Discussiones Mathematicae
Graph Theory 25 (2005) 261–266

MYCIELSKIANS AND MATCHINGS

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Abstract

It is shown in this note that some matching-related properties of graphs, such as their factor-criticality, regularizability and the existence of perfect 2-matchings, are preserved when iterating Mycielski's construction.

Keywords: Mycielskian, factor-critical graph, perfect matching, perfect 2-matching.

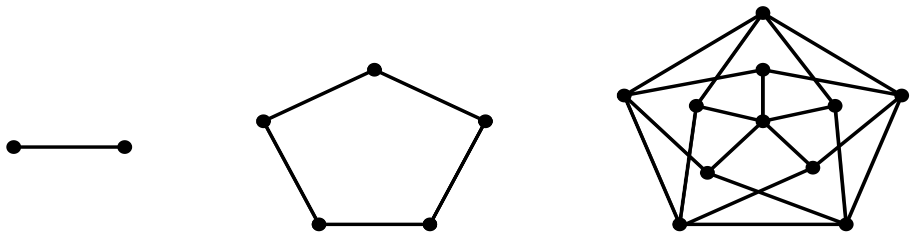
2000 Mathematics Subject Classification: 05C70, 05C75.

Members of the sequence of graphs, obtained by starting with K_2 and iterating Mycielski's construction ([4]), serve as the standard example of graphs whose clique number is much smaller than their chromatic number. Mycielski's results were extended in [2], where the fractional chromatic number of iterated Mycielskians was calculated. In the recent article [1] authors considered a number of other properties and graph-theoretic invariants of Mycielski's graphs, among others their diameter, Hamiltonicity, biclique partition number, packing number and domination number, as well as some properties of their adjacency matrices. The purpose of this note is to present some results concerning matching-related parameters and structural properties of Mycielski's graphs that have not been considered in previous papers.

All graphs considered in this paper will be simple and connected, unless explicitly stated otherwise. For all the terms and concepts not defined here, we refer the reader to the monograph [3].

For a given graph G on the vertex set $V(G) = \{v_1, \ldots, v_n\}$, we define its *Mycielskian* $\mu(G)$ as follows. The vertex set of $\mu(G)$ is $V(\mu(G)) =$

$\{X, Y, z\} = \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ with $x_iy_j \in E(\mu(G))$ if and only if $v_iv_j \in E(G)$, with $x_ix_j \in E(\mu(G))$ if and only if $v_iv_j \in E(G)$, with $y_iz \in E(\mu(G))$ for all $1 \leq i \leq n$ and with $\mu(G)$ having no other edges. We say that the edges $x_ix_j, x_iy_j, x_jy_i, y_iz$ and y_jz are *generated* by the edge $v_iv_j \in E(G)$. For a graph G, its k-th *iterated Mycielskian*, $\mu^k(G)$, is defined with $\mu^0(G) = G$, and $\mu^k(G) = \mu(\mu^{k-1}(G))$, for $k \geq 1$. As an example, we show in Figure 1 the graph K_2 and its first and second iterated Mycielskian, the graphs C_5 and the Grötzsch graf G_{11}, respectively.



K_2 $\qquad$ $C_5 = \mu(K_2)$ $\qquad$ $G_{11} = \mu^2(K_2)$

Figure 1. The graph K_2 and its first two iterated Mycielskians.

Recall that a *matching* M in G is a set of edges of G such that no two edges from M have a vertex in common. The size of any largest matching in G is called the *matching number* of G and denoted by $\nu(G)$. A matching M in G is *perfect* if every vertex of G is incident to some edge from M.

Let us first consider the case when G contains no perfect matching.

Proposition 1. *For a graph G without perfect matching, $\nu(\mu(G)) \geq 2\nu(G) + 1$.*

Proof. As there is no perfect matching in G, there must be at least one vertex, say $v_k \in V(G)$, that is not incident to any edge e from some largest matching M in G. By taking edges x_iy_j and x_jy_i for all edges $v_iv_j \in M$, and the edge y_kz we get a matching in $\mu(G)$ of size $2\nu(G) + 1$. ■

Let us now consider the more interesting case when G itself has a perfect matching. Obviously, having an odd number of vertices, $\mu(G)$ cannot contain a perfect matching, but the graph obtained from $\mu(G)$ by deleting any vertex will have one. In order to formulate this more precisely, recall that a graph G is *factor-critical* if $G - v$ has a perfect matching for any vertex $v \in V(G)$.

