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MEDIAN AND QUASI-MEDIAN DIRECT PRODUCTS OF GRAPHS

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Abstract

Median graphs are characterized among direct products of graphs on at least three vertices. Beside some trivial cases, it is shown that one component of $G \times P_3$ is median if and only if G is a tree in that the distance between any two vertices of degree at least 3 is even. In addition, some partial results considering median graphs of the form $G \times K_2$ are proved, and it is shown that the only nonbipartite quasi-median direct product is $K_3 \times K_3$.

Keywords: median graph, direct product, quasi-median graph, isometric embeddings, convexity.

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1. Introduction

Graph products and metric graph theory have developed in the last few decades, and a rich theory involving the structure and recognition of classes of graphs related to these fields has emerged, cf. the books [7, 11]. Among the most studied classes of these graphs are hypercubes (Cartesian products of K_2's) and median graphs. They are closely connected since hypercubes are median graphs and median graphs are precisely retracts of hypercubes [2]. For additional information on median graphs see [2, 5, 16, 18] as well as the survey [14] and references therein.

The distance function of the Cartesian product is the sum of the distance functions of the factors. Hence it is no surprise that this product of graphs behaves hereditarily with respect to being median. More precisely, the Cartesian product of median graphs is again median, and, given a median graph G, every retract of G is a median graph as well. Nevertheless, one can still ask some interesting questions in this respect, for instance which median graphs are Cartesian products of trees, or Cartesian products of paths. These two questions were solved by Bandelt, Burosch and Laborde [3] in terms of forbidden convex subgraphs. For the first case $K_{2,3} - e$ is the only forbidden convex subgraph, while for the second one the forbidden convex subgraphs are $K_{2,3} - e$ and $K_{1,3}$. Another related question was considered in [15] where median subgraphs of Cartesian products of two paths are characterized in several different ways.

Since the strong product and the lexicographic product of factors with at least one edge both contain K_4, no such product is a median graph.

The fourth standard product of graphs, the direct product (also known as cardinal product [10], Kronecker product [12], etc.), is more interesting with respect to median graphs. It is well known, cf. [11] that the direct product is bipartite if and only if one of the factors is bipartite and that the direct product of two connected bipartite graphs consists of two connected components.

In this paper we characterize direct products of graphs that are median graphs for the case when all factor graphs have at least three vertices. First, there are two simple cases: if both factors are paths then both components of their direct product are median; and if both factors are stars then one component of the product is also a star, and hence a median graph. But there is a more interesting instance of median graphs. If one factor is the path P_3 on three vertices, then one component of $G \times P_3$ is median if and only if G is a tree in which the distance between any two vertices of degree at least 3 is even. The case where one factor is K_2 turns out to be most difficult and we obtain two partial results. First we show that if $G \times K_2$ is a median graph then exactly one of the irreducible components of G with respect to so-called K_2-amalgamation is nonbipartite. On the other hand, for a median graph M, the direct product $(M \Box K_4) \times K_2$ is isomorphic to the Cartesian product $M \Box Q_3$, which gives a (large) family of such irreducible nonbipartite graphs. We also show that the only nonbipartite quasi-median direct product is $K_3 \times K_3$.

In the rest of this section we fix the notation and state preliminary results. Section 2 considers the case when both factors have at least three vertices, and Section 3 the case when one factor is K_2. In the last section nonbipartite quasi-median direct products are characterized.

Let $G = (V(G), E(G))$ be a connected graph. The *distance* in G between vertices u, v is denoted $d_G(u, v)$ (or shortly $d(u, v)$) and is defined as the number of edges on a shortest u, v-path. We call a tree T an *even tree*, if the distance between any two vertices of T of degree at least 3 is even.

A subgraph H of a graph G is called *isometric* if $d_H(u, v) = d_G(u, v)$ for all $u, v \in V(H)$. The set $I(u, v)$ of all vertices in G which lie on shortest paths between vertices $u, v \in V(G)$ is called an *interval.* A graph G is a *median graph* if for every triple of vertices $u, v, w \in V(G)$ there exists a unique vertex in $I(u, v) \cap I(u, w) \cap I(v, w)$. This vertex is called the *median* of u, v, w.

