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POTENTIAL FORBIDDEN TRIPLES IMPLYING HAMILTONICITY: FOR SUFFICIENTLY LARGE GRAPHS

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Abstract

In [2], Brousek characterizes all triples of connected graphs, G_1, G_2, G_3, with $G_i = K_{1,3}$ for some $i = 1, 2$, or 3, such that all $G_1G_2G_3$-free graphs contain a hamiltonian cycle. In [8], Faudree, Gould, Jacobson and Lesniak consider the problem of finding triples of graphs G_1, G_2, G_3, none of which is a $K_{1,s}$, $s \geq 3$ such that $G_1G_2G_3$-free graphs of sufficiently large order contain a hamiltonian cycle. In [6], a characterization was given of all triples G_1, G_2, G_3 with none being $K_{1,3}$, such that all $G_1G_2G_3$-free graphs are hamiltonian. This result, together with the triples given by Brousek, completely characterize the forbidden triples G_1, G_2, G_3 such that all $G_1G_2G_3$-free graphs are hamiltonian. In this paper we consider the question of which triples (including $K_{1,s}$, $s \geq 3$) of forbidden subgraphs potentially imply all sufficiently large graphs are hamiltonian. For $s \geq 4$ we characterize these families.

Keywords: hamiltonian, forbidden subgraph, claw-free, induced subgraph.

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1. Introduction

The problem of recognizing graph properties based on forbidden connected subgraphs has received considerable attention. A wide variety of properties and forbidden families have been studied. In particular, the property of being hamiltonian has been widely studied. A series of results culminated in the characterization of the pairs of forbidden subgraphs which imply all graphs free of these pairs of graphs are hamiltonian by Bedrossian [1]. In his proof, Bedrossian used a small order nonhamiltonian graph to eliminate some cases. Faudree and Gould [5] extended the collection to characterize the forbidden pairs which imply all sufficiently large ($n \geq 10$ suffices) graphs are hamiltonian.

Since the only single forbidden subgraph that implies a graph is hamiltonian is P_3 (the path on 3 vertices) and it forces the graph to be complete, the problem of all single or pairs of forbidden subgraphs implying hamiltonicity has been completely characterized, both for all graphs and for all sufficiently large graphs.

An interesting feature of both characterizations for pairs is that the claw, $K_{1,3}$, must be one of the graphs in each pair. This led naturally to the question: If we consider triples of forbidden subgraphs implying hamiltonicity, must the claw always be one of the graphs in the triple? This question was answered negatively in [8]. There, all triples containing no $K_{1,t}$, $t \geq 3$ which imply all sufficiently large graphs are hamiltonian were given. Brousek [2] gave the collection of all triples which include the claw that imply all 2-connected graphs are hamiltonian.

We follow the notation of [4]. In addition, we say a graph H is $G_1G_2G_3$-free if H does not contain G_i, $i = 1, 2, 3$ as an induced subgraph. In [6], a characterization was given of all triples G_1, G_2, G_3 with none being $K_{1,3}$, such that all $G_1G_2G_3$-free graphs are hamiltonian. Thus, the remaining case is, for sufficiently large graphs, to determine the possible triples where $G_1 = K_{1,s}$, with $s \geq 3$.

The purpose of this paper is to study those triples which include $K_{1,s}$, $s \geq 3$ such that all 2-connected graphs of sufficiently large order and free of such triples are hamiltonian. For $s \geq 4$ we characterize these triples. For $s = 3$ we present a list of triples which potentially imply hamiltonicity. The triples containing $K_{1,3}$ will be further studied in [7].

Given a cycle with an implied orientation, we write x^+ and x^- for the successor and predecessor of x on the cycle, respectively. Further, by $[x, y]$

we mean the subpath of C beginning at x and ending at y and following the orientation of C. We also use the notation $H \leq G$ to mean that H is an induced subgraph of G.

For the remainder of this paper we will assume G_1, G_2 and G_3 are connected. We define the graph $C(i,j,k)$ (see Figure 1 for $C(2,2,1)$) to be the graph obtained by identifying the end vertex of paths of lengths i, j and k, respectively. This graph may be thought of as a form of generalized claw as $K_{1,3} = C(1,1,1)$. Define the graphs $Z_i(m)$ and $J_i(m)$ to be the complete graph on m vertices ($m \geq 3$) with a path of length i or i edges joined to a single vertex of the K_m, respectively (see Figure 1 for $Z_1(m)$ and $J_2(m)$). Note that $Z_1 = Z_1(3)$ is the notation common in the literature. The book B_n is obtained by identifying an edge from each of n copies of K_3 (see Figure 1 for B_2).

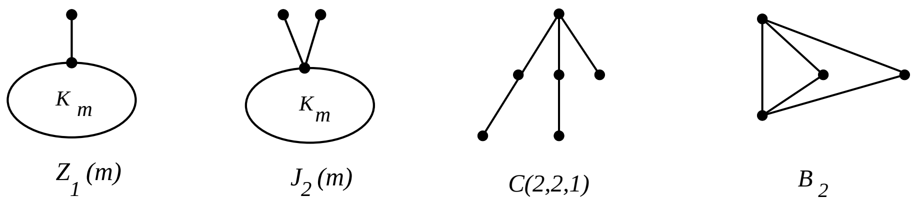

Figure 1. Common forbidden graphs.

