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CONNECTED ODD DOMINATING SETS IN GRAPHS

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Abstract

An *odd dominating set* of a simple, undirected graph $G = (V, E)$ is a set of vertices $D \subseteq V$ such that $|N[v] \cap D| \equiv 1 \bmod 2$ for all vertices $v \in V$. It is known that every graph has an odd dominating set. In this paper we consider the concept of connected odd dominating sets. We prove that the problem of deciding if a graph has a connected odd dominating set is NP-complete. We also determine the existence or non-existence of such sets in several classes of graphs. Among other results, we prove there are only 15 grid graphs that have a connected odd dominating set.

Keywords: dominating set, odd dominating set.

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1. Introduction

An *odd dominating set* of a simple, undirected graph $G = (V, E)$ is a set of vertices $D \subseteq V$ such that $|N[v] \cap D| \equiv 1 \bmod 2$ for all vertices $v \in V$, where $N[v]$ denotes the closed neighborhood of v. Odd dominating sets and the analogously defined *even dominating sets* have received considerable attention in the literature, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14]. Sutner proved that every graph contains at least one odd dominating set [14] and other proofs of this can be found in [4, 8]. Sutner also showed that deciding if a graph contains an odd dominating set of size at most k is NP-complete [14]. Caro et al. considered the size of the smallest odd dominating set in certain classes of graphs [5, 6].

A well-known variation on the concept of a dominating set is that of a *connected dominating set* (cf. [13]). In this paper, we extend this notion to odd dominating sets by examining connected odd dominating sets or *CODS*. A CODS of a graph is an odd dominating set D such that the subgraph induced by D is connected. If D is a CODS, then either D is a single vertex or the subgraph induced by D has all vertices of even degree. Thus, stars are the only trees that have CODS and, more generally, graphs having "real bridges" (bridges whose endpoints have degree at least 2) do not have CODS, a path with four vertices being the smallest example. We show that the problem of deciding if a graph has a CODS is NP-complete, whereas one can decide in polynomial time if a series-parallel graph contains a CODS. We also examine CODS in various classes of graphs such as grids, complete partite graphs and complements of powers of cycles. In particular, we prove that only 15 grid graphs have CODS.

2. Computational Aspects

2..1 NP-Completeness

Theorem 1. *It is NP-complete to decide if a graph has a CODS.*

Proof. The problem is obviously in NP. To show it is NP-hard, we do a reduction from the NP-complete 1-in-3 3SAT with no negated literals (1-in-3). A figure detailing the most intricate subgraph in the reduction is shown in Figure 1. Let F be an instance of 1-in-3 with clause set $C = c_1, c_2, \ldots, c_p$ and variable set $U = u_1, u_2, \ldots u_q$. Denote the three variables in clause c_i as u^i_1, u^i_2, u^i_3.

Construct a graph as follows. For each clause c_i create a *clause* vertex c_i and three *variable* vertices u_1^i, u_2^i, u_3^i that are adjacent to c_i. Create three *parity* vertices for this clause, x_1^i, x_2^i, x_3^i where x_1^i is adjacent to u_1^1 and u_2^i; x_2^i is adjacent to u_1^i and u_3^i; and x_3^i is adjacent to u_2^i and u_3^i. To each parity vertex x_j^i attach a *parity check* vertex p_j^i and to parity check vertex p_j^i attach a vertex q_j^i and make each q_j^i adjacent to c_i. For each x_j^i create three new vertices t_1, t_2, t_3 (superscripts omitted for clarity) so that x_j^i is adjacent to t_1, t_1 is adjacent to t_2, and t_2 is adjacent to t_3. (Note that only one of the three t type paths is shown in Figure 1.) Also attach to c_i a path w_1^i, w_2^i, w_3^i so that w_1^i is adjacent to c_i, w_2^i is adjacent to w_1^i and w_3^i is adjacent to w_2^i.