A set A in $V(G)$ is called *convex* if $I(u, v) \subseteq A$ for all $u, v \in A$ and a subgraph H in G is *convex* if its vertex set is convex. It is well known that the intersection of convex subgraphs is convex, so we may speak of the smallest convex subgraph which includes a given subgraph H of G, and we call it the *convex closure* of H in G. The following characterization of median graphs due to Bandelt, cf. [14] will be frequently used in our arguments.

Theorem 1.1. *A connected graph G is a median graph if and only if the convex closure of any isometric cycle in G is a hypercube.*

The *Cartesian product* $G \Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ where vertex (a, x) is adjacent to vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The Cartesian product of k copies of K_2 is a *hypercube* or a *k-cube* Q_k. Isometric subgraphs of hypercubes are called *partial cubes*. It is well-known that partial cubes contain no $K_{2,3}$ as an induced subgraph.

The *direct product* $G \times H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{(a, x)(b, y) \mid ab \in E(G), xy \in E(H)\}$. The direct product of connected factors is connected if and only if at least one of the factors is not bipartite [19]. We will also need the following result.

Lemma 1.2. *Let G and H be graphs and let $(a, x), (b, y)$ be vertices of $G \times H$. Then $d_{G\times H}((a, x), (b, y))$ is the smallest d such that there is an a, b-walk of length d in G and an x, y-walk of length d in H. In particular, if such walks do not exist, then (a, x) and (b, y) are in different connected components of $G \times H$.*

Lemma 1.2 was in a different form first established by Kim in [13]. However, the above (more useful) formulation is due to Abay-Asmerom and Hammack [1].

Note that if G and H are bipartite graphs, and (a, x) and (b, y) belong to the same connected component of $G \times H$, then Lemma 1.2 implies that

$$d_{G\times H}((a, x), (b, y)) = \max\{d_G(a, b), d_H(x, y)\}\,.$$

Finally, a *clique* is a maximal complete subgraph. A cutset $C \subset V(G)$ is a set of vertices for which $V(G) \setminus C$ induces a disconnected graph. If in G there are no cutsets on at most k vertices then G is called *k-connected.* By $\Delta(G)$ we denote the largest vertex degree in a graph G.

2. Connected Factors with at Least Two Edges

In this section we prove the following (our main) result.

Theorem 2.1. *Let G and H be connected graphs with at least two edges. Then $G \times H$ contains* (*as a connected component*) *a median graph precisely in the following cases:*

(i) *G and H are paths; then both components of $G \times H$ are median graphs.*

(ii) *$G = K_{1,m}$ and $H = K_{1,n}$; then the component $K_{1,mn}$ is a median graph.*

(iii) *G is an even tree and $H = P_3$; then one component is a median graph.*

Suppose first that $\Delta(G) = \Delta(H) = 2$. Then the factors are paths and cycles. If both factors are paths, then the connected components of $G \times H$ are grid graphs, that is, induced subgraphs of the Cartesian product of a path by a path. It is easy to see that then both components are median graphs. (Alternatively, one may invoke a result from [15] which asserts that a grid graph is a median graph if and only if it contains no isometric cycle of length at least 6.) The subcase when at least one of the factors is a cycle is covered by the next lemma.

Lemma 2.2. *For any $m, n \geq 3$, the components of $C_m \times P_n$ and $C_m \times C_n$ are not median graphs.*

Proof. Let x, y be adjacent vertices of the second factor graph (P_n or C_n) which we denote by A_n to simplify the notation. We claim that $V(C_m) \times \{x, y\}$ induces in $C_m \times A_n$ either an isometric cycle C_{2m} when m is odd, or two disjoint isometric cycles C_m when m is even.

To see that the cycle (resp. cycles) is (are) isometric consider first two vertices a, b of C_m such that (a, x) and (b, y) are in the same component of $C_m \times A_n$. By Lemma 1.2 we get $d_{C_m \times A_n}((a, x), (b, y)) = d_{C_m}(a, b)$ by which we infer that a shortest path between (a, x) and (b, y) is realized on $V(C_m) \times \{x, y\}$. Using an analogous distance argument in the case of two vertices a and b such that (a, x) and (b, x) are in the same component of $C_m \times A_n$ we deduce that the cycle (resp. cycles) is (are) isometric in $C_m \times A_n$.

