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## ON GRAPHS G FOR WHICH BOTH G AND $\overline{G}$ ARE CLAW-FREE

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#### Abstract

Let G be a graph with  $|V(G)| \ge 10$ . We prove that if both G and  $\overline{G}$  are claw-free, then  $\min\{\Delta(G), \Delta(\overline{G})\} \le 2$ . As a generalization of this result in the case where |V(G)| is sufficiently large, we also prove that if both G and  $\overline{G}$  are  $K_{1,t}$ -free, then  $\min\{\Delta(G), \Delta(\overline{G})\} \le r(t-1,t)-1$  where r(t-1,t) is the Ramsey number.

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## 1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by V(G), E(G) and  $\Delta(G)$  the vertex set, the edge set and the maximum degree of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by  $N_G(x)$ , and  $d_G(x) := |N_G(x)|$ .

For a subset S of V(G), the subgraph in G induced by S is denoted by  $\langle S \rangle_G$ . For a subgraph H of G,  $G - H = \langle V(G) - V(H) \rangle_G$ . For disjoint subsets S and T of V(G), we let  $E_G(S,T)$  denote the set of edges of G joining a vertex in S and a vertex in T. When S or T consists of a single vertex, say  $S = \{x\}$  or  $T = \{y\}$ , we write  $E_G(x,T)$  or  $E_G(S,y)$  for  $E_G(S,T)$ . Let  $\overline{G}$  stand for the complement of G. For positive integers s, t, let r(s,t) be the Ramsey number, i.e., the smallest value of n for which every red-blue coloring of  $K_n$  yields a red  $K_s$  or a blue  $K_t$ . A graph G is said to be  $K_{1,t}$ -free if G contains no  $K_{1,t}$  as an induced subgraph. In particular, a graph G is said to be *claw-free* if G contains no  $K_{1,3}$  as an induced subgraph.

In this paper, we are concerned with a structure of graphs G for which both G and  $\overline{G}$  are  $K_{1,t}$ -free where  $t \geq 3$ . Our results are following.

**Theorem A.** Let G be a graph with  $|V(G)| \ge 10$ . If both G and  $\overline{G}$  are claw-free, then  $\min\{\Delta(G), \Delta(\overline{G})\} \le 2$ .

**Theorem B.** Let t be an integer with  $t \ge 4$ , and let G be a graph with  $|V(G)| \ge r(t^2 - t + 2, t^2 - t + 2)$ . If both G and  $\overline{G}$  are  $K_{1,t}$ -free, then  $\min\{\Delta(G), \Delta(\overline{G})\} \le r(t-1,t)-1$ .

In Theorem A, the bound on |V(G)| is best possible. To see this, we construct a graph G of order 9 such that  $\min\{\Delta(G), \Delta(\overline{G})\} > 2$ . Let  $C = v_1 v_2 \dots v_8 v_1$  be a cycle of length 8. Let v be a new vertex. Consider the graph G = (V(G), E(G)) such that  $V(G) = V(C) \cup \{v\}$  and  $E(G) = E(C) \cup \{v_i v_j \mid 1 \le i < j \le 8, i+j \equiv 0 \pmod{4}\} \cup \{v v_{2l} \mid 1 \le l \le 4\}$ . Then |V(G)| = 9,  $\min\{\Delta(G), \Delta(\overline{G})\} > 2$ , and both G and  $\overline{G}$  are claw-free (and isomorphic). Note that the converse of Theorem A is not true. To see this, consider  $G = K_3 \cup \overline{K_m}$  where m is a large positive integer. Then  $\min\{\Delta(G), \Delta(\overline{G})\} = \Delta(G) = 2$ . However, it is obvious that  $\overline{G}$  contains  $K_{1,3}$  as an induced subgraph. Avoiding this particular case, we obtain the following corollary.

**Corollary of Theorem A.** Let G be a graph with  $|V(G)| \ge 10$ . Then the following statements are equivalent:

- (i) both G and  $\overline{G}$  are claw-free,
- (ii) either G or  $\overline{G}$  is a triangle-free graph of maximum degree at most 2.

Alternatively, the statement (ii) can be formulated as follows. (ii) either G or  $\overline{G}$  is a disjoint union of cycles of length  $l \ge 4$ , paths and isolated vertices.