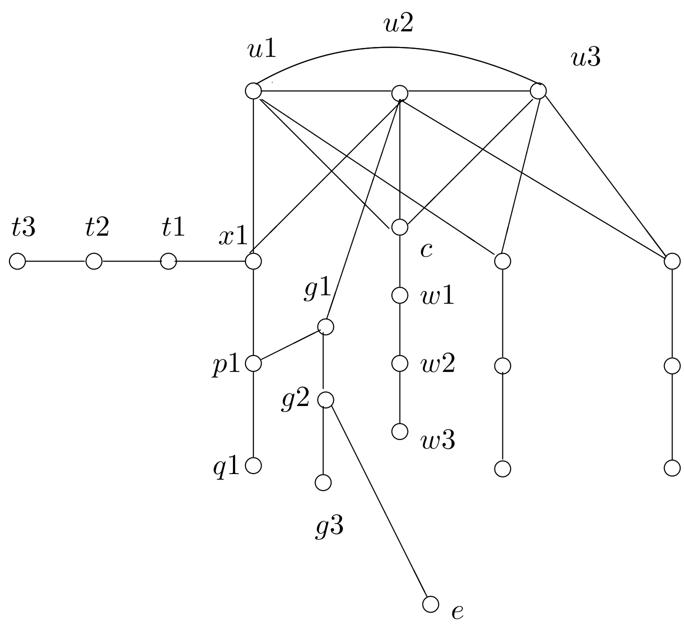

Figure 1. Gadget

Next create three vertices f_1^i, f_2^i, f_3^i so that f_1^i is adjacent to f_2^i and f_2^i is adjacent to f_3^i. Connect f_1^i to both u_3^i and p_3^i. Then create three vertices g_1^i, g_2^i, g_3^i so that g_1^i is adjacent to g_2^i and g_2^i is adjacent to g_3^i. Connect g_1^i to both u_2^i and p_1^i. Then create three vertices h_1^i, h_2^i, h_3^i so that h_1^i is adjacent to h_2^i and h_2^i is adjacent to h_3^i. Connect h_1^i to both u_1^i and p_2^i. Now create a vertex e^i that is adjacent to f_2^i, g_2^i and h_2^i. (Note that the f and h type vertices are not shown in Figure 1.)

If variable u_j appears in clauses c_i and c_k create a *consistency* vertex y_j^{ik} that is adjacent to both u_j^i and u_j^k. This is done for every pair of distinct clauses that a given variable appears in. To each y_j^{ik} attach a two vertex subgraph z_1, z_2 (omitting the subscripts, but being clear that each y_j^{ik} is attached to distinct such subgraph) where y_j^{ik} is adjacent to z_1 and z_1 is adjacent to z_2.

Connect all $3p$ variable vertices in a complete subgraph. Create a *super-vertex* a_1 and another vertex a_2 where a_1 is adjacent to a_2. Then connect a_1 to the following vertices: w_2^i, for all i; p_i^j for all i, j; q_i^j, for all i, j; f_2^i, for all i; g_2^i for all i; h_2^i for all i; f_3^i, for all i; g_3^i for all i; h_3^i for all i; and to each z_1 and t_2 vertex. Repeat the same process with another super-vertex b_1, which has a pendant vertex b_2 attached to it and make a_1 adjacent to b_1. If p is even then connect a_1 to u_j^i, for all i, j; else connect both a_1 and b_1 to u_j^i, for all i, j. Let the graph constructed so far be called G.

To complete the construction, if a_1 has odd number of neighbors other than a_2 and b_1 has an odd number of neighbors other than b_2, attach new vertex d to a_1 and b_1 with a pendant vertex d_1 connected to d. Else if a_1 has an odd number of neighbors other than a_2, then create a copy of G called G' and connect a_1 with the corresponding vertex a_1' in G'. Likewise, if b_1, rather than a_1, has an odd number of neighbors (in G) other than b_2. Let G^* denote the final graph constructed. It is easy to see that G^* can be constructed from F in polynomial time.