Theorem 2. *Let G be a graph with a perfect matching. Then the graph $\mu(G)$ is factor-critical.*

Proof. Let F be a perfect matching in G. We want to show that $\mu(G)-w$ contains a perfect matching for any $w \in V(\mu(G))$.

Consider first the case $w = z$. It is obvious that the matching F' in $\mu(G)$, containing the edges x_iy_j and x_jy_i for all $v_iv_j \in F$, is a perfect matching of $\mu(G) - z$.

Let w now be an element of X, say $w = x_k$. The vertex v_k is incident to some edge $e = v_kv_l \in F$. Form the matching F' in $\mu(G)$ by taking the edges x_iy_j and x_jy_i for all $v_iv_j \in F - e$. Extending the matching F' by the edges x_ly_k and y_lz we get a perfect matching in $\mu(G) - x_k$.

Let, finally, $w = y_l \in Y$. The vertex v_l is covered by some edge $e = v_kv_l \in F$. Taking the edges x_iy_j and x_jy_i for all $v_iv_j \in F - e$ we get a matching in $\mu(G)$, and adding the edges x_kx_l and y_kz we get a perfect matching in $\mu(G) - y_l$. ■

By similar reasoning, we can prove the following result.

Theorem 3. *Let G be a factor-critical graph. Then the graph $\mu(G)$ is also factor-critical.*

Proof. Consider a vertex $w = x_k \in X$. Let F be a matching in G that misses the vertex v_k only. By taking the edges x_iy_j and x_jy_i for all $v_iv_j \in F$, we get a matching F' in $\mu(G)$. Adding the edge y_kz, we get a matching in $\mu(G)$ that misses x_k only, i.e., a perfect matching in $\mu(G) - x_k$.

Let us now consider a matching F in G that misses some vertex $v_1 \in V(G)$ only. Then there is an odd cycle in G, $C_{2k+1} = v_1v_2 \dots v_{2k+1}v_1$, such that the edges of C_{2k+1} are alternating with respect to F, and both edges of C_{2k+1} incident to v_1 do not belong to F. The edges of $F - E(C_{2k+1})$ form a perfect matching in $F - C_{2k+1}$. Define a matching F'' in $\mu(G)$ by taking the edges x_iy_j and x_jy_i for all $v_iv_j \in F - E(C_{2k+1})$. Extending this matching by the edges $x_1y_2, x_2y_3, \dots, x_{2k}y_{2k+1}, x_{2k+1}y_1$ we get a perfect matching in $\mu(G)-z$. Extending F'' by the edges $x_1y_2, x_2y_3, \dots, x_{2k}x_{2k+1}, y_{2k+1}z$ we get a perfect matching in $\mu(G) - y_1$. Hence $\mu(G) - w$ has a perfect matching for all $w \in V(\mu(G))$, so $\mu(G)$ is factor-critical. ■

As a consequence, we get the factor-criticality of all iterated Mycielskians of a graph with perfect matching.

Corollary 4. *Let G be a graph with perfect matching. Then $\mu^k(G)$ is factor-critical, for all $k \geq 1$.* ■

Now we consider the more general case of graphs with perfect 2-matchings. A 2-*matching* of a graph G is an assignment of weights 0, 1 or 2 to the edges of G such that the sum of the weights of the edges incident with any given vertex is at most 2. A 2-matching is *perfect* if this sum is exactly 2 for all vertices of G. First we prove a simple lemma.

Lemma 5. *Let C be an odd cycle. Then every edge of $\mu(C)$ is contained in some Hamiltonian cycle of $\mu(C)$.*

Proof. Let C be an odd cycle $v_1 v_2 \ldots, v_{2k+1} v_1$. By taking the edges $x_1 x_2$, $x_2 y_3$, $y_3 x_4$, $\ldots$,$x_{2k} y_{2k+1}$, $y_{2k+1} z$, $z y_1$, $y_1 x_{2k+1}$, $x_{2k+1} y_{2k}$, $\ldots$,$x_3 y_2$, $y_2 x_1$ we get a Hamiltonian cycle in $\mu(C)$. Since this procedure can start at any vertex of C, the second claim follows too. ■

In the proof of the next theorem it is convenient to refer to edges of a 2-matching F in a graph G which have weight 1 as edges of F; therefore, by referring to a cycle in F we mean a cycle in G consisting of edges of weight 1 in F.