### Sketch of proof.

(i) $\Rightarrow$ (ii). Theorem A implies that either G or  $\overline{G}$  (say, G) has maximum degree at most 2. Then it is easy to see that G is also triangle-free: if  $\{x, y, z\} \subset V(G)$  induces a triangle in G, then this triangle is a component

of G since  $\Delta(G) \leq 2$ . Then for any vertex  $u \in V(G) \setminus \{x, y, z\}$  (which exists since  $|V(G)| \geq 10$ ), the set  $\{u, x, y, z\}$  induces a claw in  $\overline{G}$ , centerd at u. (ii) $\Leftarrow$ (i). Suppose that e.g. G is triangle-free with  $\Delta(G) \leq 2$ . Then G is claw-free since  $\Delta(G) \leq 2$  and  $\overline{G}$  is claw-free since G is triangle-free. For  $\overline{G}$  the proof is similar.

Theorem B is a similar result concerning graphs G for which both G and  $\overline{G}$  are  $K_{1,t}$ -free where  $t \geq 4$ . Now we show that there exists a graph G such that both G and  $\overline{G}$  are  $K_{1,t}$ -free and  $\min\{\Delta(G), \Delta(\overline{G})\} = r(t-1,t) - 1$ . Let R be a graph with |V(R)| = r(t-1,t) - 1 such that R does not contain  $K_{t-1}$  or  $\overline{K_t}$  as an induced subgraph. Let v be a new vertex. Consider  $G = (R+v) \cup \overline{K_{|V(G)|-r(t-1,t)}}$  where |V(G)| is sufficiently large. Then G is a graph such that both G and  $\overline{G}$  are  $K_{1,t}$ -free and  $\min\{\Delta(G), \Delta(\overline{G})\} = r(t-1,t) - 1$ .

## 2. Proof of Theorem A

By contradiction, suppose that  $\Delta(G) \geq 3$  and  $\Delta(\overline{G}) \geq 3$ . Then by the assumption that both G and  $\overline{G}$  are claw-free, G contains a subgraph A such that  $A \cong K_3$  in  $\overline{G}$ , and  $\overline{G}$  contains a subgraph B such that  $B \cong K_3$  in  $\overline{G}$ .

**Claim.** Both G and  $\overline{G}$  do not contain a subgraph which is isomorphic to  $K_4$ .

**Proof.** Suppose not. Then by symmetry, we may assume  $\overline{G} \supset K_4$ . Then G contains a subgraph S such that  $\langle V(S) \rangle_G \cong \overline{K_4}$ . Let  $V(S) = \{a, b, c, d\}$ . First suppose that G - S contains a subgraph T which is isomorphic to  $K_3$ . Let  $V(T) = \{e, f, g\}$ . Since  $\overline{G}$  is claw-free,  $E_G(x, V(T)) \neq \emptyset$  for every  $x \in V(S)$ . From |V(S)| = 4, there exists  $y \in V(T)$  such that  $|E_G(y, V(S))| \ge 2$ . By symmetry, we may assume  $ae, be \in E(G)$ . Since G is claw-free and  $\langle V(S) \rangle_G \cong \overline{K_4}$ , it follows that  $E_G(e, \{c, d\}) = \emptyset$ . By the assumption that G is claw-free,  $\langle \{e, a, b, f\} \rangle_G$  is not isomorphic to  $K_{1,3}$ . Hence by symmetry of the roles of a and b, we may assume  $af \in E(G)$ . Note that  $\langle \{a, e, f\} \rangle_G \cong K_3$ . Then by the assumption that  $\overline{G}$  is claw-free,  $E_G(c, \{a, e, f\}) \neq \emptyset$  and  $E_G(d, \{a, e, f\}) \neq \emptyset$ . This forces  $cf, df \in E(G)$ . Then  $\langle \{f, a, c, d\} \rangle_G \cong K_{1,3}$ . This is a contradiction. Hence it follows that G - S does not contain a triangle. Since G contains a triangle, we may assume that there exist  $u, v \in V(G - S)$  such that  $\langle \{a, u, v\} \rangle_G \cong K_3$ . Since  $\overline{G}$  is claw-free,  $|E_G(\{b, c, d\}, \{u, v\})| \ge 3$ . Then by the symmetry of the roles

of u and v, we may assume  $bu, cu \in E(G)$ . Then  $\langle \{u, a, b, c\} \rangle_G \cong K_{1,3}$ . This is a contradiction.

Case 1.  $V(A) \cap V(B) = \emptyset$ .