Let $C_3 = K_3$ and P_n be a path on n vertices. Let the family $N(i,j,k)$ be obtained by identifying an endvertex of each of P_{i+1}, P_{j+1} and P_{k+1} with distinct vertices of a K_3. We follow the standard that $i \geq j \geq k$. In particular, we denote the net $N = N(1,1,1)$ (see Figure 2), while other special cases have been commonly denoted in the literature as $Z_3 = N(3,0,0)$, $B = N(1,1,0)$ and $W = N(2,1,0)$. We further define the graph family $N(G_1, G_2, G_3)$ to be those graphs obtained by identifying a distinct vertex of K_3 with a distinct vertex of G_1, G_2 and G_3 respectively. If the vertex of G_i to be identified is important, we specify it as in the definition of $N(i,j,k)$. In particular, if $G_i = Z_1(m)$, for some i, then the vertex being identified from $Z_1(m)$ will always be the vertex of degree one. For our purposes, the graphs G_i ($i = 1,2,3$) will always be one of K_n, P_n, or $Z_1(m)$, and hence, there will be no ambiguity in the graph constructed.

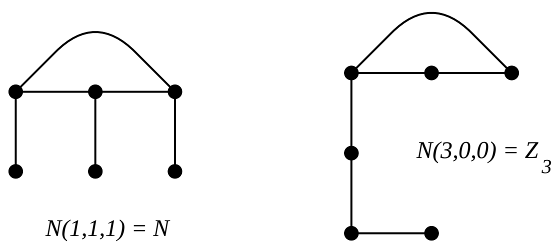

Figure 2. More common forbidden graphs.

We will need the following characterization of forbidden pairs from [5].

Theorem 1.1. *Let R and S be connected graphs $(R, S \neq P_3)$ and G a 2-connected graph of order $n \geq 10$. Then G is (R, S)-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is an induced subgraph of one of $N(1,1,1), N(3,0,0), N(2,1,0)$ or P_6.*

2. Triples Including $K_{1,s}, s \geq 4$

In this section, we characterize those triples G_1, G_2, G_3, one of which is $K_{1,s}$, $(s \geq 4)$ such that $G_1G_2G_3$-free graphs of sufficiently large order are hamiltonian. We begin by showing certain triples containing $K_{1,s}$ do imply hamiltonicity.

Theorem 2.1. *If G is a 2-connected $K_{1,s}P_4J_2(m)$-free graph ($s \geq 4$ and fixed, $m \geq 3$ and fixed) of sufficiently large order n, then G is hamiltonian.*

Proof. Observe first that there must be a vertex of degree at least $\sqrt{n-1}$, for otherwise G would have diameter at least four and an induced P_4 would result.

Using the neighborhood of such a vertex, for n sufficiently large, since G contains no induced $K_{1,s}$, by Ramsey's Theorem, G contains a $K_{l'}$ (where $l' = l'(n) > ms$). Select a largest clique K_l in G. Note that there are no vertices at distance 2 from this clique, for if there were, an induced P_4 is easily found. Thus, every vertex not in K_l is adjacent to vertices in K_l.

Let $S = V(G) - V(K_l)$ and

$$\begin{aligned} S_L &= \{v \in S \mid 1 \leq \deg_{K_l}(v) < l - (m-2)\} \text{ and} \\ S_B &= S - S_L. \end{aligned}$$

Let $x, y \in S_L$ and suppose that x and y are not adjacent. Further, without loss of generality, suppose that $\deg_{K_l}(x) \leq \deg_{K_l}(y)$. If the neighborhood $N_{K_l}(x) \not\subseteq N_{K_l}(y)$, then there exist vertices $w_1 \in N_{K_l}(x) - N_{K_l}(y)$ and $w_2 \in N_{K_l}(y) - N_{K_l}(x)$ such that w_1, x, w_2, y is an induced P_4, a contradiction. But now, x and y must have at least one common neighbor in K_l and a $J_2(m)$ results. Hence, the induced graph on S_L, $\langle S_L \rangle$, must be complete.

Now in $\langle S_B \rangle$ we select a longest path P_1. If P_1 is not all of S_B, we select a longest path in $\langle S_B - V(P_1) \rangle$, and continue this process until all of S_B is covered by these paths. It is easy to see there are at most $s - 1$ such paths, for otherwise, due to the degree condition on S_B, there would be a vertex of K_l common to the neighborhoods of all the final vertices of these paths and $K_{1,s}$ would result.

