Discussiones Mathematicae
Graph Theory 25 (2005) 267–272

ON GRAPHS G FOR WHICH BOTH G AND $\overline{G}$ ARE CLAW-FREE

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Abstract

Let G be a graph with $|V(G)| \geq 10$. We prove that if both G and $\overline{G}$ are claw-free, then $\min\{\Delta(G), \Delta(\overline{G})\} \leq 2$. As a generalization of this result in the case where $|V(G)|$ is sufficiently large, we also prove that if both G and $\overline{G}$ are $K_{1,t}$-free, then $\min\{\Delta(G), \Delta(\overline{G})\} \leq r(t-1,t)-1$ where $r(t-1,t)$ is the Ramsey number.

Keywords: claw-free, complement, maximum degree.

2000 Mathematics Subject Classification: 05C75, 05C99.

1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\Delta(G)$ the vertex set, the edge set and the maximum degree of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and $d_G(x) := |N_G(x)|$.

For a subset S of $V(G)$, the subgraph in G induced by S is denoted by $\langle S\rangle_G$. For a subgraph H of G, $G - H = \langle V(G) - V(H)\rangle_G$. For disjoint subsets S and T of $V(G)$, we let $E_G(S,T)$ denote the set of edges of G joining a vertex in S and a vertex in T. When S or T consists of a single vertex, say $S = \{x\}$ or $T = \{y\}$, we write $E_G(x,T)$ or $E_G(S,y)$ for $E_G(S,T)$. Let $\overline{G}$ stand for the complement of G. For positive integers s, t, let $r(s,t)$

be the Ramsey number, i.e., the smallest value of n for which every red-blue coloring of K_n yields a red K_s or a blue K_t. A graph G is said to be *$K_{1,t}$-free* if G contains no $K_{1,t}$ as an induced subgraph. In particular, a graph G is said to be *claw-free* if G contains no $K_{1,3}$ as an induced subgraph.

In this paper, we are concerned with a structure of graphs G for which both G and $\overline{G}$ are $K_{1,t}$-free where $t \geq 3$. Our results are following.

Theorem A. *Let G be a graph with $|V(G)| \geq 10$. If both G and $\overline{G}$ are claw-free, then* $\min\{\Delta(G), \Delta(\overline{G})\} \leq 2$.

Theorem B. *Let t be an integer with $t \geq 4$, and let G be a graph with $|V(G)| \geq r(t^2 - t + 2, t^2 - t + 2)$. If both G and $\overline{G}$ are $K_{1,t}$-free, then* $\min\{\Delta(G), \Delta(\overline{G})\} \leq r(t-1, t) - 1$.

In Theorem A, the bound on $|V(G)|$ is best possible. To see this, we construct a graph G of order 9 such that $\min\{\Delta(G), \Delta(\overline{G})\} > 2$. Let $C = v_1 v_2 \dots v_8 v_1$ be a cycle of length 8. Let v be a new vertex. Consider the graph $G = (V(G), E(G))$ such that $V(G) = V(C) \cup \{v\}$ and $E(G) = E(C) \cup \{v_i v_j \mid 1 \leq i < j \leq 8,\ i+j \equiv 0 \pmod 4\} \cup \{v v_{2l} \mid 1 \leq l \leq 4\}$. Then $|V(G)| = 9$, $\min\{\Delta(G), \Delta(\overline{G})\} > 2$, and both G and $\overline{G}$ are claw-free (and isomorphic). Note that the converse of Theorem A is not true. To see this, consider $G = K_3 \cup \overline{K_m}$ where m is a large positive integer. Then $\min\{\Delta(G), \Delta(\overline{G})\} = \Delta(G) = 2$. However, it is obvious that $\overline{G}$ contains $K_{1,3}$ as an induced subgraph. Avoiding this particular case, we obtain the following corollary.

Corollary of Theorem A. *Let G be a graph with $|V(G)| \geq 10$. Then the following statements are equivalent:*

(i) *both G and $\overline{G}$ are claw-free,*

(ii) *either G or $\overline{G}$ is a triangle-free graph of maximum degree at most* 2.