We claim that F is 1-in-3 satisfiable if and only if G^* has a connected odd dominating set. In the following, a vertex is defined to be adjacent to itself. Suppose F is 1-in-3 satisfiable, i.e., F can be satisfied so that each clause contains exactly one "true" variable. Say $\{u_1, u_2, \ldots, u_r\}$ is the set of variables that are assigned "true" in this satisfying assignment. Then a connected odd dominating set D is formed by the set of variable vertices u_j^i where u_j^i is the unique variable vertex adjacent to clause vertex c_i corresponding to the "true" variable in clause c_i, plus the appropriate two q_k^i vertices for each i (e.g., q_1^i, q_2^i if $u_1^i \in D$), plus the appropriate p_j^i vertex for each i (e.g., p_3^i if $u_1^i \in D$), plus the appropriate one of $\{f_2^i, f_3^i\}$ for each i, plus the appropriate one of $\{g_2^i, g_3^i\}$ for each i, plus the appropriate one of $\{h_2^i, h_3^i\}$ for each i, plus each of the z_1, w_2, and t_2 vertices, plus a_1 and b_1, plus d if it exists. This set of vertices is clearly connected since a_1 and b_1 are adjacent to each other (and to d) and since a_1 is adjacent to each other vertex in D (except in the case where we create a copy of G, G', in which case a_1 is also adjacent to a_1'). Note that since D contains one vertex from each u_j^i triple of vertices, then exactly one p_j^i vertex will be in D for each i, two q_j^i vertices will be in D for each i, one of $\{f_2^i, g_2^i, h_2^i\}$ will be in D for each i, and two of $\{f_3^i, g_3^i, h_3^i\}$ will be in D for each i.

It is not difficult to see that D is an odd dominating set: each u_j^i vertex is adjacent to the other u_j^i vertices and a_1 and possibly b_1 (depending on the parity of p); each c_i vertex is adjacent to three vertices in D; each w_1, w_3, t_1, t_3 vertex is adjacent to one vertex in D, each w_2 and t_2 vertex is adjacent to three vertices in D; each y_j^{ik} vertex is adjacent to either one or three vertices in D; each x_j^i is adjacent to one vertex in D; each p_j^i, q_j^i is adjacent to three vertices in D; each z_1 is adjacent to three vertices in D; each z_2 is adjacent to one vertex in D; each f_1^i, g_1^i and h_1^i vertex is adjacent to one vertex in D; each f_2^i, g_2^i and h_2^i vertex is adjacent to three vertices in D; each f_3^i, g_3^i and h_3^i vertex is adjacent to three vertices in D; each e^i is adjacent to one vertex in D; each of a_2, b_2 are adjacent to one vertex in D; and we constructed G^* so that a_1, b_1 (and d if it exists) will have an odd number of neighbors in D, since every neighbor of a_1 and of b_1 (except for a_2 and b_2) is in D.

Now suppose D is a connected odd dominating set of G^*. A few simple observations show that the u_j^i vertices in D correspond to a 1-in-3 satisfying assignment for F. Note that any pendant vertex v in G^* cannot be in D, in fact, v's neighbor must be in D. So no x_j^i vertex can be in D because each t_2 vertex must be in D. Likewise, each w_2 vertex must be in D and thus no w_1 vertex, nor any c_i vertex, can be in D. It follows that at least one of u_1^i, u_2^i, u_3^i must be in D for each clause c_i in order to dominate the vertex c_i. This is because if none of u_1^i, u_2^i, u_3^i were in D, then neither would any of q_1^i, q_2^i, q_3^i be in D (since if no u_j^i vertices were in D, each of the three p_j^i would have to be in D in order to dominate the x_j^i vertices). There cannot be exactly two vertices from u_1^i, u_2^i, u_3^i in D, else in order to avoid an x_j^i having an even number of neighbors in D, we would have to add to D the p_j^i vertex that is adjacent to x_j^i. But this implies that c_i has four neighbors in D (two u_j^i type vertices and two type q_j^i vertices). Note that we cannot have both p_j^i and q_j^i in D as this would mean that each of these are adjacent to four vertices in D since a_1 and b_1 must also be in D. Nor can there be three vertices from u_1^i, u_2^i, u_3^i in D, because this would force each of three p_j^i vertices to be in D. This in turn would force that each of f_2^i, g_2^i and h_2^i vertices be in D (since, for instance, at least one of f_2^i, f_3^i must be in D as each f_2^i and f_3^i are adjacent to both a_1 and b_1). But then e^i would have no neighbors in D (and e^i cannot be in D else D would not be connected).