Now for each path $P_i, i = 1, \ldots, t$ $(t < s)$ created above and for some spanning path of $\langle S_L \rangle$, we match the $2(t+1)$ end vertices of these paths to $2(t+1)$ distinct vertices of K_l. Note that in the special case that $V(\langle S_L \rangle)$ has only one neighbor in K_l, the fact G is 2-connected implies $V(\langle S_L \rangle)$ has a neighbor in S_B. Include that neighbor in S_L and proceed as above. Hence, G is clearly hamiltonian, completing the proof of the Theorem. ∎

Theorem 2.2. *If G is a 2-connected $K_{1,s}P_4B_2$-free graph $(s \geq 4)$ of sufficiently large order n, then G is hamiltonian.*

Proof. From Theorem 3 in [8], G being 2-connected $P_4B_2K_{2,\lceil \frac{n+1}{2} \rceil}$-free implies G is hamiltonian and $K_{1,s} \leq K_{2,\lceil \frac{n+1}{2} \rceil}$, if $s \leq \lceil \frac{n+1}{2} \rceil$, and so the result follows. ∎

Theorem 2.3. *If G is a 2-connected $K_{1,s}P_rZ_1(m)$-free graph (with $r \geq 5$, $s \geq 4, m \geq 3$ fixed) of sufficiently large order n, then G is hamiltonian.*

Proof. As before, G contains a vertex of degree at least $n^{\frac{1}{r}}$ or P_r would be an induced subgraph of G. By Ramsey's Theorem, since $K_{1,s} \not\leq G$, we see G contains $K_{l'}$ for $l' > sm$ and $l' = l'(n)$. Choose a largest clique K_l in G.

Since G is 2-connected, there exists $x \in V(G) - V(K_l)$ with x adjacent to vertices of K_l. Note that x must be nonadjacent to at most $m-2$ vertices of K_l, for otherwise a $Z_1(m)$ results.

If there exists a vertex y at distance 2 from K_l through x, since $l > sm$, then an m-clique including x along with y forms a $Z_1(m)$, again a contradiction. Thus, every vertex of $S = V - V(K_l)$ must have adjacencies in K_l. Further, S_L (defined as before) is empty, hence $S_B = S$.

As before, choose a system of longest paths $P_i, i = 1, \ldots, t$, that covers S. If $t \geq s$, since $l > s(m-2)$ we would find $K_{1,s}$ in G, a contradiction.

Thus, since the end vertices of these $t < s$ paths all have high degree ($\geq l - (m-2)$) to K_l and $l > s(m-2)$, we can match the end vertices of each of these paths to $2t$ distinct vertices of K_l and thus, G is clearly hamiltonian. ■

Note, Theorem 2.3 also holds when $r = 4$, however this triple follows from Theorem 2.1.

Theorem 2.4. *If G is a 2-connected $K_{1,s}C(l,1,1)Z_1$-free (l, s fixed, $l \geq 2$, $s \geq 4$) graph of sufficiently large order n, then G is hamiltonian.*

Proof. Suppose G is not hamiltonian. Then, from our previous result, we know that G contains a long induced path. Choose $P = P_r$ with $r > ls$ to be a longest induced path in G. Since $V(P) \neq V(G)$ and G is 2-connected, there exists a vertex $x \notin V(P)$ adjacent to a vertex on P. Say x is adjacent to v (where v is not an end vertex of P). If x is also adjacent to v^+, then since P is an induced path, we see that Z_1 results unless x is adjacent to the entire path. But if x is adjacent to all of P, since $r > ls$, a $K_{1,s}$ would result.

Now we note that if x has no adjacencies within l vertices of v (on either side), then $C(l,1,1)$ results. Hence, x must have an adjacency within every l vertices of any other adjacency on P. But $r > ls$, so again $K_{1,s} \leq G$. The only remaining possibility is that x must be adjacent to both end vertices of P.

Now suppose y is at distance 2 from P through x. Then we immediately find $C(l,1,1) \leq G$. Hence, all vertices of $V(G) - V(P)$ are at distance one from P and therefore are adjacent to only the end vertices of P.

Suppose x and y are two vertices at distance one from P. If $xy \notin E(G)$, then $C(l,1,1)$ is found using either end vertex, say w, of P along with x, y and an l vertex segment of P following w. Thus, $xy \in E(G)$ and now $\langle x, y, w, w^+ \rangle \cong Z_1$, a contradiction. ■

In order to complete the characterization of triples containing $K_{1,s}$ with $s \geq 4$, we need the families of graphs in Figure 3. For convenience, the graph $H_2 = F_1$ (see Figure 4).

We now show that the triples shown to imply hamiltonicity in Theorems 2.1 – 2.4 form a complete list.

