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DOMINATION AND LEAF DENSITY IN GRAPHS

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Abstract

The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a subset D of $V(G)$ with the property that each vertex of $V(G) - D$ is adjacent to at least one vertex of D. For a graph G with n vertices we define $\epsilon(G)$ to be the number of leaves in G minus the number of stems in G, and we define the leaf density $\zeta(G)$ to equal $\epsilon(G)/n$. We prove that for any graph G with no isolated vertex, $\gamma(G) \leq n(1-\zeta(G))/2$ and we characterize the extremal graphs for this bound. Similar results are obtained for the total domination number and the partition domination number.

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1. Introduction

A subset D of the vertex set of a graph G is a *dominating set* of G if each vertex of $V(G) - D$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A *leaf* is a vertex of degree one.

The problem of determining the domination number is NP-complete [4] and therefore much effort has been put into attaining upper and lower bounds for the domination number. An early result due to Ore [8] states that the domination number of any graph G of order n and containing no isolated vertex is at most $n/2$. Better upper bounds have been obtained for graphs with minimum degree greater than one (see Haynes *et al.* [6]). However, for some graphs with many leaves the $n/2$-bound is far from the actual value of the domination number. For instance, $\gamma(K_{1,n-1}) \le n/2$ is a rather crude bound when n is large. We give a new upper bound, which takes into account the density of the leaves.

We use the following notation. Let $L(G)$ denote the set of leaves in a graph G, and let $L(v)$ denote the set of leaves adjacent to v. A vertex that is adjacent to a leaf is called a *stem*, and the set of all stems of G will be denoted by $S(G)$. For $i = 1, \ldots, \Delta(G)$ we define

$$S_i(G) = \{v \in V(G) \,|\, v \text{ is adjacent to precisely } i \text{ leaves }\}.$$

Thus the elements of $S_i(G)$ are the vertices of G with precisely i adjacent leaves. Let $s_i(G) = |S_i(G)|$, $s(G) := |S(G)| = \sum_{i\ge 1} s_i(G)$, and $l(G) = |L(G)|$. A *corona graph* G is a graph where $s_1(G) = n/2$, i.e., each vertex is a leaf or a stem adjacent to exactly one leaf.

We introduce two new graph parameters; ϵ and ζ. First, we define

$$\epsilon(G) := \sum_{i=2}^{\Delta(G)} s_i(G)(i-1) = l(G) - s(G).$$

Secondly, we define the *leaf density* $\zeta(G)$ of a graph G by

$$\zeta(G) := \frac{\epsilon(G)}{n} = \frac{l(G) - s(G)}{n}.$$

This concept enables us to compare the leaf density of different graphs.

For any graph parameter $\mu(G)$, we may write μ whenever the graph G under consideration is given by the context.

In the following sections, we consider three different domination parameters and give upper bounds for the domination parameters in terms of the order and the leaf density. In each case, we exhibit the extremal graphs.

2. An Improvement of Ore's Theorem

We shall use the following two classical theorems on domination.

Theorem 2.1 [8]**.** *If G is a graph with no isolated vertex, then $\gamma(G) \le n/2$.*

Theorem 2.2. *For any graph G with no isolated vertex, $\gamma(G) = n/2$ if and only if each component of G is a 4-cycle or a corona graph.*

The above theorem was proved independently by [9] and [3]. Our result is as follows.

Theorem 2.3. *Let G denote any graph with no isolated vertex. Then*

$$\gamma(G) \leq \frac{n-\epsilon}{2} = (1-\zeta)\frac{n}{2}, \tag{1}$$

and equality holds, if and only if, each component of G is a 4-cycle or a connected graph in which each vertex is a leaf or a stem.

Proof. Let G' denote the subgraph of G obtained by removing $|L(v)|-1$ leaves from each stem v of G. Then G' is a graph with no isolated vertex, and

$$n(G') = n(G) - \sum_{i=2}^{\Delta(G)} s_i(i-1) = n(G) - \epsilon(G).$$

From Theorem 2.1 we obtain $\gamma(G') \leq n(G')/2 = (n(G) - \epsilon(G))/2$. Let D be a $\gamma(G')$-set which contains all stems of G' and no leaves of G'. Then D is also a dominating set of G, and so $\gamma(G) \leq (n(G) - \epsilon(G))/2$.