First suppose that there exist  $x, y \in V(B)$  such that  $N_G(x) \cap N_G(y) \cap$  $V(A) \neq \emptyset$ . We may assume that  $V(A) = \{a, b, c\}, V(B) = \{x, y, z\}$ , and  $ax, ay \in E(G)$ . Since  $\langle \{a, b, x, y\} \rangle_G \ncong K_{1,3}$ , by the symmetry of the roles of x and y, we may assume  $bx \in E(G)$ . Since  $\langle \{a, b, x\} \rangle_G \cong K_3, \langle \{a, x, y, z\} \rangle_G \ncong$  $K_{1,3}$  and  $\overline{G}$  is claw-free, it follows that  $bz \in E(G)$ . By the claim,  $xc \notin C$ E(G). Hence it follows from  $\langle \{b, c, x, z\} \rangle_G \ncong K_{1,3}$  that  $cz \in E(G)$ . Since  $\langle b, c, z \rangle_G \cong K_3$  and both G and  $\overline{G}$  are claw-free, this forces  $cy \in E(G)$ . Hence by the claim, we have  $E_G(A, B) = \{ax, ay, bx, bz, cy, cz\}$ . Let  $v \in$ V(G - A - B). Since  $\overline{G}$  is claw-free, it follows that  $E_G(v, V(A)) \neq \emptyset$ . By symmetry, we may assume  $va \in E(G)$ . Suppose that  $vc \in E(G)$ . Then it follows from the claim that  $E_G(v, \{b, y\}) = \emptyset$ . Then  $\langle a, b, y, v \rangle_G \cong K_{1,3}$ , a contradiction. Thus  $vc \notin E(G)$ . We can similarly obtain  $vb \notin E(G)$ . Since  $\langle \{a, v, x, c\} \rangle_G \ncong K_{1,3}$  and  $\langle \{a, v, y, b\} \rangle_G \ncong K_{1,3}$ , it follows that  $vx, vy \in$ E(G). Since  $\overline{G}$  is claw-free,  $E_G(z, \{v, a, x\}) \neq \emptyset$ . This forces  $vz \in E(G)$ . Then  $\langle \{v, x, y, z\} \rangle_G \cong K_{1,3}$ , a contradiction. Thus we may assume that there exist no two vertices  $x, y \in V(B)$  such that  $N_G(x) \cap N_G(y) \cap V(A) \neq \emptyset$ . Since  $\overline{G}$  is claw-free, we may assume that  $V(A) = \{a, b, c\}, V(B) = \{x, y, z\}$ , and  $E_G(V(A), V(B)) = \{ax, by, cz\}$ . Take  $v \in V(G - A - B)$ . Since  $\overline{G}$  is clawfree,  $E_G(v, V(A)) \neq \emptyset$ . By symmetry, we may assume  $av \in E(G)$ . By the claim,  $N_G(v) \not\supseteq V(A)$ . By symmetry, we may assume  $bv \notin E(G)$ . Since  $\langle \{a, v, x, b\} \rangle_G \ncong K_{1,3}, vx \in E(G)$ . Then  $\langle \{x, v, a\} \rangle_G \cong K_3$ . This forces  $vy, vz \in E(G)$  because  $\overline{G}$  is claw-free. Then  $\langle \{v, x, y, z\} \rangle_G \cong K_{1,3}$ . This is a contradiction.

Case 2.  $V(A) \cap V(B) \neq \emptyset$ .

We may assume that  $V(A) = \{a, b, c\}$  and  $V(B) = \{a, x, y\}$ . Since both G and  $\overline{G}$  are claw-free, it follows that either  $E_G(\{b, c\}, \{x, y\}) = \{bx, cy\}$  or  $E_G(\{b, c\}, \{x, y\}) = \{cx, by\}$ . By symmetry, we may assume  $E_G(\{b, c\}, \{x, y\}) = \{bx, cy\}$ . First suppose that there exist two vertices  $u, v \in V(G - A - B)$  such that either  $|E_G(b, \{u, v\})| = 2$  or  $|E_G(c, \{u, v\})| = 2$ . By symmetry, we may assume  $|E_G(c, \{u, v\})| = 2$ . Suppose that  $av \in E(G)$ . Then  $xv \in E(G)$  because  $\langle \{a, c, v\} \rangle_G \cong K_3$ . Also by the claim,  $vb \notin E(G)$ . Since  $\langle \{v\} \cup V(B) \rangle_G \ncong K_{1,3}, vy \notin E(G)$ . Then  $\langle \{c, v, y, b\} \rangle_G \cong K_{1,3}$ . This is a contradiction. Thus we have  $va \notin E(G)$ . By the symmetry of the roles of u and

