equiv 0 \pmod 3$ and $\gamma_{\times 2}(P_n) = 2\lceil \frac{n}{3} \rceil$ otherwise.

Now we establish similar results for the exact doubly dominating sets in cycles and paths.

Proposition 10. *A cycle C_n has an exact doubly dominating set if and only if $n \equiv 0 \pmod 3$. If this holds the size of any such set is $2n/3$.*

Proof. Let S be an exact doubly dominating set of a cycle C_n. By Corollary 6, we have $|S| = 2n/3$ and so $n \equiv 0 \pmod 3$. Conversely, assume the vertices of C_n are labelled $v_1, v_2, \ldots, v_n, v_1$. If $n \equiv 0 \pmod 3$, then it is easy to check that the set $\{v_i, v_{i+1} \mid i \equiv 1(\text{mod } 3), 1 \le i \le n-1\}$ is an exact doubly dominating set of C_n. ■

Proposition 11. *A path P_n has an exact doubly dominating set if and only if $n \equiv 2 \pmod 3$. If this holds the size of any such set is $2(n+1)/3$.*

Proof. If $n = 2$ the fact is obvious, so let us assume $n \ge 3$. Let S be an exact doubly dominating set of a path P_n. Note that for every vertex v of degree 2, either v or its two neighbours are in S. So $V - S$ is an independent set, and $N(v) \cap N(w) = \emptyset$ for any two $v, w \in V - S$. By Observation 2, every vertex of S has exactly one neighbour in $V - S$. Thus $|S| - 2 = 2|V - S|$ and so $n = |S| + |V - S| = 3|V - S| + 2$.

Conversely, assume that the vertices of P_n are labelled $v_1, v_2, \ldots, v_n$. If $n \equiv 2 \pmod 3$ then it is easy to check that the set $\{v_i, v_{i+1} \mid i \equiv 1(\text{mod } 3), 1 \le i \le n-1\}$ is an exact doubly dominating set of P_n. ■

Chellali and Haynes [4] established the following upper bound for the double domination number:

Theorem 12 ([4])**.** *Every graph G without isolated vertices satisfies*

$$\gamma_{\times 2}(G) \le n - \delta + 1.$$

Theorem 13. *Let G be a graph that admits an exact doubly dominating set S. Then $|S| = n - \delta + 1$ if and only if either $G = tK_2$ with $t \ge 1$, if $\delta = 1$, or $G = K_n$ with $n \ge 3$ otherwise.*

Proof. Let S be an exact doubly dominating set of G such that $|S| = n - \delta + 1$. If $\delta = 1$, then $|S| = n$. Since S induces a 1-regular subgraph, G itself is 1-regular, i.e., $G = tK_2$ with $t \ge 1$. Now assume that $\delta \ge 2$. Let v be a vertex of S. Then $V - S$ contains all the neighbours of v except one, and so $deg_G(v) - 1 \le |V - S| = n - (n - \delta + 1) = \delta - 1$. Thus all the vertices of S have the same degree δ, and $|V - S| = \delta - 1$. Let u be a vertex of $N(v) \cap S$. Then u is adjacent to all the vertices of $V - S$ and hence at this point every vertex of $V - S$ is doubly dominated by u and v. Thus $S = \{u, v\}$ and all the vertices of $V - S$ are mutually adjacent. So G is a complete graph. ■

Next, we consider nontrivial trees. A vertex of degree 1 is called a *leaf*, and a *support vertex* is any vertex adjacent to a leaf. It is easy to see that a star with at least three vertices is an example of a tree that does not admit an exact doubly dominating set. The following observation generalizes this remark.

Observation 14.

- If a graph G has a leaf, then any doubly dominating set of G contains this leaf and its neighbour.
- If a graph G has an exact doubly dominating set, then every support vertex is adjacent to exactly one leaf, and no two support vertices are adjacent.

We now define recursively a collection $\mathcal{T}$ of trees, where each tree $T \in \mathcal{T}$ has two distinguished subsets $A(T), B(T)$ of vertices. First, $\mathcal{T}$ contains any tree T_1 with two vertices x, y, and for such a tree we set $A(T_1) = \{x, y\}$ and $B(T_1) = \{y\}$. Next, if T' is any tree in $\mathcal{T}$, then we put in $\mathcal{T}$ any tree T that can be obtained from T' by any of the following two operations:

Type-1 *operation*: Attach a path $P_3 = uvw$, with $u, v, w \notin V(T')$, by adding an edge from w to one vertex of $A(T')$. Set $A(T) = A(T') \cup \{u, v\}$ and $B(T) = B(T') \cup \{u\}$.

Type-2 *operation*: Attach a path $P_5 = a_1a_2a_3a_4a_5$, with $a_1, a_2, a_3, a_4, a_5 \notin V(T')$, by adding an edge from a_3 to one vertex of $V(T') - A(T')$. Set $A(T) = A(T') \cup \{a_1, a_2, a_4, a_5\}$ and $B(T) = B(T') \cup \{a_1, a_5\}$.