Alternatively, the statement (ii) can be formulated as follows.
(ii) *either G or $\overline{G}$ is a disjoint union of cycles of length $l \geq 4$, paths and isolated vertices.*

Sketch of proof.
(i)$\Rightarrow$(ii). Theorem A implies that either G or $\overline{G}$ (say, G) has maximum degree at most 2. Then it is easy to see that G is also triangle-free: if $\{x, y, z\} \subset V(G)$ induces a triangle in G, then this triangle is a component

of G since $\Delta(G) \leq 2$. Then for any vertex $u \in V(G)\backslash\{x, y, z\}$ (which exists since $|V(G)| \geq 10$), the set $\{u, x, y, z\}$ induces a claw in $\overline{G}$, centerd at u.
(ii)$\Leftarrow$(i). Suppose that e.g. G is triangle-free with $\Delta(G) \leq 2$. Then G is claw-free since $\Delta(G) \leq 2$ and $\overline{G}$ is claw-free since G is triangle-free. For $\overline{G}$ the proof is similar. ■

Theorem B is a similar result concerning graphs G for which both G and $\overline{G}$ are $K_{1,t}$-free where $t \geq 4$. Now we show that there exists a graph G such that both G and $\overline{G}$ are $K_{1,t}$-free and $\min\{\Delta(G), \Delta(\overline{G})\} = r(t-1, t) - 1$. Let R be a graph with $|V(R)| = r(t-1, t) - 1$ such that R does not contain K_{t-1} or $\overline{K_t}$ as an induced subgraph. Let v be a new vertex. Consider $G = (R + v) \cup \overline{K_{|V(G)|-r(t-1,t)}}$ where $|V(G)|$ is sufficiently large. Then G is a graph such that both G and $\overline{G}$ are $K_{1,t}$-free and $\min\{\Delta(G), \Delta(\overline{G})\} = r(t-1, t) - 1$.

2. Proof of Theorem A

By contradiction, suppose that $\Delta(G) \geq 3$ and $\Delta(\overline{G}) \geq 3$. Then by the assumption that both G and $\overline{G}$ are claw-free, G contains a subgraph A such that $A \cong K_3$ in G, and $\overline{G}$ contains a subgraph B such that $B \cong K_3$ in $\overline{G}$.

Claim. *Both G and $\overline{G}$ do not contain a subgraph which is isomorphic to K_4.*

Proof. Suppose not. Then by symmetry, we may assume $\overline{G} \supset K_4$. Then G contains a subgraph S such that $\langle V(S)\rangle_G \cong \overline{K_4}$. Let $V(S) = \{a, b, c, d\}$. First suppose that $G - S$ contains a subgraph T which is isomorphic to K_3. Let $V(T) = \{e, f, g\}$. Since $\overline{G}$ is claw-free, $E_G(x, V(T)) \neq \emptyset$ for every $x \in V(S)$. From $|V(S)| = 4$, there exists $y \in V(T)$ such that $|E_G(y, V(S))| \geq 2$. By symmetry, we may assume $ae, be \in E(G)$. Since G is claw-free and $\langle V(S)\rangle_G \cong \overline{K_4}$, it follows that $E_G(e, \{c, d\}) = \emptyset$. By the assumption that G is claw-free, $\langle\{e, a, b, f\}\rangle_G$ is not isomorphic to $K_{1,3}$. Hence by symmetry of the roles of a and b, we may assume $af \in E(G)$. Note that $\langle\{a, e, f\}\rangle_G \cong K_3$. Then by the assumption that $\overline{G}$ is claw-free, $E_G(c, \{a, e, f\}) \neq \emptyset$ and $E_G(d, \{a, e, f\}) \neq \emptyset$. This forces $cf, df \in E(G)$. Then $\langle\{f, a, c, d\}\rangle_G \cong K_{1,3}$. This is a contradiction. Hence it follows that $G - S$ does not contain a triangle. Since G contains a triangle, we may assume that there exist $u, v \in V(G - S)$ such that $\langle\{a, u, v\}\rangle_G \cong K_3$. Since $\overline{G}$ is claw-free, $|E_G(\{b, c, d\}, \{u, v\})| \geq 3$. Then by the symmetry of the roles of u and v, we may assume $bu, cu \in E(G)$. Then $\langle\{u, a, b, c\}\rangle_G \cong K_{1,3}$. This is a contradiction. ■

Case 1. $V(A) \cap V(B) = \emptyset$.