v, we can similarly have  $ua \notin E(G)$ . Then since  $\langle \{c, a, u, v\} \rangle_G \ncong K_{1,3}$ , it follows that  $uv \in E(G)$ . Since  $\langle \{c, a, v, y\} \rangle_G \not\cong K_{1,3}$  and  $\langle \{c, a, u, y\} \rangle_G \not\cong K_{1,3}$ , it follows that  $uy, vy \in E(G)$ . Then  $\langle \{c, u, v, y \rangle_G \cong K_4$ , which contradicts the claim. Thus we may assume that there exist no two vertices  $u, v \in V(G - A - B)$  such that  $|E_G(b, \{u, v\})| = 2$  or  $|E_G(c, \{u, v\})| = 2$ 2. Then  $|E_G(a, V(G - A - B))| \geq 3$  because  $|V(G)| \geq 10$  (note that  $E_G(z, V(A)) \neq \emptyset$  for every  $z \in V(G - A - B)$  since  $\overline{G}$  is claw-free). Let  $u, v, w \in N_G(a) \cap V(G - A - B)$ . Since  $\langle \{a, u, v, w\} \rangle_G \ncong K_{1,3}$ , we may assume  $uv \in E(G)$ . Since  $\langle \{a, u, v\} \rangle_G \cong K_3$  and  $\langle \{a, x, y\} \rangle_G \cong \overline{K_3}$ , either  $E_G(\{x, y\}, \{u, v\}) = \{vy, ux\}$  or  $E_G(\{x, y\}, \{u, v\}) = \{vx, uy\}$ . By symmetry, we may assume  $E_G(\{x, y\}, \{u, v\}) = \{vy, ux\}$ . If  $\langle \{c, a, v\} \rangle_G \cong K_3$ , then  $xv \in E(G)$ , which implies  $\langle \{v, a, x, y\} \rangle_G \cong K_{1,3}$ , a contradiction. Hence  $vc \notin E(G)$ . We can similarly have  $ub \notin E(G)$ . Suppose that  $E_G(w, \{u, v\}) = \emptyset$ . Then since  $\langle \{a, b, u, w\} \rangle_G \ncong K_{1,3}$  and  $\langle \{a, c, v, w\} \rangle_G \ncong$  $K_{1,3}$ , this forces  $wb, wc \in E(G)$ , which contradicts the claim. Thus we may assume  $E_G(w, \{u, v\}) \neq \emptyset$ . By the claim, note that  $|E_G(w, \{u, v\})| \leq 1$ . By symmetry, we may assume  $E_G(w, \{u, v\}) = \{wv\}$ . Then  $xw \in E(G)$  because  $\langle \{a, w, v\} \rangle_G \cong K_3$ . Since  $\langle \{a, b, u, w\} \rangle_G \ncong K_{1,3}$ , this forces  $bw \in E(G)$ . Then we have  $yw \in E(G)$  because  $\langle \{a, b, w\} \rangle_G \cong K_3$ . Then  $\langle \{w, a, x, y\} \rangle_G \cong$  $K_{1,3}$ . This is a contradiction. This completes the proof of Theorem A.

## 3. Proof of Theorem B

By contradiction, suppose that  $\Delta(G) \geq r(t-1,t)$  and  $\Delta(\overline{G}) \geq r(t-1,t)$ . Then both G and  $\overline{G}$  contain  $K_t$  because both G and  $\overline{G}$  are  $K_{1,t}$ -free. Let A be a subgraph of G such that  $A \cong K_t$ . Since  $|V(G)| \geq r(t^2 - t + 2, t^2 - t + 2)$ , by symmetry, we may assume that G contains  $\overline{K_{t^2-t+2}}$  as an induced subgraph. Hence there exists a subgraph H of G such that  $\langle V(H) \rangle_G \cong \overline{K_{t^2-t+1}}$  and  $V(A) \cap V(H) = \emptyset$ . Since  $\overline{G}$  is  $K_{1,t}$ -free,  $E_G(x, V(A)) \neq \emptyset$  for every  $x \in$ V(H). Since |V(H)| = t(t-1) + 1, there exists a  $v \in V(A)$  such that  $|E_G(v, V(H))| \geq t$ . This implies that  $\langle \{v\} \cup V(H) \rangle_G$  contains  $K_{1,t}$  as a induced subgraph. This is a contradiction. This completes the proof of Theorem B.

**Remark.** Theorem A implies as an immediate corollary that the graph property "both G and  $\overline{G}$  are claw-free" is stable under the closure for claw-free graphs, i.e., if G has the property, then cl(G) has the property as well (see e.g. the survey paper [1]).

# References

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