Finally, it is easily seen that the y_j^{ij} consistency vertices force that if a variable appears in more than one clause, the corresponding variable vertices in each clause are either both in D or both not in D. Thus the u_j^i vertices in D correspond to a 1-in-3 satisfying assignment for F. ■

2..2 Series-Parallel graphs

Proposition 2. *Let G be a series-parallel graph with n vertices. There is an $O(n)$ time algorithm to decide if G contains a CODS.*

Proof. The algorithm is similar to that given in [2] to compute the smallest odd dominating set in a graph. As in that algorithm, we begin by computing a binary tree T that describes the series and parallel constructions used to build G. That is, each non-leaf node of T represents either a series or parallel construction of the two series-parallel graphs that are its children as well as which vertices are the terminals of the resulting graph. Such a tree is called the *parse tree* of the series-parallel graph [2]. The remainder of the algorithm is a simple dynamic programming algorithm, whose correctness follows by induction (the details of which are straightforward and omitted). Working from the leaf level of T upwards to the root level, we process a node in T as follows. We store at each node v of T additional information as we process T, namely, whether there exists a CODS of the series-parallel graph G_v that is described by the subtree of T whose root is v and also which of the terminals of G_v can possibly be in such a CODS. Let y and z be the children of v in T. Let y_1, y_2 be the two terminals of G_y and z_1, z_2 be the two terminals of G_z.

If v represents a series construction of G_y with G_z (with y_2 identified with z_1), then there is a CODS of G_v if and only if there is a CODS of G_y containing y_2 and a CODS of G_z containing z_1. We must record at v whether or not there is a CODS of G_v containing y_1 (which is the case if there is a CODS of G_v and there is a CODS of G_y containing y_1) and if there is a CODS of G_v containing z_2 (which is the case if there is a CODS of G_v and there is a CODS of G_z containing z_2).

If v represents a parallel construction of G_y and G_z, then G_v has a CODS if and only if G_y has a CODS containing y_1 and y_2 and G_z has a CODS containing z_1 and z_2, in which case G_v has a CODS containing both its terminals. Note that we do not need to consider the case when both G_y and G_z are P_2's, since G is required to be a simple graph. ■

2..3 k-exclusive graphs

k-exclusive graphs were defined in [6].

Definition. A graph is *k-exclusive* if its vertices can be ordered $v_1, \ldots, v_n$ such that for every $j > k$, v_j is the unique neighbor in $\{v_j, \ldots, v_n\}$ of at least one vertex in $\{v_1, \ldots, v_{j-1}\}$.

Note that this class contains several well-known classes of graphs including the k^{th} power of paths and cycles and grids of dimension $k \times m$. The following was proved in [6].

Proposition 3. *Let G be a k-exclusive graph with vertex ordering $v_1, \ldots, v_n$ realizing the k-exclusiveness. Then for every $j \geq 1$, the following hold:*

1. *At most k vertices in $\{v_1, \ldots, v_j\}$ have neighbors in $\{v_{j+1}, \ldots, v_n\}$.*
2. *Vertex v_j is adjacent to at most k vertices preceding it.*

We note that k-exclusive graphs have pathwidth k (this follows from (1) of Proposition 3) and hence also treewidth k.

We now describe the CODS algorithm for k-exclusive graphs.

Proposition 4. *Let G be a k-exclusive graph with n vertices and its k-exclusive vertex ordering given. Then a CODS can be found or be determined not to exist in time $O(2^k k^3 n)$.*

Proof. Let $v_1, \ldots, v_n$ be a vertex order realizing the k-exclusiveness of G. Using an "exhaustive-search" strategy, we shall construct a candidate dominating set, D, by considering all possible 2^k combinations of vertices $\{v_1, \ldots, v_k\}$. Let f denote the characteristic function of D. For each combination of these k vertices, we verify that each vertex v in $\{v_1, \ldots, v_k\}$ that has no neighbor in $\{v_{k+1}, \ldots, v_n\}$ satisfies $|N[v] \cap D| \equiv b \bmod 2$. If this congruence is satisfied for all such vertices in $\{v_1, \ldots, v_k\}$, we can continue. Otherwise, the initial combination is illegal and the next combination is tested. If the initial combination is legal, we consider (in increasing order of index) vertex v_j, which is the unique neighbor in $\{v_j, \ldots, v_n\}$ of at least one vertex in $\{v_1, \ldots, v_{j-1}\}$ and, by Proposition 3, of at most k vertices, in $\{v_1, \ldots, v_{j-1}\}$. The possible value, 0 or 1, of $f(v_j)$ is completely determined by those vertices in $\{v_1, \ldots, v_{j-1}\}$ for which v_j is the unique neighbor in the set $\{v_j, \ldots, v_n\}$. Add v_j to D (or not) if its addition (omission) properly satisfies the parity domination constraints for these vertices and the requirement that D be connected. If so, proceed to v_{j+1}, otherwise consider the next initial combination. If this algorithm does not terminate with a successful construction of D, then we infer no CODS exists.