Theorem 6. *Let G be a graph with a perfect 2-matching. Then the graph $\mu(G)$ also has a perfect 2-matching.*

Proof. Take a perfect 2-matching F of the graph G. If F contains at least one edge of weight 2, say $e = v_k v_l$, we construct a perfect 2-matching F' of $\mu(G)$ as follows. Assign the weight 1 to the edges $zy_k, y_k x_l, x_l x_k, x_k y_l$ and $y_l z$ generated by e. Assign the weight 0 to all other edges of the form $y_i z$ and $x_i x_j$. Finally, to the edges $x_i y_j$ and $x_j y_i$ assign the weight of the edge $v_i v_j$ in F. In this way, for any odd cycle in F there will be a corresponding even cycle in F' of double length; for any even cycle in F there will be a pair of even cycles of the same length in F', and for any other edge of weight 2 in F there will be two edges of the same weight in F'. Hence, F' is a perfect 2-matching.

If there are no edges of weight 2 in F, but F contains at least one even cycle C, we construct a perfect 2-matching F_1 for G by taking all the edges of $F - C$ and from C we take only every second edge, but with the weight 2. Now we consider F_1 and proceed as in the previous case.

Consider, finally, the case when F consists of odd cycles only. Take such an odd cycle and denote it by C'. Then the subgraph of $\mu(G)$ induced by $\mu(C')$ contains a Hamiltonian cycle C''. By assigning the weight 1 to all edges of C'', the weight 1 to all edges x_iy_j and x_jy_i such that v_iv_j has the weight 1 in F, and the weight 0 to all other edges of $\mu(G)$, we get a perfect 2-matching of $\mu(G)$. ∎

Corollary 7. *If a graph G contains a perfect 2-matching, then $\mu^k(G)$ also contains a perfect 2-matching, for all $k \geq 0$.* ∎

The graphs with perfect 2-matchings are related to regularizable graphs, and regularizability is another matching-related property that is preserved by iterating Mycielski's construction. A graph G is *regularizable* if it is possible to replace every edge of G by some non-empty set of parallel edges so as to obtain a regular graph. For example, the Grötzsch graph G_{11} is regularizable: Replacing each edge of the form y_iz by two parallel edges and each edge x_iy_j by four parallel edges, we get a regular (but not simple) graph of degree 10. We shall use the following characterization of regularizable graphs.

Theorem 8. *A graph G is regularizable if and only if for each edge e of G there exists a perfect 2-matching of G in which e has weight 1 or 2.* ∎

For the details of the proof, we refer the reader to [3], p. 218.

Theorem 9. *If G is a regularizable graph, then $\mu^k(G)$ is also regularizable, for all $k \geq 0$.*

Proof. For a given edge $e' \in E(\mu(G))$ we consider the edge $e \in E(G)$ that generates e'. If there is a perfect 2-matching F of G such that e has the weight 2 in F, we construct a perfect 2-matching F' in $\mu(G)$ as follows: We assign the weight 1 to all edges generated by e (hence also to e'). To the edges of the form x_iy_j we assign the weight of the edge v_iv_j in F, and to all other edges of $\mu(G)$ we assign the weight 0.

If the edge e' has weight 1 in all perfect 2-matchings of G, we consider a perfect 2-matching M of G such that the weight of e in M is equal to 1. Such a perfect 2-matching M must exist, since G is regularizable. The edge e is then contained in some odd cycle $C \subseteq M$, and the claim now follows from Lemma 5 and Theorem 6. ∎

As a consequence, we get regularizability of all iterated Mycielskians of a 1-extendable graph. (A graph G is 1-*extendable* if every edge of G is contained in some perfect matching of G.)

Corollary 10. *Let G be a* 1*-extendable graph. Then $\mu^k(G)$ is regularizable, for all $k \geq 0$.*

Proof. This follows from the fact that every perfect matching is also a perfect 2-matching in G. Now, from Theorem 8 it follows that G itself is regularizable, and by Theorem 9 so are all its iterated Mycielskians. ∎

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Received 15 July 2003
Revised 23 July 2004