Now, suppose that $\gamma(G) = (n(G)-\epsilon(G))/2$. Let D be a $\gamma(G')$-set which contains all stems of G' and no leaves of G'. Then D is also a dominating set of G, and $|D| < n(G')/2$ would imply $\gamma(G) < (n(G) - \epsilon(G))/2$, a contradiction. Hence we must have $\gamma(G') = n(G')/2$, which, by Theorem 2.2, implies that each component H of G' is either a 4-cycle or a corona graph.

If H is a 4-cycle, then H is also a 4-cycle component of G, and if H is a corona graph, then H corresponds to a component in G in which every vertex is a leaf or a stem.

Now for the converse. Let $H_1, \ldots, H_k$ denote the components of G. Then $\epsilon(G) = \epsilon(H_1) + \cdots + \epsilon(H_k)$, and showing $\gamma(G) = (n(G) - \epsilon(G))/2$ is equivalent to showing $\gamma(H_j) = (n(H_j) - \epsilon(H_j))/2$ for every $j \in \{1, \ldots, k\}$.

If $H_j = C_4$, then $\epsilon(H_j) = 0$ and $\gamma(H_j) = 2 = (n(H_j) - \epsilon(H_j))/2$, and we have the desired equality. Now suppose that every vertex of H_j is a leaf or a stem. If $H_j = K_2$, then $\epsilon(H_j) = 0$ and $\gamma(H_j) = 1 = (n(H_j) - \epsilon(H_j))/2$.

Otherwise, if $H_j \neq K_2$, then every vertex of H_j is either a leaf or a stem vertex, but not both. This implies $n(H_j) = l(H_j) + s(H_j)$. Let D denote a $\gamma(H_j)$-set. We may w.l.o.g. assume $S(H_j) \subseteq D$. On the other hand, $S(H_j)$ is a dominating set, and so $D = S(H_j)$. Hence

$$\begin{aligned}\gamma(H_j) &= s(H_j)\\ &= \frac{l(H_j) + s(H_j) - (l(H_j) - s(H_j))}{2} = \frac{n(H_j) - \epsilon(H_j)}{2},\end{aligned}$$

and we have the desired equality. This completes the proof. ■

3. Total Domination and Leaf Density

In this section we give an upper bound of the total domination number γ_t in terms of the number of vertices and the leaf density. A subset S of the vertex set $V(G)$ of a graph G is a *total dominating set* of G if every vertex of $V(G)$ is adjacent to some vertex of S. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a total dominating set of G.

The 2-*corona of a graph* H is the graph of order $3n(H)$ obtained from H by attaching a K_2 at each vertex of H. If a graph G is a 2-corona of some graph H, then G is said to be a 2-*corona graph*. Clearly, a 2-corona graph G has total domination number equal to $2n(G)/3$.

We use the notion of leaf density to extend the two following theorems on total domination.

Theorem 3.1 [2]. *Let G denote a connected graph of order $n \geq 3$. Then $\gamma_t(G) \leq 2n/3$.*

Theorem 3.2 [1]. *Let G denote a connected graph of order $n \geq 3$. Then $\gamma_t(G) = 2n/3$ if and only if $G \in \{C_3, C_6\}$ or G is a 2-corona graph.*

Our result is as follows.

Theorem 3.3. *Let G denote a connected graph of order $n \geq 3$. If $G = K_{1,n-1}$ then $\gamma_t(G) = 2$, otherwise*

$$\gamma_t(G) \leq \frac{2}{3}(n - \epsilon) = \frac{2}{3}n(1 - \zeta), \tag{2}$$

and equality holds, if and only if, $G \in \{C_3, C_6\}$ *or* G *can be constructed from a* 2*-corona graph* H *by attaching some (possibly none) leaves at the stems of* H*.*

Proof. Obviously, $\gamma_t(K_{1,n-1}) = 2$, so we may assume $G \neq K_{1,n-1}$. Let G' denote the subgraph of G obtained by removing $|L(v)| - 1$ leaves from each stem v of G. Then G' is a connected graph, and

$$n(G') = n(G) - \sum_{i=2}^{\Delta(G)} s_i(i-1) = n(G) - \epsilon(G).$$

Since $G \neq K_{1,n-1}$, the graph G' must contain at least three vertices, and so Theorem 3.1 implies $\gamma_t(G') \leq 2n(G')/3 = 2(n(G) - \epsilon(G))/3$. Let D be a $\gamma_t(G')$-set. Since D must dominate the leaves of G' from the stems of G', it follows that D contains all stems of G and so D is a total dominating set of G. Thus, $\gamma_t(G) \leq |D| \leq 2(n(G) - \epsilon(G))/3$.