We easily see that unless $m = 4$ the convex closure of the cycle (resp. cycles) induced by $V(C_m) \times \{x, y\}$ cannot be a hypercube which by Theorem 1.1 implies that $C_m \times A_n$ is not a median graph. Finally, if $m = 4$, then every component of $C_4 \times A_n$ contains a $K_{2,3}$. ■

In fact, invoking Lemma 1.2 one can show that the Djoković-Winkler relation Θ [8, 20] is not transitive in the graphs of Lemma 2.2, thus they are not even partial cubes.

Assume now that $\Delta(G) \geq 3$ and suppose that G or H is not bipartite. Then $G \times H$ is connected. Let u be a vertex of G of degree at least 3 and let u_1, u_2, u_3 be three neighbors of u. Let x, y, z be vertices of H such that y is adjacent to x and z (such vertices exist because H is connected and has at least two edges). Now $(x, u), (y, u_1), (y, u_2), (y, u_3), (z, u)$ induce a $K_{2,3}$, thus $G \times H$ is not a median graph. In the rest we may thus assume that G and H are bipartite.

Assume that $P_4 \subseteq H$ and let x, y, z, w induce a P_4 in H. Then, as above, (x, u), (y, u_1), (y, u_2), (y, u_3), and (z, u) induce a $K_{2,3}$. In addition, (y, u), (z, u_1), (z, u_2), (z, u_3), and (w, u) induce another $K_{2,3}$ of $G \times H$. From Lemma 1.2 we infer that (u, x) and (u, y) belong to different connected components of $G \times H$ and so each of the two components contains a $K_{2,3}$. Thus, if a component of $G \times H$ is supposed to be a median graph, the only possibility left is that G is bipartite with $\Delta(G) \geq 3$, and that $H = K_{1,n}$.

Assume first $n \geq 3$. Then if $P_4 \subseteq G$ then we conclude as above (reversing the roles of G and H) that no component of $G \times H$ is a median graph. Thus, in this case we only need to consider the product $K_{1,m} \times K_{1,n}$. Note that the components of $G \times H$ are $K_{1,mn}$ and $K_{m,n}$, thus the first one is a median graph, the other is not.

Suppose now $n = 2$, in other words we consider the product $G \times P_3$, where G is a bipartite graph with $\Delta(G) \geq 3$. This last case is settled by the following lemma. Recall that a tree is an even tree, if the distance between any two vertices of degree at least 3 is even.

Lemma 2.3. *Let G be a bipartite graph with $\Delta(G) \geq 3$. Then the components of $G \times P_3$ are not median graphs except if G is an even tree, when exactly one component is a median graph.*

Proof. Let C be a shortest (even) cycle of G. If $C = C_4$ then every component of $G \times P_3$ contains a $K_{2,3}$. Assume thus $C = C_{2k}$, $k \geq 3$. Since C is a shortest cycle, it must also be isometric. Thus, using Lemma 1.2, we observe that both components of $G \times P_3$ contain C_{2k} as an isometric subgraph. But we see just as in the proof of Lemma 2.2 that the components of $G \times P_3$ are not median graphs (in fact, not even partial cubes).

Suppose next that G is a tree and that u and v are its vertices of degree at least 3 where $d(u, v)$ is odd. Let x, y, z be vertices of P_3 with y adjacent to

x and z, let u_1, u_2, u_3 be neighbors of u and v_1, v_2, v_3 neighbors of v. Now, (u,x), (u_1,y), (u_2,y), (u_3,y), (u,z) as well as (v,x), (v_1,y), (v_2,y), (v_3,y), (v,z) induce $K_{2,3}$. Moreover, by Lemma 1.2 we infer that (u,x) and (v,x) belong to different components of $G \times P_3$. Thus no component of $G \times P_3$ is a median graph.