Testing each initial combination takes $O(k^2)$ time. As we can maintain during the algorithm the value $p(v) \equiv |N[v] \cap D| \bmod 2$ for every vertex v already visited, we can decide "legality" of the value of $f(v_j)$ in $O(k)$ time: updating the $p(v)$ values that may be changed can be done in $O(k)$ time as v_j has at most k neighbors. Thus the running time of the algorithm is as claimed. ■

3. Combinatorial Aspects

3..1 Grid graphs

Much of the attention on even and odd dominating sets has focused on grid graphs [2, 6, 9, 10, 11]. We show in this section that only a handful of grids have CODS.

Denote by $G_{m,n}$ the grid with m rows and n columns, where $n \geq m$. The vertices of the grid are the pairs (i, j), $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

Theorem 5. *For all $7 < m \leq n$, the grid $G_{m,n}$ does not have a CODS. There are precisely 15 finite grids that have CODS.*

Proof. We first prove that for all $21 \leq m \leq n$, the grid $G_{m,n}$ does not have a CODS. Fix a grid $G_{m,n}$ with $m \geq 21$. Let $A = \{(1, 1), (2, 1), \ldots, (21, 1)\}$ denote the first 21 vertices in the first column of $G_{m,n}$. We will show that for any possible choice of a subset $B \subset A$, and any choice of $D \subset V(G_{m,n})$ satisfying $D \cap A = B$, then D is not a CODS. Consequently, $G_{m,n}$ has no CODS. Notice that there are precisely 2^{21} choices for B.

Fix a choice of B. We claim that if D is an odd dominating set of $G_{m,n}$ which satisfies $D \cap A = B$ then the membership in D of all pairs (i, j) for $j = 2, \ldots, 21$ and $i = 1, \ldots, 22 - j$ is determined. Indeed, for $j = 2$ we have that for all $i = 1, \ldots, 20$, the vertex $(i, 2)$ is in D if and only if $|D \cap \{(i, 1), (i - 1, 1), (i + 1, 1)\}|$ is even. Similarly, for $j = 3$ we have that for all $i = 1, \ldots, 19$, the vertex $(i, 3)$ is in D if and only if $|D \cap \{(i, 2), (i - 1, 2), (i + 1, 2), (i, 1)\}|$ is even, and so on until $j = 21$. We have shown that the membership in D of all the 231 vertices in the "lower left" triangle with side length 21 is determined.

Let $C \subset D$ denote the vertices of this "lower left" triangle that belong to D. Namely, $C = \{(i,j) \in D : 1 \le i \le 21, 1 \le j \le 22 - i\}$. Notice that $B \subset C$ and notice that B determines C uniquely. We claim that the subgraph of $G_{m,n}$ induced by C has (at least one) connected component W consisting of vertices not belonging to the external diagonal of C. Namely, W consists only of vertices from $\{(i,j) : 1 \le i \le 20, 1 \le j \le 21 - i\}$. This implies that W is also a connected component of the subgraph of $G_{m,n}$ induced by D. In particular, D does not induce a connected subgraph of $G_{m,n}$, completing the proof.

A computer program was used to verify our last claim. [The program can be found on the second author's web site.] The program generates all 2^{21} possible B (the function "nextB" in the program). For any B generated by our program, the program computes C (the function "buildC"). Then, the program checks whether any vertex of C is reachable from the external diagonal (the function "checkC"). Any vertex of C not reachable from the external diagonal demonstrates the existence of W, and hence such a B is discarded. If all vertices of C are reachable from the boundary, then C is output (the function "printC"). It turns out that the choice of the constant 21 is the lowest number that causes the program to output nothing. Figure 2 demonstrates a plausible B and C that have no W in case we try to replace 21 by 20. Notice that all vertices of C are reachable from the external diagonal. In fact, this example is *unique* up to transposing the columns and rows, (namely, the output of the program in the case of the constant 20 consists of two plausible C: the one from Figure 2 and its transpose).