Suppose $\gamma_t(G) = 2(n - \epsilon)/3$. Then we must have $\gamma_t(G') = 2n(G')/3$ and, by Theorem 3.2, either $G' \in \{C_3, C_6\}$ or G' is a 2-corona graph. In the former case we find that $G \in \{C_3, C_6\}$ and in the latter case we find that G can be constructed from the 2-corona graph G' by attaching some (possibly none) leaves at the stems of G'.

Conversely, if $G \in \{C_3, C_6\}$, then we clearly obtain equality in (2). Now suppose that G can be constructed from a 2-corona graph H by attaching some (possibly none) leaves at the stems of H. Then $\gamma_t(G) = \gamma_t(H) = 2n(H)/3$ and $n(H) = n(G) - \epsilon(G)$. This completes the proof. ■

4. Partition Domination and Leaf Density

In this section we give an upper bound of the k-partition domination number $\gamma(G, \pi_k)$ in terms of the number of vertices and the leaf density. The concept of partition domination was introduced by Hartnell and Vestergaard [5]. Other references on this topic include [7], [10] and [11].

By a *k-partition* ($k = 2, 3, \ldots$) of $V(G)$ we shall mean pairwise disjoint subsets $V_1, V_2 \ldots, V_k \subseteq V(G)$ such that $V_1 \cup V_2 \cdots \cup V_k = V(G)$. Note that some of the subsets $V_1, \ldots, V_k$ may be empty. If $V_i \neq \varnothing$, then a set $D_i \subseteq V(G)$ is called a dominating set for V_i if each vertex of V_i not in D_i has a neighbour in D_i. The domination number $\gamma_G(V_i)$ is the smallest cardinality of a dominating set of V_i. We define $\gamma_G(\varnothing) = 0$.

The *k-partition domination number* $\gamma(G, \pi_k)$ of a graph G with respect to a k-partition π_k is defined to be the number

$$\gamma(G, \pi_k) = \gamma(G) + \sum_{i=1}^{k} \gamma_G(V_i).$$

Since any dominating set of G is also a dominating set for V_i, we obtain the following.

Observation 4.1. For any graph G and k-partition π_k, we have $\gamma(G, \pi_k) \leq (k+1)\gamma(G)$.

Together, Observation 4.1 and Theorem 2.3 imply the following result.

Corollary 4.2. *Let k denote any positive integer greater than one and let G denote any graph with no isolated vertex. Then*

$$\gamma(G, \pi_k) \leq (k+1)\frac{n-\epsilon}{2} = (k+1)(1-\zeta)\frac{n}{2}.$$

For $k = 2$, the extremal graphs of the bound stated in Corollary 4.2 are given in Theorem 4.3 below. An example of an extremal graph is given in Figure 1.

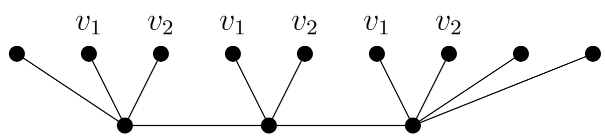

Figure 1. Let all the vertices labelled v_1 be contained in V_1 and let all the vertices labelled v_2 be contained in V_2. The unlabelled vertices may be arbitrarily distributed among V_1 and V_2. With this partition we obtain $\gamma(G, \pi_2) = 3s = 3(n-\epsilon)/2$.

Theorem 4.3. *Let G denote any graph with no isolated vertex. Then*

$$\gamma(G, \pi_2) = \frac{3\,(n-\epsilon)}{2} = 3(1-\zeta)\frac{n}{2}$$

if and only if each component H of G is either a K_2 with one vertex in each partition set V_1 and V_2, or H satisfies (i) *and* (ii):

(i) *Every vertex in H is either a stem or a leaf, and*

(ii) *for every $v \in S(H)$, we must have $L(v) \cap V_1 \neq \varnothing$ and $L(v) \cap V_2 \neq \varnothing$.*