Assume finally that G is an even tree. We wish to show that exactly one component of $G \times P_3$ is a median graph. We are going to show this by induction on the number of vertices of G. Note that the smallest even tree with $\Delta \geq 3$ is $K_{1,3}$, and that one component of $K_{1,3} \times P_3$ is a star $K_{1,6}$ and the other is $K_{2,3}$, which proves the induction basis. Now, let G be an even tree on more than 4 vertices. Let u be an arbitrary vertex of G of degree at least 3. We claim that the component K containing (u,y) is a median graph. (That the other component is not median we easily see noting again that we have a $K_{2,3}$ in it.) Let v be an arbitrary vertex of G. By Lemma 1.2 we see that $(v,y) \in K$ if $d(u,v)$ is even, while $(v,x) \in K$ and $(v,z) \in K$ if $d(u,v)$ is odd. Let w be an arbitrary vertex of G of degree one and let w' be its neighbor. Clearly, $G' = G - w$ is an even tree and by induction the component K' of $G' \times P_3$ containing (u,y) is a median graph. Let $d(u,w)$ be odd. Then $d(u,w')$ is even, thus $(w',y) \in K'$. But then K is obtained from K' by adding two vertices of degree one (w,x) and (w,z), thus K is median since K' is median. Let $d(u,w)$ be even. Then $d(u,w')$ is odd, and thus in this case (w',x) and (w',z) belong to K'. Since $d(u,w')$ is odd, w' has exactly one neighbor w'' in G with $w'' \neq w$. Thus the vertices (w',x) and (w',z) are vertices of degree one of K' and it follows easily that K is median graph. ■

By Lemma 2.3 the proof of Theorem 2.1 is complete. We are thus left with the case when one factor is K_2.

3. Median Graphs $G \times K_2$

The first immediate observation in this case is the following. If a graph G is bipartite, then $G \times K_2$ consists of two connected components both isomorphic to G. Therefore in this case $G \times K_2$ has two median components if and only if G is a median graph.

Let us now focus to the nontrivial case when G is nonbipartite. We need some further definitions to be used only in this section. We say that G can be obtained from graphs G_1 and G_2 by a K_2-*amalgam* (over an edge

uv) if there exist induced subgraphs $G_1' \simeq G_1$ and $G_2' \simeq G_2$ of G, such that $V(G_1') \cap V(G_2') = \{u, v\}$ and $G_1' \cup G_2' = G$. (Note that this in particular implies that there are no edges between $G_1 \setminus \{u, v\}$ and $G_2 \setminus \{u, v\}$.) A graph G is called K_2*-irreducible* if it cannot be obtained from two subgraphs on at least three vertices by a K_2-amalgam. Furthermore, in this section we will simplify the notation of vertices $(x, i) \in G \times K_2$ by setting $x_i := (x, i)$, where $i \in \{0, 1\}$.

In determining the conditions for factor G so that $G \times K_2$ is a median graph, we first show the meaning of K_2-amalgams in median graphs.

Lemma 3.1. *A graph G is a median graph if and only if it can be obtained from K_2-irreducible median graphs by a sequence of K_2-amalgams.*

Proof. Suppose that a median graph G can be obtained from two graphs G_1 and G_2 by a K_2-amalgam over an edge uv. Since G_1 and G_2 are convex subgraphs of G, they are obviously median graphs.

On the other hand, in the case when G_1 and G_2 are median graphs, we have to consider the following two cases:

Case A. Suppose that x, y, z are vertices of the same subgraph G_1 or G_2. As G_1 and G_2 are convex, median subgraphs, the vertices x, y, z possess a unique median in G.

Case B. Suppose that x, y, z are not vertices of the same subgraph G_1 or G_2. Without loss of generality we may assume that $x, y \in G_1$ and $z \in G_2$. As G_2 is median, it is bipartite and therefore $d(z, u) = d(z, v) \pm 1$. We may assume that $d(z, u) = d(z, v) - 1$. Obviously, $I(u, x) = I(z, x) \cap V(G_1)$, and $I(u, y) = I(z, y) \cap V(G_1)$, therefore the unique median $m(x, y, u)$ of the vertices x, y, u is also a median of the vertices x, y, z. Since G_1 is convex, $I(x, y) \subseteq G_1$, hence the median $m(x, y, z)$ must be in G_1, and $m(x, y, u)$ is also unique for the vertices x, y, z. ■

Lemma 3.2. *Let $G \times K_2$ be a median graph. If $K_3 \subseteq G$, then $K_4 \subseteq G$.*

Proof. Let u, v, w be the vertices of a $K_3 \subseteq G$. Then $u_0, v_0, w_0, u_1, v_1, w_1$ induce an isometric cycle C_6 in $G \times K_2$, whose convex closure is Q_3. Therefore there exists a vertex $z_0 \in G \times K_2$, adjacent to vertices u_1, v_1, w_1. Hence u, v, w, z induce a K_4 in G. ■

The following lemma will lead to a (partial) characterization of median graphs of the form $G \times K_2$.