Finally, we need to show how to reduce the constant 21 to the constant 8, as stated in the theorem. For each *fixed* $m = 1, \ldots, 20$ we perform the following algorithm. Consider a grid $G_{m,\infty}$. Notice the obvious fact that any odd dominating set D of $G_{m,\infty}$ is determined by the vertices of D belonging to the first column. Thus, there are precisely 2^m possible choices for D. Fix a subset B of the vertices of the first column that corresponds to the vertices of D belonging to the first column. We now sequentially construct, using our program, the vertices of D belonging to columns $2, 3, 4, \ldots$. We do this until we reach a column consisting of no vertices of D (i.e., a column of zeroes, which we call a *null* column). Indeed, an easy periodicity argument shows that we must always eventually reach a null column (our program shows that the index of the null column never exceeds 2000 in case $m \le 20$).

Figure 2

```
11101100000111001110
1011110111010100101
100010110111010010
10011010000011001
1001011110011000
100101101001011
10011000100101
1000110010011
101101001000
11110100101
0010110011
011010000
01011111
0101100
011001
00111
1100
111
01
1
```

Figure 3

```
11011011
11111111
01011010
11000011
10000001
11111111
00000000
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Once the null column is reached, there is no point to continue since we are searching for connected D. Thus, when reaching the null column we check whether the resulting D is connected. If so, we output D, but only in the case where the index of the null column is greater than m (to avoid multiplicities we assume m is the smaller dimension). In case the index of the null column is equal to m, we output D only if the column before the null column is completely within D (since, in this case, the null column is allowed to be part of the grid). Our program functions "nextSmallB", "nextColumn", "checkSmallC" and "PrintSmallC" perform the operations mentioned here. It turns out that the program outputs nothing for $m = 8, \ldots, 20$.

For $m = 1, \ldots, 7$ the program outputs all grids that have CODS, and their respective CODS (in some cases there are more than one CODS). In fact, the following grids have CODS: $G_{1,1}$, $G_{1,2}$, $G_{1,3}$, $G_{2,2}$, $G_{2,3}$, $G_{2,4}$, $G_{3,4}$, $G_{3,5}$, $G_{3,6}$, $G_{4,4}$, $G_{4,5}$, $G_{6,7}$, $G_{6,8}$, $G_{6,9}$, $G_{7,8}$. Figure 3 shows one of the two possible CODS of $G_{7,8}$. ∎

Notice that an immediate corollary of the theorem is that if either m or n (or both) is infinite, then the resulting infinite grid has no CODS.

A detailed analysis of length of the periodicity of the related recurrence for even dominating sets of grids can be found in [9, 11].

3..2 Complements of powers of cycles

Odd dominating sets in powers of cycles and their complements were studied in [6]. Denote these as C_n^k and $\overline{C_n^k}$, respectively, where $n \geq 3$, $k \geq 1$. It is easy to see that C_n^k is either complete or an Eulerian graph and thus always has a CODS, likewise $\overline{C_n^k}$ is either isolated or Eulerian when n is odd (notice that $\overline{C_n^k}$ is connected only for $n \geq 2k+3$). The situation is not so clear when n is even; the following represents a partial characterization.

Theorem 6. *Let n, k be positive integers such that $n \geq 2k+3$. The graph $\overline{C_n^k}$ has a CODS in the following cases:*

(1) $\overline{C_n^k}$ *has a CODS if* $n \equiv 1 \bmod 2$.

(2) $\overline{C_n^k}$ *has a CODS if* $n \equiv 2 \bmod 4$.

(3) $\overline{C_n^1}$ *has a CODS and* $\overline{C_n^2}$ *has a CODS.*

(4) $\overline{C_n^k}$ *and* $\overline{C_n^{k+1}}$ *have CODS if k is odd and* $n \equiv 0 \bmod 2(k+1)$.

(5) $\overline{C_{12k}^{4k}}$ *has a CODS.*

Proof. 1. If $n \equiv 1 \bmod 2$ then the graph is Eulerian and we simply take $V(G)$ as a CODS.