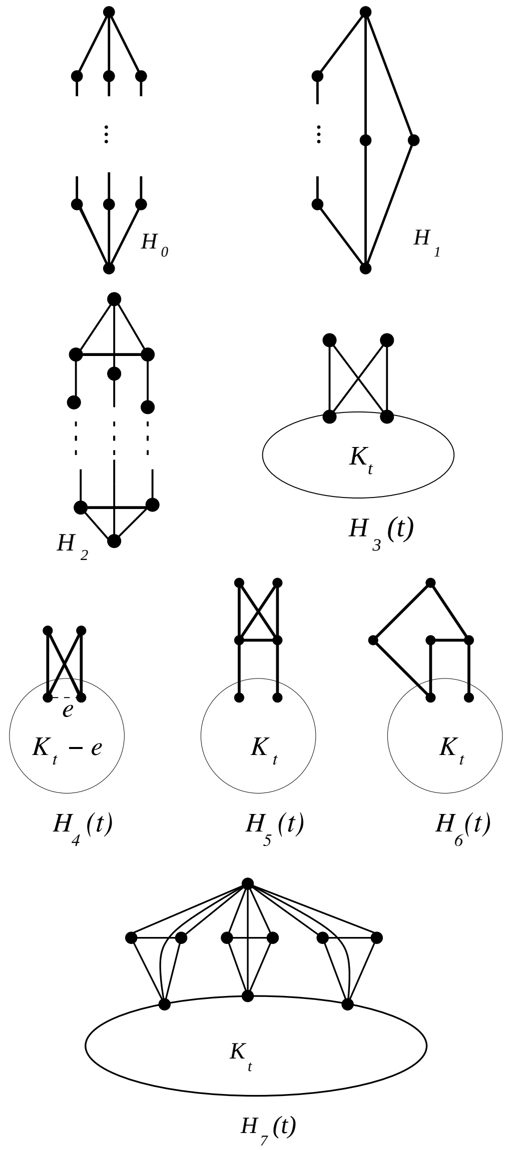

Figure 3. More nonhamiltonian graphs.

Theorem 2.5. *If G is a 2-connected graph of sufficiently large order which is $G_1G_2G_3$-free where $G_1G_2G_3$ are one of the following triples:*

(a) $K_{1,s}, P_4, J_2(m)$; $s \geq 4, m \geq 3$,

(b) $K_{1,s}, P_4, B_2$; $s \geq 4$,

(c) $K_{1,s}, P_r, Z_1(m)$; $r \geq 5, s \geq 4, m \geq 3$,

(d) $K_{1,s}, C(l,1,1), Z_1(3) = Z_1$; $l \geq 2, s \geq 4$

or $G_1G_2G_3$ is a triple of induced subgraphs of one of these triples, then G is hamiltonian. Furthermore, these are the only possible triples that contain $K_{1,s}, s \geq 4$.

Proof. We know each of these triples implies hamiltonicity by Theorems 2.1 – 2.4. Thus, we need only show there are no other possibilities.

Since the graphs H_0–H_7 of Figure 3 are all $K_{1,s}$-free ($s \geq 4$) non-hamiltonian, we may assume without loss of generality $G_2 \leq H_0$. Thus, $P_4 \leq G_2 \leq C(i,j,k)$. Further, since $P_4 \not\leq H_3$ and $P_4 \not\leq H_4$, we see that $G_3 \leq H_3$ and $G_3 \leq H_4$. This implies that $K_r \leq G_3 \leq J_2(m)$, for $r \geq 3$ and some $m \geq 3$, or else $G_3 \leq B_2$.

Since in either case $K_3 \leq G_3$ and $G_3 \not\leq H_1$ then $G_2 \leq H_1$. Hence, as $G_2 \leq H_0$, we see that $G_2 \leq C(l,1,1)$, for some $l \geq 2$. Thus, either G_2 is a path P_k, $k \geq 4$, or $G_2 = C(l,1,1)$, that is $P_k \leq G_2 \leq C(l,1,1)$.

Case 1. Suppose $G_2 = P_r$, $r \geq 6$.

Since $P_6 \not\leq H_4$, $P_6 \not\leq H_5$ and $P_6 \not\leq H_6$, then $G_3 \leq H_4$, $G_3 \leq H_5$ and $G_3 \leq H_6$. But then, $G_3 \leq Z_1(m)$ for some $m \geq 3$. This yields triple (c), when $r \geq 6$.

Case 2. Suppose $G_2 = P_5$.

Note H_5 is $K_{1,s}P_5J_2(m)$-free, where $s \geq 4$. Thus, the triple $K_{1,s}, P_5, J_2(m)$ is excluded from consideration. Next consider H_7, which is $K_{1,4}P_5B_2$-free, excluding this triple from consideration. Now consider H_4, H_5 which are $K_{1,4}$, P_5-free. This implies G_3 is a subgraph of both H_4 and H_5, hence $G_3 \leq Z_1(m)$, $m \geq 3$. This completes case (c).

Case 3. Suppose $G_2 = P_4$.

Since H_3 and H_4 are $K_{1,s}P_4$-free, we see that $G_3 \leq H_3$ and $G_3 \leq H_4$. Thus, $G_3 \leq J_2(m)$ for some $m \geq 3$ or $G_3 \leq B_2$. Hence, we obtain the triples of (a) and (b).

Case 4. Suppose $G_2 = C(l,1,1), l \geq 2$.

Now $G_2 \not\leq H_2$, $G_2 \not\leq H_3$ and $G_3 \not\leq H_4$ thus, $G_3 \leq H_2$, $G_3 \leq H_3$ and $G_3 \leq H_4$. Hence, using H_2, we see that $K_3 \leq G_3$ and thus, $\omega(G_3) = 3$. But then, using H_2 and H_3 or H_4, we see that $G_3 \leq Z_1$, and we obtain family (d). ■

3. Determining Families of Triples Including $K_{1,3}$

In this section the graphs of Figures 4, 5 and 6 represent families of $K_{1,3}$-free nonhamiltonian graphs. Note that $F_1 = H_2$. For $i = 2, 3, 5, 6, 7, 8, 9$ we denote by $F_i(t)$ the graph from the family F_i for fixed t, ($t \geq 3$ for $i = 2, 3$ and $t \geq 1$ for $i = 5, 6, \ldots, 9$ respectively). Note that in $F_i(t)$, $i = 5, \ldots, 9$, the vertices at distance one from the K_t are in fact adjacent to all vertices of the K_t.