Lemma 3.3. *Let $G \times K_2$ be a median graph. Then G can be obtained from a nonbipartite K_2-irreducible graph and a set of bipartite graphs by a sequence of K_2-amalgams.*

Proof. Suppose that G can be obtained from two nonbipartite graphs G_1 and G_2 by a K_2-amalgam over an edge uv. We need to show that this is not possible. For this sake we first claim that $G \times K_2$ contains an even isometric cycle $\widehat{C}$ which intersects both connected components of $(G \setminus \{u,v\}) \times K_2$.

Let d_1 and d_2 be the distance functions in $G_1 \times K_2$ and $G_2 \times K_2$, respectively. Let $D = \{u_0, u_1, v_0, v_1\}$ and let w and z be nonadjacent vertices from D that minimize $d_1(w,z) + d_2(w,z)$ over all such pairs. Let P_1 be a shortest path from w to z in $G_1 \times K_2$ and let P_2 be a shortest path between w and z in $G_2 \times K_2$. Since G_1 and G_2 are nonbipartite graphs, $G_1 \times K_2$ and $G_2 \times K_2$ are connected and so P_1 and P_2 exist. We now define $\widehat{C}$ as the cycle induced by the vertices of $P_1 \cup P_2$.

Clearly, $\widehat{C}$ intersects both connected components of $(G \setminus \{u,v\}) \times K_2$, so we need to show that $\widehat{C}$ is isometric. For this it suffices to show that for a vertex $x \in P_1$ and a vertex $y \in P_2$ the distance between them on $\widehat{C}$ is the same as the distance in $G \times K_2$. Let R be an arbitrary x,y-shortest path in $G \times K_2$. If R contains w or z we are done because $\widehat{C}$ contains w and z. Note that R contains at least one vertex from D. So suppose that R contains $\overline{w}$ adjacent to one of w or z, say z. Let $k = d(x,z)$ and $\ell = d(z,y)$, which are the distances on $\widehat{C}$ as well. Because the subpath of R from x to $\overline{w}$ and the subpath of $\widehat{C}$ from x to z are shortest paths and z and $\overline{w}$ are adjacent, we have $k - 1 \leq d(x,\overline{w}) \leq k+1$. As the lengths of all paths between a pair of vertices of $G \times K_2$ have the same parity by the definition of the direct product, the distance $d(x,\overline{w})$ is either $k-1$ or $k+1$. Analogously, the distance $d(\overline{w},y)$ is either $\ell - 1$ or $\ell + 1$.

Note that $d(x,\overline{w}) = k+1$ and $d(\overline{w},y) = \ell + 1$ cannot happen since in that case R would not be a shortest x,y-path. Also, there is nothing to show if $d(x,y) = d(x,\overline{w}) + d(\overline{w},y) = k + \ell$. So it remains to consider the case $d(x,\overline{w}) = k-1$ and $d(\overline{w},y) = \ell - 1$. Then

$$\begin{aligned}
d_1(w,z) + d_2(w,z) &= d(w,x) + d(x,z) + d(z,y) + d(y,w) = \\
&= d(w,x) + k + \ell + d(y,w) \\
&= d(w,x) + d(x,\overline{w}) + 1 + d(\overline{w},y) + 1 + d(y,w) \\
&= d_1(w,\overline{w}) + d_2(w,\overline{w}) + 2\,.
\end{aligned}$$

This contradicts the way we have selected w and z and so we have proved our claim that $G \times K_2$ contains an even isometric cycle $\widehat{C}$ which intersects both connected components of $(G \setminus \{u, v\}) \times K_2$.