Proof. First, suppose that $\gamma(G, \pi_2) = 3(n-\epsilon)/2$. Then we must have $\gamma(G) = (n-\epsilon)/2$, $\gamma_G(V_1) = (n-\epsilon)/2$ and $\gamma_G(V_2) = (n-\epsilon)/2$. Now, by Theorem 2.3, each component H of G is either a 4-cycle or every vertex of H is a stem or a leaf. Moreover, $\gamma_H(V_1') = \gamma(V_2') = (n(H) - \epsilon(H))/2$. For $H = C_4$, we obtain $\gamma(H, \pi_2') = 4 < 6 = 3(n(H) - \epsilon(H))/2$, a contradiction. If $H = K_2$, then $\gamma(H, \pi_2') = 3(n(H) - \epsilon(H))/2 = 3$ if and only if H has one vertex in V_1 and the other in V_2, and we are done. If $H \neq K_2$, then every vertex of G is either a stem or a leaf. This proves (i).

Assume that there is a vertex $v \in S(H)$ such that $L(v) \subseteq V_1'$ or $L(v) \subseteq V_2'$, say $L(v) \subseteq V_1'$. Let $k := |L(v)|$. If the graph $H - L(v)$ only consists of the vertex v, then we obtain a contradiction with $\gamma_H(V_1') = (n(H) - \epsilon(H))/2$. It follows that $H - L(v)$ contains no isolated vertices, and therefore $\gamma(H - L(v)) \leq (n(H - L(v)) - \epsilon(H - L(v)))/2$. If the vertex v is a leaf in $H - L(v)$, then $l(H - L(v)) = l(H) - k + 1$ and $s(H - L(v)) \leq s(H)$, otherwise $l(H - L(v)) = l(H) - k$ and $s(H - L(v)) = s(H) - 1$. In any case $\epsilon(H - L(v)) \geq \epsilon(H) - k + 1$, and, since $\gamma_{H - L(v))}(V_2') \leq \gamma(H - L(v))$, we obtain

$$\begin{aligned}\gamma_H(V_2') \leq \gamma_{H-L(v)}(V_2') &\leq \frac{n(H - L(v)) - \epsilon(H - L(v))}{2}\\ &\leq \frac{n(H) - k - (\epsilon(H) - k + 1)}{2} < \frac{n(H) - \epsilon(H)}{2},\end{aligned}$$

which is a contradiction. This proves (ii).

For the converse we need to show $\gamma(H, \pi_2') = (n(H) - \epsilon(H))/2$ for each component H of G. If $H = K_2$ with one vertex in each partition set V_1 and V_2, then clearly the desired equality holds. Suppose that H satisfies (i) and (ii). Then $S(H)$ is a minimum dominating set of $V(H), V_1'$, and V_2', that is, $\gamma(H, \pi_2') = 3s(H)$. Since every vertex is either a leaf or a stem we obtain $l(H) + s(H) = n(H)$, which implies $n(H) = 2s(H) + (l(H) - s(H)) = 2s(H) + \epsilon(H)$, and $\gamma(H, \pi_2') = 3s(H) = 3(n(H) - \epsilon(H))/2$. This completes the proof. ■

The following generalization of Theorem 4.3 may be obtained by a similar proof.

Theorem 4.4. *Let G denote any graph with no isolated vertex, and let k denote any integer greater than two. Then*

$$\gamma(G,\pi_k) = (k+1)\frac{(n-\epsilon)}{2} = (k+1)(1-\zeta)\frac{n}{2}$$

if and only if each component H of G satisfies (i) *and* (ii).

(i) *Every vertex in H is either a stem or a leaf, and*

(ii) *for every $v \in S(H)$ and $j \in \{1,2,\ldots,k\}$, $L(v) \cap V_j \neq \varnothing$.*

Hartnell and Vestergaard [5] gave another upper bound of $\gamma(G,\pi_2)$.

Theorem 4.5 [5]**.** *If G is a connected graph of order $n \geq 3$, then $\gamma(G,\pi_2) \leq 5n/4$.*

Now the question is which of the two bounds presented in Theorem 4.5 and Corollary 4.2 is better. Calculations show that

$$3(1-\zeta)\frac{n}{2} < \frac{5}{4}n \iff \zeta > \frac{1}{6}.$$

Hence we have obtained a better bound of $\gamma(G,\pi_2)$ for graphs with leaf density $\zeta > 1/6$.

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