Let $\mathcal{A}$ be the collection of triples $G_1G_2G_3$ with $G_1 = K_{1,3}$ so that 2-connected $G_1G_2G_3$-free graphs of sufficiently large order are hamiltonian. We use the families of graphs of Figures 4, 5 and 6 to arrive at a restricted class of triples which contains $\mathcal{A}$. Due to the size of this class, we continue the study of these triples in [7]. Note that the case that no G_i, $i = 1, 2, 3$, is equal to a star was characterized in [8].

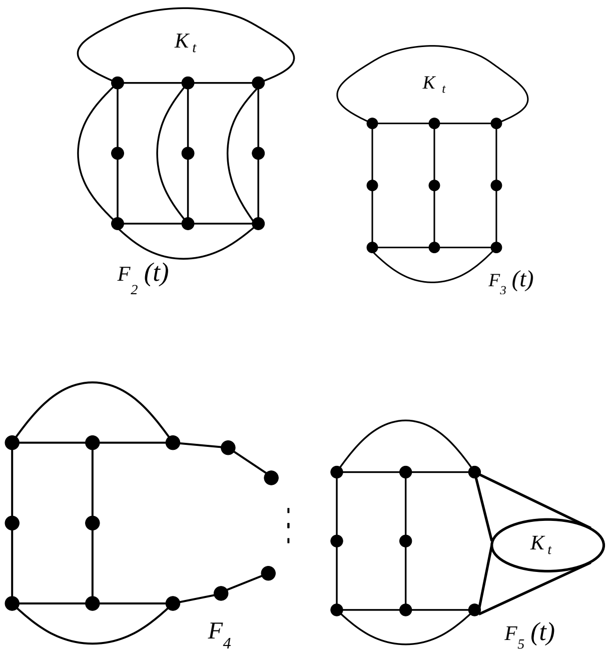

Figure 4. Forbidden families F_1 through F_5.

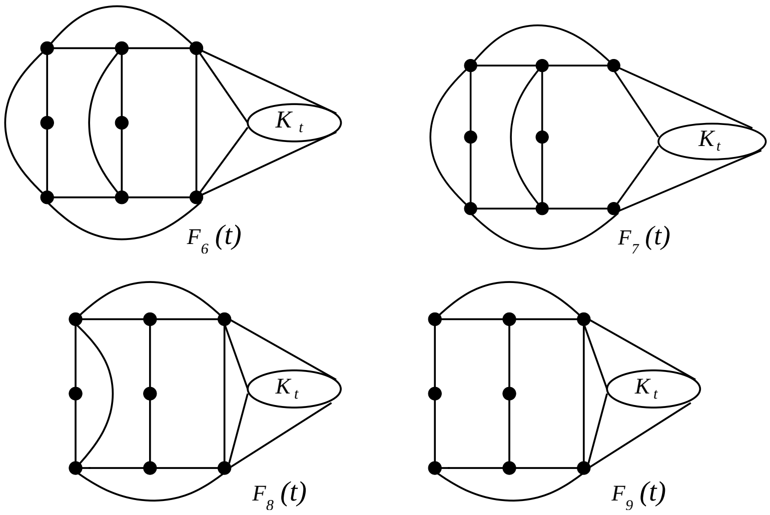

Figure 5. Forbidden families F_6 to F_9.

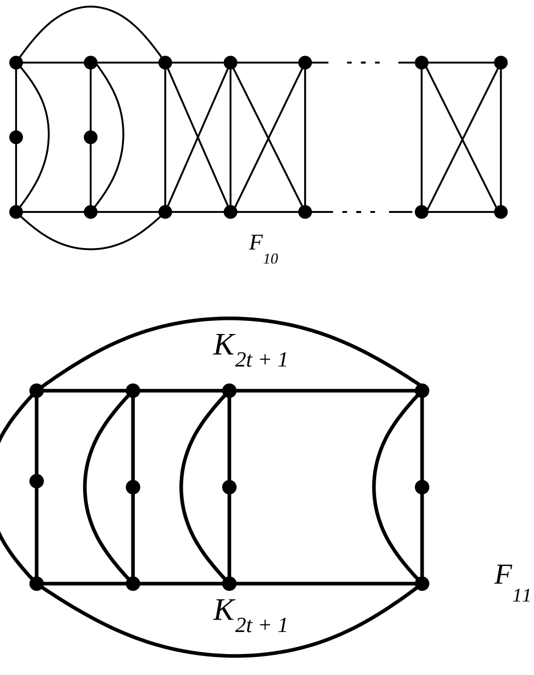

Figure 6. Forbidden families F_{10} and F_{11}.

Without loss of generality, we may assume $G_2 \leq F_1$. This implies $G_2 \leq N(i,j,k), i \geq j \geq k \geq 0$, where possibly $G_2 = P_l, l \geq 4$. If $l \leq 6$, then $K_{1,3}P_l$ implies G is hamiltonian. Now, based on the different structures of G_2, we determine the possibilities for G_3. First we present three Lemmas which will help expedite the cases. Throughout this section we consider only 2-connected $G_1G_2G_3$-free graphs G.

