A TANDEM VERSION OF THE COPS AND ROBBER GAME PLAYED ON PRODUCTS OF GRAPHS

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Abstract

In this version of the Cops and Robber game, the cops move in tandems, or pairs, such that they are at distance at most one from each other after every move. The problem is to determine, for a given graph G, the minimum number of tandems sufficient to guarantee a win for the cops. We investigate this game on three graph products, the Cartesian, categorical and strong products.

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1. Introduction

The game of Cops and Robber is played on a reflexive graph, i.e., a graph with a loop at every vertex. The cops choose vertices to occupy, then the robber chooses a vertex. The two sides then move alternately, where a move is to slide along an edge or along a loop, i.e., pass. Both sides have perfect information, and the cops win if any of the cops and the robber occupy the

same vertex at the same time. Graphs on which one cop suffices to win are called *copwin* graphs and are characterized in [7, 8]. The minimum number of cops that suffice to win on a graph G is the *copnumber* of G, denoted $c(G)$. The game has been considered on infinite graphs but, in this paper, we only consider finite graphs.

We use $a \sim b$ to indicate that vertex a is adjacent to vertex b ($a \neq b$), and $a \simeq b$ if a is adjacent or equal to b. For $x \in V(G)$, $N(x) = \{y \mid y \sim x\}$ is the *open neighborhood* of x and $N[x] = N(x) \cup \{x\}$ is the *closed neighborhood*. A subgraph H is a *retract* of a graph G if there is a homomorphism $f : V(G) \to V(H)$ such that $f(x) = x$, for all $x \in V(H)$. Note that, since G is reflexive, a homomorphism can send two adjacent vertices to the same vertex.

Sometimes we need to consider the situation where the cops are playing on a retract H, while the robber is playing on the full graph G. If r is the vertex occupied by the robber and f is a fixed retraction map of G onto H, then we refer to $f(r)$ as the robber's *image*. A vertex u of a graph G is *c-dominated* if there exists a vertex v in G such that $N[u] \subseteq N[v]$; also we say that v *c-dominates* u. A vertex u of a graph G is *o-dominated* if there exists a vertex v in G such that $N(u) \subseteq N(v)$. We say that the vertex v *o-dominates* u. (In the literature of Cops and Robber, c-dominated vertices were first called irreducible, because of the similarities between copwin graphs and dismantlable, partially ordered sets. Later, they were called either corners or pitfalls, because the robber really should stay away from them. Here we adopt, but adapt, the notation of [1]. To avoid confusion with the usual notions of domination, we use o- and c- to make specific the references to the open and closed neighborhoods.)

In [2, 3], we propose a variation of the game in which the cop side consists of two cops. The cops must be within distance one of each other after every move. A graph on which two cops playing in tandem in this way can win is said to be *tandem-win*. A strategy that can be used by a tandem of cops to win on a tandem-win graph is given in [2, 3] and is called a *Tandem-win Strategy*. The minimum number of tandems that suffice to win on a graph G is the *tandem number* of G, denoted $T(G)$. Hence, if H is a tandem-win graph, then $T(H) = 1$.

In particular, a 4-cycle shows that a tandem of cops can win on more graphs than a single cop. Copwin graphs are characterized in [7] by an elimination order (see [1]) $(v_0, v_1, \ldots, v_n)$, where each v_i is c-dominated in $G - \{v_0, v_1, \ldots, v_{i-1}\}$. In [2, 3] we show that if G has an elimination order

(*tandem-win decomposition*) $(v_0, v_1, \ldots, v_n)$, where each v_i is o-dominated in $G - \{v_0, v_1, \ldots, v_{i-1}\}$, then G is tandem-win. This is a characterization of triangle-free tandem-win graphs (which turn out to be bipartite), but not of tandem-win graphs in general.

Graph products with two factors can be represented by 3×3 matrices called *edge matrices* as introduced by Imrich & Izbicki [4]. The rows and columns correspond to the first and second factors respectively. The rows and columns each receive one of three labels: E indicating adjacency of the vertices of the corresponding factor, N indicating nonadjacency, and Δ indicating that the vertices are the same. The entries of the matrix are also E, N, and Δ, representing the adjacency relations between the vertices of the product. It should be noted that if the relationship in both factors is Δ, then the corresponding matrix entry is also Δ since the two vertices are the same.

$$
\begin{array}{c}
\begin{array}{ccc} E & \Delta & N \end{array} \\
\begin{array}{c} E \\ \Delta \\ N \end{array}
\left(
\begin{array}{ccc}
- & - & - \\
- & \Delta & - \\
- & - & -
\end{array}
\right)
\end{array}
$$