2. If $n \equiv 2 \bmod 4$ then label the vertices $0, 1, 2, \ldots, n-1$ and take for the CODS the set of vertices with even label. Notice that $|D| = n/2$ is odd and, by symmetry, every vertex is not a neighbor of an even number of vertices in D. Hence, D is an odd dominating set. Furthermore, D induces a connected subgraph as D has the following Hamiltonian cycle: $0, n/2+1, 2, n/2+3, \ldots, n/2-3, n-2, n/2-1, 0$.

3. We must show that $\overline{C_n^1}$ has a CODS and $\overline{C_n^2}$ has a CODS. By the previous cases, we only need to check the case $n \equiv 0 \bmod 4$. Label the vertices $0, 1, 2, \ldots, n-1$ and take for the CODS the set of vertices $D = \{z : z \equiv 0, 1 \bmod 4\}$. It is straightforward to verify that D is a CODS.

4. We must show that $\overline{C_n^k}$ and $\overline{C_n^{k+1}}$ have CODS if k is odd and $n \equiv 0 \bmod 2(k+1)$. Indeed, label the vertices $0, 1, 2, \ldots, n-1$ and consider this

labeling over $Z_{2(k+1)}$. Take the CODS to be the set of vertices $D = \{z : z \equiv 0, 1, 2, \ldots, k \bmod 2(k+1)\}$. It is not hard to verify that D is a CODS.

5. We must show that $\overline{C_{12k}^{4k}}$ has a CODS. Label the vertices $0, 1, \ldots, 12k-1$ and consider the set of vertices $D = V - \{0, 4k, 8k\}$. So $|D| = 12k-3$ and the verification that D is indeed a CODS is easy. ■

Let 01^n denote the sequence consisting of one "0" followed by n "1"'s and in general let a^n denote the sequence a repeated n times.

Theorem 7. *Let n and k be positive integers, where $n > 8k$. Suppose k is odd and n is even. Let $t = gcd(n-2(k+1), k+1, n)$. If $(n-2(k+1))/t$ is odd then the sequence $(01^{t-1})^{n/t}$ is the characteristic function of a CODS of $\overline{C_n^k}$ and $\overline{C_n^{k+1}}$.*

We provide two examples before giving the proof (in the first example $n \leq 8k$, but the idea is the same).

Example 1. $n = 12$, $k = 3$. Then $t = gcd(12-8, 12, 4) = 4$ and $(12-8)/4 = 4/4 = 1$. Thus, 011101110111 is a CODS for $\overline{C_{12}^3}$ and $\overline{C_{12}^4}$.

Example 2. $n = 36$, $k = 3$. Then $t = gcd(n-2(k+1), k+1, n) = gcd(28, 4, 36) = 4$ and $(n-2(k+1))/t = 28/4 = 7$ is odd. Hence $(0111)^9$ is a CODS for $\overline{C_{36}^3}$ and $\overline{C_{36}^4}$.

***Proof of Theorem* 7.** Observe that $n-2(k+1)$ is just one less than the degree of $\overline{C_n^k}$. We also observe that the length of the sequence 01^{t-1} is even (as n and $k+1$ are even) and thus $t-1$ is odd and so the number of "1"'s in $(n-2(k+1))/t$ repetitions of the sequence is $(n-2(k+1))(t-1)/t$, which is odd.

Take $D = V - \{z : z \equiv 0 \bmod t\}$. We claim D is a CODS of $\overline{C_n^k}$. We first show the odd-domination property. The vertex v_0 is adjacent to precisely $(n-2(k+1))/t$ repetitions of 01^{t-1} plus an extra 0 as its degree is $n-2(k+1)+1$ and as $t|(k+1)$. So the first edge emanating from v_0, going clockwise around the cycle, is to a vertex v_j, where $j \equiv 0 \bmod t$ and hence v_j is not in D. For the same reason, the last edge out of v_0 is to a vertex v_r, where $r \equiv 0 \bmod t$, and thus v_r is not in D. Hence v_0 has an odd number of "1"'s in its closed neighborhood, $(n-2(k+1))(t-1)/t$ to be exact. The vertices $v_1, \ldots, v_{t-1}$ are not adjacent to the first 0 that is adjacent to v_0 (which is v_j), but each are adjacent to two additional "1 "'s, as compared to v_0 (one of these additional ""1"'s is themselves).

As to $\overline{C_n^{k+1}}$, it follows that, in comparison with v_0 in $\overline{C_n^k}$, v_0 in $\overline{C_n^{k+1}}$ loses the first and the last "0" in its neighborhood (denoted v_j and v_r above) and so, as before, v_0's closed neighborhood contains an odd number of members of D, while each of $v_1, \ldots, v_{t-1}$ add one "0" (v_r) and lose one "1" (relative to v_0), but they themselves add "1" to their closed neighborhood and again have an odd number of (closed) neighbors in D.