Lemma 3.1. *If G_3 is an induced subgraph of all of the graphs in $\{F_2, F_3, F_6\}$ then either*

(a) *$G_3 \leq G_1$ where G is $K_{1,3}G_1$-free implies G is hamiltonian or*

(b) *the clique number $\omega(G_3) \geq 4$.*

Proof. If $\omega(G_3) \leq 2$, then by the cycle structure of F_2 and F_3, G_3 must be a path. Since there are no induced $K_{1,3}$ and F_2 contains no induced P_7, it follows that $G_3 \leq P_6$. But $K_{1,3}P_6$-free graphs are hamiltonian by Theorem 1.1.

If $\omega(G_3) = 3$, then G_3 contains at most one K_3, since the distance between two distinct K_3 in F_2 is at most one and it is more than one in F_3. Also note that there are no cycles other than K_3 in G_3, since F_2 has only 4-cycles as other induced cycles, while F_3 has only 6-cycles as other induced cycles. Thus, $G_3 \leq N(i,j,k)$ where $i,j,k \geq 0$.

If $i,j,k > 0$, then $G_3 \leq N(2,1,1)$ by F_2 or F_3 and by F_6 it follows that $G_3 \leq N(1,1,1)$, hence we are again done by Theorem 1.1. If $k = 0$ and $i,j > 0$, then by F_3, $j = 1$ and by F_6, $i \leq 2$. Thus, $G_3 \leq N(2,1,0)$ and we are done by Theorem 1.1. If $j = k = 0$ and $i > 0$, then F_2 implies that $i \leq 3$ and so $G_3 \leq N(3,0,0)$ and we are again done by Theorem 1.1. Thus, either $\omega(G_3) \geq 4$ or we have a pair of graphs implying G is hamiltonian. ■

Lemma 3.2. *If G is a 2-connected non-hamiltonian $K_{1,3}G_3$-free graph of sufficiently large order n and G_3 is an induced subgraph of each of the graphs of $\{F_2, F_3, F_5, F_6\}$ or $\{F_2, F_3, F_6, F_7\}$, then $G_3 \leq Z_3(m)$, $m \geq 4$.*

Proof. By Lemma 3.1, $\omega(G_3) \geq 4$. Since G_3 is an induced subgraph of F_5 and F_6 (or F_6 and F_7) containing a K_4, it follows that $G_3 \leq Z_t(m)$, with $m \geq 4$ and $G_3 \leq F_2$ implies that $t \leq 3$. ■

Lemma 3.3. *If G is a 2-connected non-hamiltonian $K_{1,3}G_3$-free graph of sufficiently large order n and G_3 is an induced subgraph of each of the graphs in $\{F_2, F_3, F_5, F_6, F_{10}\}$, then $G_3 \leq Z_2(4)$.*

Proof. By Lemma 3.1, $\omega(G_3) \geq 4$, and since $G_3 \leq F_{10}$, we see that $\omega(G_3) \leq 4$, so $\omega(G_3) = 4$. Lemma 3.2 now implies that $G_3 \leq Z_3(4)$ and by considering F_{10} it follows that $G_3 \leq Z_2(4)$. ■

For Propositions 3.1 – 3.7 of this Section, we assume that $G_2 = N(i,j,k)$ for certain values of $i \geq j \geq k$ and $G_1 = K_{1,3}$.

Proposition 3.1. *If $k \geq 2$, then $K_{1,3}G_3$ implies G is hamiltonian.*

Proof. If G is $K_{1,3}$-free and non-hamiltonian and $k \geq 2$, then we have that $G_2 \geq N(2,2,2)$. Since $F_2 - F_{10}$ are all $K_{1,3}N(2,2,2)$-free, G_3 must be an induced subgraph of each of them. But then F_4 implies $\omega(G_3) \leq 3$. Now by Lemma 3.1 we are done. ■

Thus, we next need to consider the cases where $k = 1$ or $k = 0$.

Proposition 3.2. *Suppose $k = 1$ and $j > 1$. Then,*

(a) *$G_3 \leq Z_2(4)$ when $j \geq 3$ and*

(b) *$G_3 \leq Z_3(m)$, with $m \geq 4$, when $j = 2$.*

Proof. Since each of F_2, F_3, F_5, F_6 are $K_{1,3}N(i,3,1)G_3$-free, if $j \geq 3$, apply Lemma 3.3 and if $j = 2$, apply Lemma 3.2. ■

The graph $H_2(l_1,l_2,l_3)$ ($l_i \geq 3$ for $i = 1,2,3$) is two copies of K_3 with corresponding vertices joined by P_{l_i}'s whose endvertices are identified with the corresponding vertices of the two copies of K_3. Note that this graph is just one particular member of the family $H_2 = F_1$.