Since $G \times K_2$ is median, the convex closure of the isometric cycle $\widehat{C}$ of length $2k$ is a k-cube Q. As D is a cutset of $G \times K_2$, we infer that $X = D \cap Q$ is also a cutset of Q. Any Q_k, $k \geq 5$, is 5-connected, but X has at most four vertices, so k must be less than five. As Q_4 is 4-connected, and any cutset consisting of four vertices is independent, we conclude that $k < 4$, because D is not independent.

Consider now the case $k = 3$. Since cutsets of Q_3 of size 3 are independent and no subset of D on three vertices is such, all the vertices of D lie in Q. Since the distance between any two vertices of Q is at most 3, there exist vertices $x_0 \in G_1 \times K_2$ and $y_0 \in G_2 \times K_2$ both adjacent to u_1 and v_1 (thus x_1 and y_1 are both adjacent to u_0 and v_0). The vertices x, u, v and y, u, v induce subgraphs isomorphic to K_3, which are, by Lemma 3.2, subgraphs in K_4. Therefore there exist vertices $x' \in G_1$ and $y' \in G_2$, both adjacent to u and v. Vertices $x_1', x_1, y_1', y_1, u_0, v_0$ induce a subgraph $K_{2,3}$ in $G \times K_2$, which contradicts the fact that $G \times K_2$ is a median graph. Hence $k < 3$.

If $k = 2$, there exist vertices $x \in G_1$ and $y \in G_2$ both adjacent to u and v, which leads to the former case and the proof is complete. ∎

Combining Lemma 3.1 with Lemma 3.3 we get

Theorem 3.4. *A graph $G \times K_2$ is a median graph if and only if G can be obtained from a nonbipartite K_2-irreducible graph H such that $H \times K_2$ is a median graph and a set of median graphs by a sequence of K_2-amalgams.*

Hence by Corollary 3.4, the problem of characterizing graphs G such that $G \times K_2$ is a median graph, reduces to the case when G is K_2-irreducible nonbipartite graph. In the following result we present a large family of such graphs. More precisely, we show that graphs of the form $G = K_4 \square M$ where M is an arbitrary median graph yield median graphs, that is $G \times K_2 = M \square Q_3$. Unfortunately, this family does not characterize graphs for which $G \times K_2$ is a median graph. Consider for instance the Greenwood-Gleason graph G which can be described as the graph obtained from Q_5 by identifying all its antipodal vertices. Then G has 16 vertices and cannot be represented as a Cartesian product with K_4, but $G \times K_2$ is isomorphic to the hypercube Q_5.

Proposition 3.5. *Let M be an arbitrary median graph. Then $(M\Box K_4)\times K_2$ is isomorphic to the median graph $M\Box Q_3$.*

Proof. Set $G = M\Box K_4$. Note that $K_4 \times K_2$ is isomorphic to Q_3. Let $e = uv$ be an arbitrary edge of a median graph M, and let K^u, K^v be the corresponding copies of K_4 in G. Then one can readily check that edges between $K^u \times K_2$ and $K^v \times K_2$ in $G \times K_2$ form a matching which induces an isomorphism between the 3-cubes. Moreover, if $u' \in K^u$ and $v' \in K^v$ are two adjacent vertices in $M\Box K_4$, then u'_0 (resp. u'_1) is adjacent to v'_1 (resp. v'_0) in $G \times K_2$. Since M is bipartite, we easily derive that $(M\Box K_4) \times K_2$ is isomorphic to $M\Box Q_3$. ∎

Let us conclude this section with the following open problem.

Problem 3.6. *Characterize nonbipartite K_2-irreducible graphs G such that $G \times K_2$ is a median graph.*

4. Quasi-Median Direct Products

Quasi-median graphs were introduced by Mulder [17] as a nonbipartite generalization of median graphs — bipartite quasi-median graphs are precisely median graphs. It turns out that most of the rich structure theory on median graphs (retractions, expansions, amalgamations, etc.) can be extended in a natural way to quasi-median graphs [4, 17]. For recent results on quasi-median graphs see [6], cf. also references therein.

As the original definition of quasi-median graphs is somewhat involved, we shall introduce them through the following result due to Bandelt, Mulder and Wilkeit.