The edge matrices of the Cartesian, categorical and strong products of two graphs G and H are given.

$$\text{Cartesian}: G \,\square\, H \quad \text{Categorical}: G \times H \quad \text{Strong}: G \boxtimes H$$

$$
\begin{pmatrix}
N & E & N \\
E & \Delta & N \\
N & N & N
\end{pmatrix}
\qquad
\begin{pmatrix}
E & N & N \\
N & \Delta & N \\
N & N & N
\end{pmatrix}
\qquad
\begin{pmatrix}
E & E & N \\
E & \Delta & N \\
N & N & N
\end{pmatrix}
$$

In this paper, we present results regarding tandem-win graphs and these three products, and compare them with results known for copnumbers.

2.   Cartesian Product

Tošić [9] and, independently, Maamoun and Meyniel [5] prove that if G and H are graphs with copnumbers $c(G)$ and $c(H)$, then $c(G \square H) \leq c(G) + c(H)$. It follows that $c(\square_{i=1}^{n} G_i) \leq \sum_{i=1}^{n} c(G_i)$.

Following their proof, it is easy to show a similar result.

Observation 1. $T(G \square H) \leq T(G) + T(H)$.

The Cartesian product of two copwin graphs has copnumber at least 2 since it contains a 4-cycle as a retract, and hence, is not copwin. However, in a special case we can say more.

For all $v \in V(G)$, let $T \cdot v$ be the subgraph of $T \square G$ whose vertices have second coordinate v.

Theorem 2. *The Cartesian product of a tree T and a copwin graph G is tandem-win.*

Proof. We show that two cops playing in tandem have a winning strategy on $T \square G$. This strategy has two phases. During the first phase, the cops move to capture the robber's image on $T \cdot v$, for some $v \in V(G)$.

Having captured the robber's image, the second phase is to capture the actual robber.

(a) If the robber moves from a vertex in $T \cdot u$ to a vertex in $T \cdot u$, $u \in V(G)$, and the result of this move is that at least one of the cops is no longer on the robber's image, then the cops move so as to capture the image.

(b) If the robber moves from a vertex in $T \cdot u$ to a vertex in $T \cdot u$ and at least one of the cops remains on the robber's image, then the cops are able to move according to their winning strategy in G.

(c) Finally, if the robber moves from $T \cdot u$ to $T \cdot w$, $u, w \in V(G)$, then the cops remain on the robber's image and are then able to move according to their winning strategy in G.

Note that the robber can play a move of type (a) only a finite number of consecutive times since he will eventually come to a leaf. If the robber only plays moves of type (c), he will be captured on G. If the robber plays a move of type (b), he has passed on G and given the cops a free move on G. The robber can then play finitely many moves of type (a), all passes in G for the cops, before having to play a move of type (b) or (c), a resumption of the game on G. ∎

3.   Categorical Product

In [6] it is proven that the copnumber of the categorical product of two copwin graphs is at most 3 and, more generally, if G and H are connected, non-bipartite graphs with $c(H) \geq c(G)$ and $c(H) \geq 2$, then

$c(G \times H) \leq 2c(G) + c(H) - 1$. Here we show that two tandems suffice on the categorical product of certain tandem-win graphs.

Let G and H be triangle-free tandem-win graphs, each having at least one cycle. A graph G has a 'special' tandem-win decomposition by o-dominated vertices if (1) leaves are retracted (as o-dominated vertices) before any other vertices, and then (2) the o-dominated vertices are retracted. See Figure 1 for an example. Note that the subgraph formed by the last four vertices in the decomposition is a 4-cycle.

Lemma 3. *Let G be a triangle-free tandem-win graph with at least one cycle. If G has a special tandem-win decomposition, then the subgraph formed by the last four vertices in this decomposition can be chosen to be a 4-cycle.*

Proof. If not, consider the retraction from a subgraph G' in the tandem-win decomposition to a subgraph G'' in the decomposition which resulted in a tree. Let the o-dominated vertex be x with o-dominating vertex y. There are no leaves (since any leaves were retracted first and this decomposition does not introduce any leaves). So G' contains a cycle, but G'' does not. Now any cycle in G' must include x since G'' is a tree. Therefore G' is isomorphic to $K_{2,m}$, $m \geq 2$. Suppose the independent sets are $\{1, 2\}$ and $\{c_1, c_2, \ldots, c_m\}$. We can alter the decomposition to first eliminate members of the set $\{c_1, c_2, \ldots, c_m\}$ until a subgraph isomorphic to $K_{2,2}$ remains. (See Figure 1 when $m = 6$.) ∎
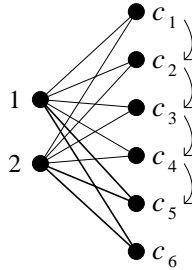

Figure 1: $K_{2,6}$ reduces to $K_{2,2}$.