Proposition 3.3. *Suppose $k = j = 1$, then*

(a) *If $i \geq 4$, then $G_3 \leq Z_3(m)$, $m \geq 4$.*

(b) *If $i = 3$, then $G_3 \leq Z_3(m)$, $m \geq 4$ or $G_3 \leq N(K_m, K_3, P_1)$, $m \geq 4$ or $G_3 \leq N(K_3, P_2, P_2)$.*

(c) *If $i = 2$, then $G_3 \leq F_6(m)$.*

(d) *If $i = 1$, then $G_2 = N(1,1,1)$ and $K_{1,3}N(1,1,1)$-free implies hamiltonicity.*

Proof. Suppose $i \geq 4$. Since F_2, F_3, F_5 and F_6 are all $K_{1,3}N(4,1,1)$-free, by Lemma 3.1, $\omega(G_3) \geq 4$, and then Lemma 3.2 implies $G_3 \leq Z_3(m)$, $m \geq 4$.

If $i = 3$, we note that F_2, F_5, F_6 and F_7 are all $K_{1,3}N(3,1,1)$-free. Suppose $\omega(G_3) = 3$ and G_3 contains more than one K_3. Then F_2 and F_5 imply G_3 contains only two K_3 and these two K_3 share a vertex. Thus, $G_3 \leq N(K_3, 1, 1)$.

Suppose $w(G_3) \geq 4$. By considering F_6 and F_7 we see that at most one vertex, say w, of the large clique may have adjacencies outside the clique. If w has one adjacency outside the clique, then F_2 and F_5 imply $G_3 \leq Z_3(m),\ m \geq 4$. If w has more than one adjacency outside the clique, then F_2 implies the degree outside the clique is exactly two and those two vertices must be adjacent. The family F_2 implies there can be only one of these two with additional adjacencies. Then F_5 and F_7 imply the extension beyond these two vertices can be at most one edge from one vertex, hence $G_3 \leq N(K_m, 1, 0),\ m \geq 4$.

If $i = 2$, since F_6 is $K_{1,3}N(2,1,1)$-free, we conclude that $G_3 \leq F_6(m)$. If $i = 1$, apply Theorem 1.1. ∎

Proposition 3.4. *Suppose $k = 0$ and $j \geq 3$, then $G_3 \leq Z_2(4)$.*

Proof. If $j \geq 3$, the families of graphs F_2, F_3, F_5, F_6 and F_{10} are all $K_{1,3}N(3,3,0)$-free, so by Lemma 3.1, $\omega(G_3) \geq 4$ and using family F_{10} and Lemma 3.3, it follows that $\omega(G_3) = 4$, and thus, $G_3 \leq Z_2(4)$. ∎

Proposition 3.5. *Suppose $k = 0$ and $j = 2$, then*

(a) *If $i \geq 3$, then $G_3 \leq Z_3(m)$, $m \geq 4$.*

(b) *If $i = 2$, then $G_3 \leq P_7$ or*

$$G_3 = C_6 \text{ if } \omega(G_3) = 2, \text{ or}$$
$$G_3 \leq H_2(3,3,3) \text{ or}$$
$$G_3 \leq N(4,0,0), \text{ if } \omega(G_3) = 3 \text{ or}$$
$$G_3 \leq Z_4(m), \text{ with } m \geq 4 \text{ if } \omega(G_3) \geq 4.$$

Proof. (a) If $j = 2$ and $i \geq 3$, again F_2, F_3, F_5, F_6 and F_7 are $K_{1,3}N(3,2,0)$-free, so by Lemma 3.1, $\omega(G_3) \geq 4$ and by Lemma 3.2, we see that $G_3 \leq Z_3(m), m \geq 4$.

(b) If $j = 2$ and $i = 2$, then only families F_3 and F_5 are $N(2,2,0)$-free. First suppose that $\omega(G_3) = 2$. Then we see that $G_3 \leq P_7$ or $G_3 = C_6$. Suppose $\omega(G_3) = 3$. Now if G_3 contains two K_3, then from F_3 we see they are disjoint and we get that $G_3 \leq H_2(3,3,3)$. If G_3 contains only one K_3, then F_3 implies $G_3 \leq N(4,0,0)$ or $G_3 \leq N(3,1,0)$, or $G_3 \leq N(2,1,1)$. But then note that $N(3,1,0)$ and $N(2,1,1)$ are subgraphs of $H_2(3,3,3)$. Finally, if $\omega(G_3) \geq 4$, then F_3 imply $G_3 \leq Z_4(m)$. ■

Proposition 3.6. *Suppose $k = 0$ and $j = 1$, then*

(a) *If $i \geq 4$, then $G_3 \leq P_6$ if $\omega(G_3) = 2$ or*

$$G_3 \leq Z_3(m) \text{ if } \omega(G_3) \geq 3.$$

(b) *If $i = 3$, then $G_3 \leq P_6$ if $\omega(G_3) = 2$, or*

$$G_3 \leq N(K_m, K_3, P_2) \text{ or}$$

$$G_3 \leq N(Z_1(m), P_3, P_1) \text{ if } \omega(G_3) \geq 3.$$

(c) *If $1 \leq i \leq 2$, then $G_2 \leq N(2,1,0)$, and $K_{1,3}N(2,1,0)$-free implies hamiltonicity.*

Proof.

