Discussiones Mathematicae
Graph Theory 25 (2005) 211–215

NOTE ON PARTITIONS OF PLANAR GRAPHS

Izak Broere

Bonita S. Wilson

Rand Afrikaans University
Johannesburg, Republic of South Africa

e-mail: ib@rau.ac.za

and

Jozef Bucko

Department of Applied Mathematics
Faculty of Economics, Technical University
B. Němcovej, 040 01 Košice, Slovak Republic

e-mail: bucko@tuke.sk

Abstract

Chartrand and Kronk in 1969 showed that there are planar graphs whose vertices cannot be partitioned into two parts inducing acyclic subgraphs. In this note we show that the same is true even in the case when one of the partition classes is required to be triangle-free only.

Keywords: planar graph, hereditary property of graphs, forest and triangle-free graph.

2000 Mathematics Subject Classification: 05C15.

In this note we investigate the generalized colourings of planar graphs. G. Chartrand and H.H. Kronk in 1969 (see [4]) showed that there are planar graphs whose vertices cannot be partitioned into two parts inducing acyclic subgraphs. We will show that the same is true even in the case when one of the partition classes is required to be triangle-free only. The result was obtained in a seminar held at Fort Wayne in 2001 together with L. Beineke, P. Hamburger, P. Mihók and C. Vandell.

A *graph property* is a non-empty isomorphism-closed subset of the class of all finite simple graphs $\mathcal{I}$. A property $\mathcal{P}$ of graphs is called *additive* if it is closed under the disjoint union of graphs and *hereditary* if it is closed under taking subgraphs. We list some additive hereditary graph properties.

$$\begin{aligned}
\mathcal{O} &= \{G \in \mathcal{I} : G \text{ is edgeless, i.e., } E(G) = \emptyset\},\\
\mathcal{D}_k &= \{G \in \mathcal{I} : G \text{ is } k\text{-degenerate, i.e., the minimum degree } \delta(H) \leq k\\
&\qquad \text{for each } H \subseteq G\},\\
\mathcal{T}_k &= \{G \in \mathcal{I} : G \text{ contains no subgraph homeomorphic to } K_{k+2} \text{ or}\\
&\qquad K_{\lfloor \frac{k+3}{2} \rfloor, \lceil \frac{k+3}{2} \rceil}\},\\
\mathcal{I}_k &= \{G \in \mathcal{I} : G \text{ does not contain } K_{k+2}\}.
\end{aligned}$$

We follow the notation and terminology of M. Borowiecki et al. in [2]. It is easy to see that for $k = 1$ the properties $\mathcal{D}_1$ and $\mathcal{T}_1$ are equal to each other i.e., "to be a forest", $\mathcal{I}_1$ denotes the class of triangle-free graphs and $\mathcal{T}_3$ is, by Kuratowski's Theorem (see [6]), the class of planar graphs.

A property can be often conveniently described in terms of its *minimal forbidden subgraphs*: The set of *minimal forbidden subgraphs* of a property $\mathcal{P}$ is defined by:

$$\boldsymbol{F}(\mathcal{P}) = \Big\{G \in \mathcal{I} : G \notin \mathcal{P} \text{ but each proper subgraph } H \text{ of } G \text{ belongs to } \mathcal{P}\Big\}.$$

For properties $\mathcal{P}_1$ and $\mathcal{P}_2$ a *vertex* $(\mathcal{P}_1, \mathcal{P}_2)$*-partition* of a graph G is a partition V_1, V_2 of $V(G)$ such that for each i the induced subgraph $G[V_i]$ has property $\mathcal{P}_i$. For given properties $\mathcal{P}_1$ and $\mathcal{P}_2$ we define their *product* by $\mathcal{P}_1 \circ \mathcal{P}_2 = \{G \in \mathcal{I} : G$ has a vertex $(\mathcal{P}_1, \mathcal{P}_2)$-partition$\}$.

In this paper we present some results of the form $\mathcal{T}_3 \not\subseteq \mathcal{P} \circ \mathcal{Q}$ by showing the existence of suitable planar graphs which do not admit the required $(\mathcal{P}, \mathcal{Q})$-partition for some well-known properties $\mathcal{P}$ and $\mathcal{Q}$. The motivation and some affirmative results on partitioning planar graphs may be found in [1, 3, 7, 8].

Let $\mathcal{R}$ be the additive hereditary graph property with $\boldsymbol{F}(\mathcal{R}) = \{C_3, C_4\}$. A recent result of Kaiser and Škrekovski (see [5]) states that $\mathcal{T}_3 \subseteq \mathcal{R} \circ \mathcal{R}$. They also showed the existence of planar graphs with the property that each vertex partition into two parts has a monochromatic cycle of length 3, 4 or 5, so that this result is, in some sense, best possible.

Our main result can be stated as follows:

Theorem 1. *There is a planar graph H which does not have a vertex $(\mathcal{D}_1, \mathcal{I}_1)$-partition, i.e.,*

$$\mathcal{T}_3 \nsubseteq \mathcal{D}_1 \circ \mathcal{I}_1.$$

Proof. Let G be the graph illustrated in Figure 1, clearly G is planar, that is, $G \in \mathcal{T}_3$ and let C_5 denote the cycle around the central vertex c. Suppose there is a vertex $(\mathcal{D}_1, \mathcal{I}_1)$-partition (colouring) of G into 1 and 2 (with colours 1 and 2 inducing a forest and a triangle-free graph respectively) and suppose vertices a and b are both coloured with 1 in this colouring. Then the two vertices d and e are forced to be of colour 2.