Let $V(G) = \{a_1, a_2, \ldots, a_{g+4}\}$ have a special decomposition ending in a 4-cycle so that $G = a_1 \cup a_2 \cup \cdots \cup a_g \cup C_4$. Similarly $H = b_1 \cup b_2 \cup \cdots \cup b_h \cup C_4$, where $V(H) = \{b_1, b_2, \ldots, b_{h+4}\}$, has a special decomposition ending in a 4-cycle.

Lemma 4. *Let G and H be triangle-free tandem-win graphs, each having at least one cycle. If y o-dominates x in H ($y \not\sim x$) and G has a special tandem-win decomposition, then $G \times H$ reduces to $G \times (H - x)$ by retraction of o-dominated vertices.*

Proof. Inductively, suppose a_i is o-dominated by $c_i \in G - \cup_{j<i} a_j$. If $(b, z) \in N((a_i, x))$, then $b \sim a_i$ and $z \sim x$. It follows that $b \sim c_i$ and $z \sim y$. Thus $(b, z) \sim (c_i, y)$ or, equivalently, $N((a_i, x)) \subseteq N((c_i, y))$ in $(G - \cup_{j<i} a_j) \times H$. ∎

Theorem 5. *Let G and H be triangle-free tandem-win graphs, each having at least one cycle. If G and H have special tandem-win decompositions, then $T(G \times H) = 2$.*

Proof. Apply Lemma 4 to the graph $G \times H$ to obtain $G \times C_4$, and again to obtain $C_4 \times C_4$. Now $C_4 \times C_4$ reduces to $e \times C_4$, where $e \in E(G)$, with another application. Thus $T(G \times H) = T(e \times C_4)$. Finally notice $e \times C_4 \cong 2C_4$, and so 2 tandems are required. ∎

4.   Strong Product

In [7], Nowakowski and Winkler show that the strong product of a finite number of copwin graphs is copwin. An analogous result for the copnumbers of graphs, due to Neufeld and Nowakowski [6], bounds the copnumber of the strong product of two graphs in terms of the individual copnumbers of these graphs. Surprisingly, we were not able to find tight bounds even if the graphs are all 4-cycles. Theorems 7 and 8 give $2^{n-1} \geq T(C_4) > n$.

Theorem 6. *The strong product of a copwin graph G and a tandem-win graph H is tandem-win.*

Proof. Let G be a copwin graph and let H be a tandem-win graph. Let h be the projection map from $G \boxtimes H$ onto H, and let g be the projection map from $G \boxtimes H$ onto G. For all $x \in V(G)$, let $x \cdot H$ be the subgraph of $G \boxtimes H$ whose vertices have first coordinate x. Thus if both cops are located on $x \cdot H$, then they project to the same image x under the map g. So the cops first play on $G \boxtimes H$ so that, after each move, their positions project to the same image on G. Since G is copwin, the image of the robber is captured

by both cops on G. The cops then play a composition of moves so that they stay with the image of the robber under g and play the Tandem-win Strategy on H under h. Since two cops playing in tandem have a winning strategy, $G \boxtimes H$ is tandem-win.   ∎

Example. This example refers to Figure 2. For $j \in \{0,1,2,3\}$, the c-dominated vertex $(0,j)$ can be retracted onto the c-dominating vertex $(1,j)$, and then the c-dominated vertex $(1,j)$ can be retracted onto the c-dominating vertex $(2,j)$. This leaves the 4-cycle given by the vertices $(2,k)$, $k \in \{0,1,2,3\}$, which is known to be tandem-win.
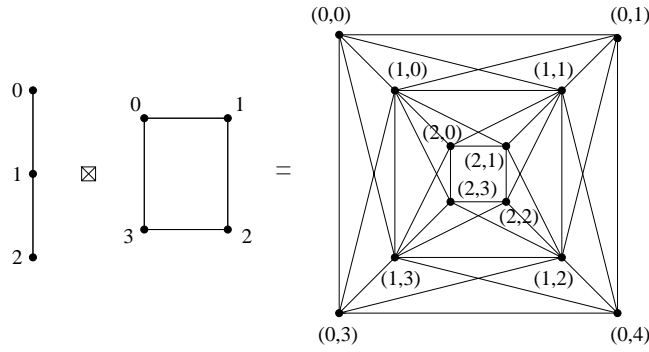

Figure 2: An illustration of Theorem 6.