Lemma 15. *If $T \in \mathcal{T}$, then:*

(a) *$A(T)$ is the unique exact doubly dominating set of T.*

(b) *$B(T)$ is a neighbourhood packing set of T.*

(c) *$|A(T)| = 2\gamma(T)$.*

Proof. Consider any $T \in \mathcal{T}$. So T can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees of $\mathcal{T}$, where T_1 is the tree with two vertices, $T = T_k$, and, if $1 \leq i \leq k - 1$, the tree T_{i+1} is obtained from T_i by one of the two operations. We prove (a) by induction on k. If $k = 1$, then $A(T)$ is obviously the unique exact doubly dominating set of T. Assume now that $k \geq 2$ holds for T and that the result holds for all trees in $\mathcal{T}$ that can be constructed by a sequence of length at most $k-1$. Let $T' = T_{k-1}$. We distinguish between two cases.

Case 1. T is obtained from T' by using the Type-1 operation. Note that $A(T)$ is an exact doubly dominating set of T since, by the induction hypothesis, $A(T')$ is an exact doubly dominating set of T' and u, v and the neighbour of w in T' are in $A(T)$. Now let S be any exact doubly dominating set of T. By Observation 14, we have $\{u, v\} \subset S$, and consequently $w \notin S$ (for otherwise v would be dominated three times by S). If x is any vertex in $V(T')$, then x is not dominated by any of u, v, so $S - \{u, v\}$ is an exact doubly dominating set of T'. By the inductive hypothesis $A(T')$ is the unique such set, so $S - \{u, v\} = A(T')$, and so $S = A(T)$, which shows the unicity anounced in (a).

Case 2. T is obtained from T' by using the Type-2 operation. Note that $A(T)$ is an exact doubly dominating set of T since, by the induction hypothesis, $A(T')$ is an exact doubly dominating set of T' and the neighbour of a_3 in T' is not in $A(T')$ while a_1, a_2, a_4, a_5 are in $A(T)$. Now let S be any exact doubly dominating set of T. By Observation 14, we have $\{a_1, a_2, a_4, a_5\} \subseteq S$, and consequently $a_3 \notin S$. If x is any vertex in $V(T')$, then x is not dominated by any of a_1, a_2, a_4, a_5, so $S - \{a_1, a_2, a_4, a_5\}$ is an exact doubly dominating set of T'. By the inductive hypothesis we have $S - \{a_1, a_2, a_4, a_5\} = A(T')$, and so $S = A(T)$. So (a) is proved.

It is a routine matter to check item (b). Note that the tree T_1 with two vertices has $|A(T_1)| = 2$ and $|B(T_1)| = 1$; moreover, each operation adds twice as many vertices to $A(T)$ as to $B(T)$, so $|A(T)| = 2|B(T)|$ holds for every tree $T \in \mathcal{T}$. It follows from this and from (a) and (b) that $\gamma_{\times 2}(T) \leq |A(T)| = 2|B(T)| \leq 2\gamma(T)$, and we have equality throughout by Corollary 9. This proves part (c) and concludes the proof of the lemma. ■

We now are ready to give a constructive characterization of trees with an exact doubly dominating sets.

Theorem 16. *Let T be a tree. Then T has an exact doubly dominating set if and only if $T \in \mathcal{T}$.*

Proof. First suppose that $T \in \mathcal{T}$. Then Lemma 15 implies that T has an exact doubly dominating set. Conversely, assume that T is a tree that has an exact doubly dominating set S, and let n be the order of T. Clearly, $n \geq 2$. If $n = 2$, then T is in $\mathcal{T}$. Observation 14 implies that $n \in \{3, 4\}$ is impossible and that $n = 5$ implies that T is a path on 5 vertices, which is in $\mathcal{T}$ since it can be obtained from T_1 by the Type-1 operation.

Now assume that $n \geq 6$ and that every tree T' of order n' with $2 \leq n' < n$ such that T' has an exact doubly dominating set is in $\mathcal{T}$. Root T at a vertex r. Let u be a leaf at maximum distance from r, let v be the parent of u in the rooted tree, and let w be the parent of v. By Observation 14, u is the unique child of v, $\{u, v\} \subseteq S$, $w \notin S$, and w is neither a support vertex nor a leaf. This implies that every child of w is a support vertex. Furthermore w has at most two children, for otherwise w would be dominated at least 3 times by S, a contradiction. So $w \neq r$. Let z be the parent of w in the rooted tree.

Suppose that w has exactly one child in the rooted tree. Let $T' = T - \{u, v, w\}$. Since $\{u, v\} \subseteq S$ and $w \notin S$, we have $z \in S$ so that w is dominated twice by S. Moreover, $S - \{u, v\}$ is an exact doubly dominating set of T'. By the inductive hypothesis, we have $T' \in \mathcal{T}$ and, by Lemma 15, $S - \{u, v\} = A(T')$ is the unique exact doubly dominating set of T'. Thus T can be obtained from T' by using Type-1 operation (with the path uvw and since $z \in A(T')$), so $T \in \mathcal{T}$.