(a) If $j = 1$ and $i \geq 4$, families F_6, F_7, F_8, F_9 and F_{11} are $K_{1,3}N(4,1,0)$-free and so $G_3 \leq P_6$ if $\omega(G_3) = 2$. If $\omega(G_3) \geq 3$, by examining the largest common subgraphs of F_6, F_7, F_8, F_9, and F_{11}, we see that $G_3 \leq Z_3(m)$.

(b) If $j = 1$ and $i = 3$, families F_6, F_7, F_8 are $K_{1,3}N(3,1,0)$-free and so $G_3 \leq P_6$ if $\omega(G_3) = 2$. By examing the largest common subgraphs of F_6, F_7, F_8, the other graphs are immediate.

(c) If $j = 1$ and $i = 2$, we note that all $K_{1,3}N(2,1,0)$-free graphs are hamiltonian by Theorem 1.1. ■

Proposition 3.7. *Suppose $k = 0$ and $j = 0$, then*

(a) *If $i \geq 5$, then $G_3 \leq Z_3(m)$, $m \geq 4$.*

(b) *If $i = 4$, then $G_3 \leq F_2(3)$.*

(c) *If $0 \leq i \leq 3$, then $G_2 \leq Z_3$ and $K_{1,3}G_2$ is sufficient to imply hamiltonicity.*

Proof.

(a) If $j = 0$ and $i \geq 5$, then F_2, F_3, F_6 and F_7 are all $K_{1,3}N(5,0,0)$-free and so by Lemma 3.2, $G_3 \leq Z_3(m)$, $m \geq 4$.

(b) If $j = 0$ and $i = 4$, $G_3 \leq F_2$, as F_2 and F_{11} are the only $K_{1,3}N(4,0,0)$-free families.

(c) If $j = 0$ and $i = 3$, then all $K_{1,3}N(3,0,0)$-free graphs of order $n \geq 10$ are hamiltonian by Theorem 1.1.

All other cases for i lead directly to G_2 being one of the graphs of Theorem 1.1 and hence, no new triples result. ■

We next consider the situation when $G_2 = P_l$, for $l \geq 7$.

Theorem 3.1. *Suppose $G_2 = P_l$, $l \geq 7$.*

(a) *If $l = 7$, then $G_3 \leq F_2(3)$ or*

$$G_3 \leq N(K_m, K_3, P_1) \text{ or}$$

$$G_3 \leq N(Z_1, 1, 0).$$

(b) *If $l \geq 8$, then $G_3 \leq Z_3(m)$, where $m \geq 4$.*

Proof. If $l = 7$, an argument similar to earlier ones involving the number of copies of K_3 in G_3 produces the result. If $l \geq 8$, then since F_2, F_3, F_5, F_6 and F_7 must contain G_3, applying Lemma 3.2 we obtain the result. ■

We end this section by summarizing the potential triples determined in this section.

In 2-connected Claw, $N(i,j,k), G_3$-Free with $i \geq j \geq k \geq 1$	
i, j, k	Possible Maximal Third Graph(s) G_3
$k \geq 2$	No new triples
$k = 1, j \geq 3$	$Z_2(4)$
$k = 1, j = 2$	$Z_3(m),\ m \geq 4$
$k = j = 1\ i \geq 4$	$Z_3(m),\ m \geq 4$
$k = j = 1\ i = 3$	$Z_3(m),\ \ N(K_m, P_2, P_1),\ m \geq 4,\ \ N(K_3, P_2, P_2)$
$k = j = 1,\ i = 2$	$F_2(m),\ m \geq 4$
$k = j = 1,\ i = 1$	No new triples

In 2-connected Claw, $N(i,j,0), G_3$-Free with $i \geq j$	
$i, j, 0$	Possible Maximal Third Graph(s) G_3
$j \geq 3$	$Z_2(4)$
$j = 2, i \geq 3$	$Z_3(m),\ m \geq 4$
$j = 2,\ i = 2$	if $\omega(G_3) = 2$: P_7, C_6
$j = 2,\ i = 2$	if $\omega(G_3) = 3$: $H_2(3,3,3)$, $N(4,0,0)$
$j = 2,\ i = 2$	if $\omega(G_3) \geq 4$: $Z_4(m)$, $m \geq 4$
$j = 1,\ i \geq 4$	if $\omega(G_3) = 2$: No new triples
$j = 1,\ i \geq 4$	if $\omega(G_3) \geq 3$: $N(Z_1(m), P_2, P_1)$
$j = 1,\ i = 3$	if $\omega(G_3) = 2$: no new triples
$j = 1,\ i = 3$	if $\omega(G_3) \geq 3$: $N(K_m, K_3, P_2)$, $N(Z_1(m), P_3, P_1)$
$j = 1,\ 1 \leq i \leq 2$	No new triples
$j = 0,\ i \geq 5$	$Z_3(m),\ m \geq 4$
$j = 0,\ i = 4$	$F_2(3)$
$j = 0,\ 0 \leq i \leq 3$	No new triples

In 2-connected Claw, P_t, G_3-Free	
t	Possible Third Graph(s) G_3
$t \geq 8$	$Z_3(m),\ m \geq 4$
$t = 7$	$F_2(3)$, $N(K_m, K_3, P_1)$, $N(Z_1, P_2, P_1)$
$t \leq 6$	No new triples

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