Theorem 4.1 [4]**.** *A graph G is a quasi-median graph if and only if every clique in G is gated and all the sets U_{ab} are convex.*

For the definition of quasi-median graphs as well as the definition of sets U_{ab} and gatedness we refer to [4]. With the intention to give more insight into quasi-median graphs, the graphs in which every clique is gated were studied in [9], and we will make use of a characterization of these graphs. A graph G is called *clique-gated* if every clique in G is gated, that is, if for every vertex $u \in V(G)$ and every clique C in G there exists a vertex x in C such that $d(u,x) = d(u,y) - 1$ for all $y \in C - x$. It is obvious that if such a vertex exists, it must be unique. By Theorem 4.1 every quasi-median graph is clique-gated.

The *triangle property* in the theorem below is defined as follows. For every edge $ab \in E(G)$ and every vertex $u \in V(G)$ such that $d(u,a) = d(u,b) = k \geq 2$ there exists a vertex $w \in V(G)$ which is adjacent to a and to b, so that $d(u,w) = k-1$.

Theorem 4.2 [9]**.** *A connected graph G is clique-gated if and only if G does not contain $K_4 - e$ as an induced subgraph and G satisfies the triangle property.*

Next we shall characterize direct product graphs that are clique-gated. First we need the following easy result.

Lemma 4.3. *A triangle-free graph is clique-gated if and only if it is bipartite.*

So by Lemma 4.3 we are left with the case when $G \times H$ has a triangle. Using the definition of the direct product we immediately derive that $G \times H$ has a triangle if and only if both G and H have a triangle. Let a, b, c (respectively x, y, z) be vertices that induce a triangle in G (respectively H). If $G = K_3$ and $H = K_3$, then $G \times H$ is clique-gated, since

$$K_3 \times K_3 = K_3 \Box K_3$$

holds. We will show that this is the only nonbipartite case of a clique-gated graph obtainable as a direct product.

Theorem 4.4. *The direct product $G \times H$ of connected graphs G and H is clique-gated if and only if*

(i) *one of the factors is bipartite, or*

(ii) $G = H = K_3$.

Proof. Let $G \times H$ be a clique-gated graph, and let us suppose that besides the triangle with vertices a, b, c we have another vertex u in G. Since G and H are connected graphs, we may assume without loss of generality that u is adjacent to c. We distinguish three cases.

Case A. First, let u be adjacent also to a and to b, so that we have K_4 in G. Then $(a,x), (b,y), (c,z)$ and (u,x) form an induced subgraph $K_4 - e$ in $G \times H$. By Theorem 4.2 it follows that $G \times H$ is not clique-gated.

Case B. The second possibility is that u is adjacent just to one of the vertices a or b. Since both cases are essentially the same (we get an induced $K_4 - e$ in G), we choose u to be adjacent to b. Again, the same four vertices as in the former case form an induced $K_4 - e$ in $G \times H$.

Case C. The remaining case is that u is adjacent only with c. Then, the vertices (a, x) and (b, z) are adjacent in $G \times H$ and

$$d((u, y), (a, x)) = d((u, y), (b, z)) = 2.$$

We now apply the triangle property from Theorem 4.2 for the edge $(a, x)(b, z)$ and a vertex (u, y), and deduce that there must be another vertex in $G \times H$ which is adjacent to $(u, y), (a, x)$ and (b, z). Such a vertex does not exist in the part of the graph that we have constructed so far, therefore we must have another vertex in $G \times H$ which obeys this property. By the definition of direct product this is only possible when in H there is another vertex which is adjacent with x, y and z. We then use *Case A* and conclude that $G \times H$ is not clique-gated. ∎

By combining the above theorem and the results considering median graphs we obtain the following characterization.

Theorem 4.5. *Let G and H be connected graphs on at least three vertices. Then $G \times H$ contains (as a connected component) a median graph precisely in the following cases:*

(i) *G and H are paths; then both components of $G \times H$ are median graphs.*

(ii) *$G = K_{1,m}$ and $H = K_{1,n}$; then the component $K_{1,mn}$ is a median graph.*

(iii) *G is an even tree and $H = P_3$; then one component is a median graph.*

Furthermore, $G \times H$ is a nonbipartite quasi-median graph only if $G = H = K_3$.

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