If the vertex f is of colour 1, then b_1, b_2 are of colour 2 and a_1, a_2 have to be of colour 1. But then we have a triangle of colour 1 in this colouring. Therefore, the vertex f cannot have colour 1 (see Figure 1).

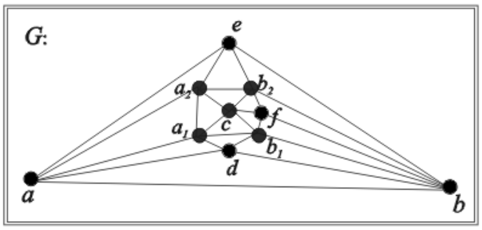

Figure 1. The graph G.

If the vertex a_1 is of colour 1, then b_1 and a_2 have to be 2 and b_2 and c have to be 1. But then we have a 5-cycle of colour 1 in this colouring. Hence the vertex a_1 is of colour 2 and therefore b_1 is of colour 1.

If the vertex a_2 is of colour 2, then b_2 and c have to be 1. But then we have a 4-cycle of colour 1 in this colouring. Therefore, the vertex a_2 is of colour 1. Then c has to be 2 and b_2 has to be 1. But then we have again a 4-cycle of colour 1 in this colouring.

We conclude that the graph $G \in \mathcal{T}_3$ does not have a vertex $(\mathcal{D}_1, \mathcal{I}_1)$-partition in which vertices a and b are both of colour 1.

Now consider the graph H consisting of the complete graph K_4 with a copy of G glued on every edge of K_4 in such a way that edge ab of each copy of G is glued on the corresponding edge xy of the graph K_4. In Figure 2 we illustrate the construction of the graph H partially, by showing a copy of the graph G placed on one edge of K_4. Clearly H is a planar graph, that is, $H \in \mathcal{T}_3$ with 52 vertices and 144 edges.

If H has a vertex $(\mathcal{D}_1, \mathcal{I}_1)$-partition (colouring) into colours 1 and 2 respectively, then two of the K_4 vertices must be of colour 1 and the remaining two in colour 2. This forces a copy of G to be $(\mathcal{D}_1, \mathcal{I}_1)$-partitioned with vertices a and b both 1, which is impossible. Hence $H \notin (\mathcal{D}_1, \mathcal{I}_1)$ and we conclude that $\mathcal{T}_3 \not\subseteq \mathcal{D}_1 \circ \mathcal{I}_1$. ■

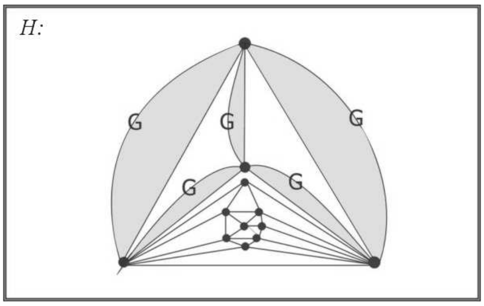

Figure 2. A (partial) illustration of the construction of the graph H.

Let us remark that all the cycles forced to be of colour 1 in the argument of the proof of Theorem 1 are C_3's C_4's and C_5's. Therefore the property $\mathcal{D}_1$ can be enlarged to an arbitrary property $\mathcal{P}$ with $\mathcal{R} \subseteq \mathcal{P}$ and one can use the same proof to show that, in each such case, the product $\mathcal{P} \circ \mathcal{I}_1$ is not an upperbound for $\mathcal{T}_3$, the class of planar graphs.

Corollary 2. *Let* $\mathcal{P}$ *be a hereditary property of graphs with* $\boldsymbol{F}(\mathcal{P}) \supseteq \{C_3, C_4, C_5\}$ *then* $\mathcal{T}_3 \not\subseteq \mathcal{P} \circ \mathcal{I}_1$.

References

[1] K. Appel and W. Haken, *Every planar graph is four colourable*, Illinois J. Math. **21** (1977) 429–567.

[2] M. Borowiecki, I. Broere, M. Frick, P. Mihók and G. Semanišin, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.

[3] M. Borowiecki, I. Broere and P. Mihók, *Minimal reducible bounds for planar graphs*, Discrete Math. **212** (2000) 19–27.

[4] G. Chartrand and H. H. Kronk, *The point arboricity of planar graphs*, J. London Math. Soc. **44** (1969) 612–616.

[5] T. Kaiser and R. Škrekovski, *Planar graph colorings without short monochromatic cycles*, J. Graph Theory **46** (2004) 25–38.

[6] K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math. **15** (1930) 271–283.

[7] P. Mihók, *Minimal reducible bound for outerplanar and planar graphs*, Discrete Math. **150** (1996) 431–435.

[8] C. Thomassen, *Decomposing a planar graph into degenerate graphs*, J. Combin. Theory (B) **65** (1995) 305–314.

Received 5 December 2003
Revised 3 November 2004