First suppose that there exist $x, y \in V(B)$ such that $N_G(x) \cap N_G(y) \cap V(A) \neq \emptyset$. We may assume that $V(A) = \{a, b, c\}$, $V(B) = \{x, y, z\}$, and $ax, ay \in E(G)$. Since $\langle\{a, b, x, y\}\rangle_G \ncong K_{1,3}$, by the symmetry of the roles of x and y, we may assume $bx \in E(G)$. Since $\langle\{a, b, x\}\rangle_G \cong K_3$, $\langle\{a, x, y, z\}\rangle_G \ncong K_{1,3}$ and $\overline{G}$ is claw-free, it follows that $bz \in E(G)$. By the claim, $xc \notin E(G)$. Hence it follows from $\langle\{b, c, x, z\}\rangle_G \ncong K_{1,3}$ that $cz \in E(G)$. Since $\langle b, c, z\rangle_G \cong K_3$ and both G and $\overline{G}$ are claw-free, this forces $cy \in E(G)$. Hence by the claim, we have $E_G(A, B) = \{ax, ay, bx, bz, cy, cz\}$. Let $v \in V(G - A - B)$. Since $\overline{G}$ is claw-free, it follows that $E_G(v, V(A)) \neq \emptyset$. By symmetry, we may assume $va \in E(G)$. Suppose that $vc \in E(G)$. Then it follows from the claim that $E_G(v, \{b, y\}) = \emptyset$. Then $\langle a, b, y, v\rangle_G \cong K_{1,3}$, a contradiction. Thus $vc \notin E(G)$. We can similarly obtain $vb \notin E(G)$. Since $\langle\{a, v, x, c\}\rangle_G \ncong K_{1,3}$ and $\langle\{a, v, y, b\}\rangle_G \ncong K_{1,3}$, it follows that $vx, vy \in E(G)$. Since $\overline{G}$ is claw-free, $E_G(z, \{v, a, x\}) \neq \emptyset$. This forces $vz \in E(G)$. Then $\langle\{v, x, y, z\}\rangle_G \cong K_{1,3}$, a contradiction. Thus we may assume that there exist no two vertices $x, y \in V(B)$ such that $N_G(x) \cap N_G(y) \cap V(A) \neq \emptyset$. Since $\overline{G}$ is claw-free, we may assume that $V(A) = \{a, b, c\}$, $V(B) = \{x, y, z\}$, and $E_G(V(A), V(B)) = \{ax, by, cz\}$. Take $v \in V(G - A - B)$. Since $\overline{G}$ is claw-free, $E_G(v, V(A)) \neq \emptyset$. By symmetry, we may assume $av \in E(G)$. By the claim, $N_G(v) \not\supseteq V(A)$. By symmetry, we may assume $bv \notin E(G)$. Since $\langle\{a, v, x, b\}\rangle_G \ncong K_{1,3}$, $vx \in E(G)$. Then $\langle\{x, v, a\}\rangle_G \cong K_3$. This forces $vy, vz \in E(G)$ because $\overline{G}$ is claw-free. Then $\langle\{v, x, y, z\}\rangle_G \cong K_{1,3}$. This is a contradiction.

Case 2. $V(A) \cap V(B) \neq \emptyset$.

We may assume that $V(A) = \{a, b, c\}$ and $V(B) = \{a, x, y\}$. Since both G and $\overline{G}$ are claw-free, it follows that either $E_G(\{b, c\}, \{x, y\}) = \{bx, cy\}$ or $E_G(\{b, c\}, \{x, y\}) = \{cx, by\}$. By symmetry, we may assume $E_G(\{b, c\}, \{x, y\}) = \{bx, cy\}$. First suppose that there exist two vertices $u, v \in V(G - A - B)$ such that either $|E_G(b, \{u, v\})| = 2$ or $|E_G(c, \{u, v\})| = 2$. By symmetry, we may assume $|E_G(c, \{u, v\})| = 2$. Suppose that $av \in E(G)$. Then $xv \in E(G)$ because $\langle\{a, c, v\}\rangle_G \cong K_3$. Also by the claim, $vb \notin E(G)$. Since $\langle\{v\} \cup V(B)\rangle_G \ncong K_{1,3}$, $vy \notin E(G)$. Then $\langle\{c, v, y, b\}\rangle_G \cong K_{1,3}$. This is a contradiction. Thus we have $va \notin E(G)$. By the symmetry of the roles of u and