Note that the strong product of a tandem-win graph and a finite collection of copwin graphs is tandem-win.

We now bound the tandem number of the strong product of a finite number of tandem-win graphs.

Theorem 7. *Let G_i, $i = 1, 2, \ldots, n$, be a finite collection of graphs with $T(G_i) = 1$, for all i. Then $T(\boxtimes_{i=1}^{n} G_i) \leq 2^{n-1}$.*

Proof. Consider the projections of $\boxtimes_{i=1}^{n} G_i$ onto the G_i, $i = 1, 2, \ldots, n$, and we will assign the cops' positions so that the projections of all the cops lie on a single edge in each G_i.

Consider the following assignments. The cops' positions in $\boxtimes_{i=1}^{n} G_i$ are (c_{2j}, c_{2j+1}), $j = 0, 1, \ldots, 2^{n-1} - 1$, the 2^{n-1} tandems. We will represent (c_{2j}, c_{2j+1}) by $((2j)_2, \overline{(2j)_2})$ where $(2j)_2 = b_1 b_2 \cdots b_n$ is the base 2 representation of $2j$, $\overline{(2j)_2}$ is the complement of $(2j)_2$, and leading zeros are permitted. Note that the $(2j)_2$ are distinct, as are the $\overline{(2j)_2}$. Hence $(2j)_2$ and $\overline{(2j)_2}$,

$j = 1, \ldots, 2^{n-1} - 1$, exhaust all of the integers m, where $1 \leq m < 2^n - 1$. We consider $(c_0, c_1) = (00 \cdots 0, 11 \cdots 1)$ as a reference pair.

Thus if in $(2j)_2 = b_1 b_2 \cdots b_n$, $b_i = 0$, then c_{2j} is projected to the same position on G_i as c_0. Otherwise c_{2j} is projected to the same position on G_i as c_1.

The cops follow the robber on each projection until he is captured on all n projections. It must be shown that the robber has been captured on $\boxtimes_{i=1}^{n} G_i$.

Consider the binary number $R = x_1 x_2 \cdots x_n$, again with leading zeros permitted, with $x_i = 0$ if the projection of the robber on G_i is captured by cop c_0, and $x_i = 1$ otherwise. Consider now the pair (c_{2j}, c_{2j+1}) which has $(2j)_2 = x_1 x_2 \ldots x_n$ or $\overline{(2j)_2} = x_1 x_2 \cdots x_n$.

Suppose first $(2j)_2 = x_1 x_2 \cdots x_n$. If $x_i = 0$, then R projects onto c_0, but c_{2j} is on c_0 in G_i. If $x_i = 1$, then R projects onto c_1, but c_{2j} is on c_1 in G_i. Hence c_{2j} captures the robber on $\boxtimes_{i=1}^{n} G_i$. Similarly, c_{2j+1} captures the robber on $\boxtimes_{i=1}^{n} G_i$ if, instead, $\overline{(2j)_2} = x_1 x_2 \cdots x_n$. ∎

Example. When $n = 3$, $T(\boxtimes_{i=1}^{n} G_i) \leq 2^{n-1} = 4$. In Table 1, the cops' positions are shown in pairs, or tandems, along with the corresponding base 2 representations. The projections onto each of G_1, G_2 and G_3 are also indicated.

Table 1. $T(\boxtimes_{i=1}^{3} G_i) \leq 4$.

	projection onto		
	G_1	G_2	G_3
	$\downarrow$	$\downarrow$	$\downarrow$
c_0	0	0	0
c_1	1	1	1
c_2	0	1	0
c_3	1	0	1
c_4	1	0	0
c_5	0	1	1
c_6	1	1	0
c_7	0	0	1

Theorem 8. *If $G_i \cong C_4$, for $i = 1, 2, \ldots, 2n$, then $T(\boxtimes_{i=1}^{2n} G_i) > n$.*

Proof. Suppose n tandems of cops choose their vertices. The robber then chooses a vertex so that, on G_i, the robber's projection is two away from the projection of cop c_i. In one move, no cop can capture the robber on all the projections, and thus not on $\boxtimes_{i=1}^{2n} C_4$. Thereafter, the robber moves to maintain these distances. See Figure 3. Hence $T(\boxtimes_{i=1}^{2n} C_4) > n$. ∎
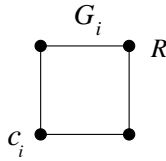

Figure 3: The robber is not captured on $\boxtimes_{i=1}^{2n} G_i$.

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