Now suppose that w has exactly two children v, v' in the rooted tree. Let T_w be the subtree of T induced by w and its descendants, rooted at w. By Observation 14, each child of w has exactly one child, and we call u' the child of v', so T_w is a path on five vertices $uvwv'u'$ with central vertex w. Moreover, by Observation 14, we have $\{u, v, u', v'\} \subseteq S$, $w \notin S$, and $z \notin S$ since w is dominated twice in S by v, v'. Thus z is doubly dominated by $S \cap V(T')$ and consequently $S \cap V(T')$ is an exact doubly dominating set of T'. By the inductive hypothesis, we have $T' \in \mathcal{T}$ and, by Lemma 15, $S \cap V(T') = A(T')$ is the unique exact doubly dominating set of T'. Thus T can be obtained from T' by using Type-2 operation (with the path $uvwv'u'$ and since $z \notin A(T')$), so $T \in \mathcal{T}$. This completes the proof of the theorem. ■

The proof of the theorem suggests a polynomial-time algorithm which, given a tree T with n vertices, decides whether T is in $\mathcal{T}$ and, if it is, returns the set $A(T)$. Here is an outline of the algorithm. If T is a path on 2 or 5 vertices, answer $T \in \mathcal{T}$, return the obvious set $A(T)$, and stop. Else, if either $n \leq 5$ or T is a star, answer $T \notin \mathcal{T}$ and stop. Now suppose $n \geq 6$. Pick a vertex r, root the tree T at r, and pick a vertex u at maximum distance from r. Let v be the parent of u in the rooted tree and w be the parent of v. If either v has at least two children, or w has at least three children, or w has exactly two children and its second child has either zero or at least two children, then return the answer $T \notin \mathcal{T}$ and stop. Else, let z be the parent of w. If w has exactly one child, call the algorithm recursively on the tree $T' = T - \{u, v, w\}$; if the answer to the recursive call is $T' \in \mathcal{T}$ and $z \in A(T')$, then answer $T \in \mathcal{T}$, return $A(T) = A(T') \cup \{u, v\}$, and stop, else answer $T \notin \mathcal{T}$ and stop. If w has exactly two children v, v', call the algorithm recursively on the tree $T' = T - \{u, v, w, v', u'\}$ (where u' is the child of v'); if the answer to the recursive call is $T' \in \mathcal{T}$ and $z \notin A(T')$, then answer $T \in \mathcal{T}$, return $A(T) = A(T') \cup \{u, v, u', v'\}$ and stop, else answer $T \notin \mathcal{T}$ and stop.

Next, we give a necessary and sufficient condition for the existence of an exact doubly dominating set in a connected cubic graph. Recall that a matching in a graph $G = (V, E)$ is *perfect* if its size is $|V|/2$. With any perfect matching $M = \{e_1, e_2, \ldots, e_{n/2}\}$ of a graph G we associate a simple graph denoted by $G_M = (V_M, E_M)$ where each edge $e_i \in M$ is represented by a vertex in V_M and two vertices of V_M are adjacent if the corresponding edges in M are joined by an edge in G. A graph is an *equitable bipartite* graph if its vertex set can be partitioned into two independent sets S_1 and S_2 such that $|S_1| = |S_2|$, and in this case (S_1, S_2) is called an *equitable bipartition* of G.

Theorem 17. *Let G be a connected cubic graph. Then G has an exact doubly dominating set if and only if G has a perfect matching M such that the associated graph G_M is an equitable bipartite graph.*

Proof. Let G be a connected cubic graph with an exact doubly dominating set S. So S induces a 1-regular graph, whose edges form a matching M_1, and every vertex of S has two neighbours in $V - S$. Since every vertex of $V - S$ has exactly two neighbours in S, the subgraph induced by $V - S$ is 1-regular, and its edges form a matching M_2. Thus G admits a perfect matching $M = M_1 \cup M_2$. Each edge of $E - M$ joins a vertex of S with a vertex of $V - S$, and the bipartite subgraph $(S, V - S; E - M)$ is 2-regular, so $|S| = |V - S|$, and so $|M_1| = |M_2|$. It follows that the graph G_M associated with M is an equitable bipartite graph with equitable bipartition (M_1, M_2).

Conversely, let M be a perfect matching of a connected cubic graph G such that the associated graph G_M is equitable bipartite, with equitable bipartition (A, B). Let A_M (resp. B_M) be the vertices of G that are contained in the edges corresponding to the vertices of A (resp. B). Since A (resp. B)

is independent in G_M, the subgraph of G induced by A_M (resp. by B_M) is 1-regular. This also implies that every vertex of A_M (resp. of B_M) has two neighbours in B_M (resp. in A_M) since G is a cubic graph. Consequently, A_M and B_M are two disjoint exact doubly dominating sets of G. This completes the proof. ■

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