Finally, D is connected since $n > 8k$ and the fact that $t \geq 2$ implies that the subgraph induced by D has minimum degree at least half its order. ■

It is plausible that Theorem 7 holds for all $n \geq 2k+5$. One might guess that $n \geq 2k+3$ is a necessary and sufficient condition for $\overline{C_n^k}$ to have a CODS. This however is not the case, since $\overline{C_{16}^5}$, $\overline{C_{16}^6}$ and $\overline{C_{24}^{10}}$ have no CODS, as we verified using a computer.

3..3 Complete partite graphs

Proposition 8. *A complete q-partite graph $K_{a_1,\ldots a_q}$, $q \geq 2$ has a CODS if and only if*

(a) $(a_1, \ldots, a_q) \equiv (0, 0, \ldots, 0) (\text{mod } 2)$; *or*

(b) *there exist distinct i, j, k such that* $(a_i, a_j, a_k) = (1, 1, 1) (\text{mod } 2)$; *or*

(c) $a_i = 1$ *for some* $1 \leq i \leq q$.

Proof. Clearly the graphs in (a), (b) and (c) have CODS. Now assuming our graph is not one of (a), (b) or (c), then it must have one or two vertex classes with odd cardinality at least 3. Since vertices in the same class are transitive, each class is either completely in or completely out of an odd dominating set. In case there is precisely one class with odd cardinality at least 3, it is easily checked that whether this class is in or out of a dominating set, such a dominating set is not a CODS. Similarly, if there are two classes with odd cardinality at least 3, it is easily checked that whether none, one of, or both of the classes are in a dominating set, such a dominating set is not a CODS. ■

3..4 Powers of paths

Let P_n^k denote the k^{th} power of the path on n vertices.

Proposition 9. *If $2k+1 \geq n$, then P_n^k has a CODS (a vertex of the centroid of the path). If $2k+1 < n$ then P_n^k has no CODS.*

Proof. If $2k+1 \geq n$, the vertex in position $\lfloor n/2 \rfloor$ is connected to all other vertices and hence constitutes a CODS. So we assume $2k+1 < n$. Let $v_1, \ldots, v_n$ be the vertices of the path, and suppose D is a CODS of P_n^k. Define $q(v) = 1$ if $v \in D$ and $q(v) = 0$ otherwise. Clearly $q(v_1) + q(v_2) + \cdots + q(v_{k+1}) = |N[v_1] \cap D| \equiv 1 \bmod 2$. Also $q(v_1) + q(v_2) + \cdots + q(v_{k+1}) + q(v_{k+2}) = |N[v_2] \cap D| \equiv 1 \bmod 2$, forcing $q(v_{k+2}) = 0$. And $q(v_1) + q(v_2) + \cdots + q(v_{k+2}) + q(v_{k+3}) = |N[v_3] \cap D| \equiv 1 \bmod 2$, forcing $q(v_{k+3}) = 0$. This pattern continues until $q(v_1) + q(v_2) + \cdots + q(v_{k+2}) + q(v_{2k+1}) = |N[v_{k+1}] \cap D| \equiv 1 \bmod 2$, forcing $q(v_{2k+1}) = 0$. But then the k consecutive vertices $v_{k+2}, v_{k+3}, \ldots, v_{2k+1}$ are not in D and hence D is not connected since $n > 2k+1$. ■

3..5 Graph products and cubes

Fact 10. If G has a CODS and H is Eulerian then $G \times H$ has a CODS.

For example, we can use this fact to deduce that for all $d \geq 1$, the d-dimensional cube Q_d has a CODS. Indeed, it is trivial for $d = 1$. For d even, Q_d is Eulerian. For d odd, we use the fact that $Q_{d+1} = K_2 \times Q_d$ and Fact 10.

4. Concluding Remarks and Open Problems

Characterizing which complements of powers of cycles have CODS is a problem that remains. We state three problems:

1. The general problem: For which n and k, such that $n \geq 2k+3$, does $\overline{C_n^k}$ have a CODS?
2. Is it true that $\overline{C_n^3}$ and $\overline{C_n^4}$ always have a CODS provided $n \geq 2k+3$?
3. Is it true that $\overline{C_n^k}$ has a CODS for $n \equiv 4 \bmod 8$ provided $n \geq 2k+3$?

It is of interest to resolve the complexity of the CODS problem for several classes of graphs: interval graphs, bipartite graphs, planar graphs, and partial k-trees for $k > 2$.

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