v, we can similarly have $ua \notin E(G)$. Then since $\langle\{c, a, u, v\}\rangle_G \not\cong K_{1,3}$, it follows that $uv \in E(G)$. Since $\langle\{c, a, v, y\}\rangle_G \not\cong K_{1,3}$ and $\langle\{c, a, u, y\}\rangle_G \not\cong K_{1,3}$, it follows that $uy, vy \in E(G)$. Then $\langle\{c, u, v, y\rangle_G \cong K_4$, which contradicts the claim. Thus we may assume that there exist no two vertices $u, v \in V(G - A - B)$ such that $|E_G(b, \{u, v\})| = 2$ or $|E_G(c, \{u, v\})| = 2$. Then $|E_G(a, V(G - A - B))| \geq 3$ because $|V(G)| \geq 10$ (note that $E_G(z, V(A)) \neq \emptyset$ for every $z \in V(G - A - B)$ since $\overline{G}$ is claw-free). Let $u, v, w \in N_G(a) \cap V(G - A - B)$. Since $\langle\{a, u, v, w\}\rangle_G \not\cong K_{1,3}$, we may assume $uv \in E(G)$. Since $\langle\{a, u, v\}\rangle_G \cong K_3$ and $\langle\{a, x, y\}\rangle_G \cong \overline{K_3}$, either $E_G(\{x, y\}, \{u, v\}) = \{vy, ux\}$ or $E_G(\{x, y\}, \{u, v\}) = \{vx, uy\}$. By symmetry, we may assume $E_G(\{x, y\}, \{u, v\}) = \{vy, ux\}$. If $\langle\{c, a, v\}\rangle_G \cong K_3$, then $xv \in E(G)$, which implies $\langle\{v, a, x, y\}\rangle_G \cong K_{1,3}$, a contradiction. Hence $vc \notin E(G)$. We can similarly have $ub \notin E(G)$. Suppose that $E_G(w, \{u, v\}) = \emptyset$. Then since $\langle\{a, b, u, w\}\rangle_G \not\cong K_{1,3}$ and $\langle\{a, c, v, w\}\rangle_G \not\cong K_{1,3}$, this forces $wb, wc \in E(G)$, which contradicts the claim. Thus we may assume $E_G(w, \{u, v\}) \neq \emptyset$. By the claim, note that $|E_G(w, \{u, v\})| \leq 1$. By symmetry, we may assume $E_G(w, \{u, v\}) = \{wv\}$. Then $xw \in E(G)$ because $\langle\{a, w, v\}\rangle_G \cong K_3$. Since $\langle\{a, b, u, w\}\rangle_G \not\cong K_{1,3}$, this forces $bw \in E(G)$. Then we have $yw \in E(G)$ because $\langle\{a, b, w\}\rangle_G \cong K_3$. Then $\langle\{w, a, x, y\}\rangle_G \cong K_{1,3}$. This is a contradiction. This completes the proof of Theorem A. ■

3. Proof of Theorem B

By contradiction, suppose that $\Delta(G) \geq r(t-1, t)$ and $\Delta(\overline{G}) \geq r(t-1, t)$. Then both G and $\overline{G}$ contain K_t because both G and $\overline{G}$ are $K_{1,t}$-free. Let A be a subgraph of G such that $A \cong K_t$. Since $|V(G)| \geq r(t^2-t+2, t^2-t+2)$, by symmetry, we may assume that G contains $\overline{K_{t^2-t+2}}$ as an induced subgraph. Hence there exists a subgraph H of G such that $\langle V(H)\rangle_G \cong \overline{K_{t^2-t+1}}$ and $V(A) \cap V(H) = \emptyset$. Since $\overline{G}$ is $K_{1,t}$-free, $E_G(x, V(A)) \neq \emptyset$ for every $x \in V(H)$. Since $|V(H)| = t(t-1)+1$, there exists a $v \in V(A)$ such that $|E_G(v, V(H))| \geq t$. This implies that $\langle\{v\} \cup V(H)\rangle_G$ contains $K_{1,t}$ as a induced subgraph. This is a contradiction. This completes the proof of Theorem B. ■

Remark. Theorem A implies as an immediate corollary that the graph property "both G and $\overline{G}$ are claw-free" is stable under the closure for claw-free graphs, i.e., if G has the property, then $cl(G)$ has the property as well (see e.g. the survey paper [1]).

References

[1] H.J. Broersma, Z. Ryjaček and I. Schiermeyer, *Closure concepts — a survey*, Graphs and Combin. **16** (2000) 17–48.

Received 23 November 2003
Revised 6 November 2004