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TWO VARIANTS OF THE SIZE RAMSEY NUMBER

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Abstract

Given a graph H and an integer $r \geq 2$, let $G \to (H, r)$ denote the Ramsey property of a graph G, that is, every r-coloring of the edges of G results in a monochromatic copy of H. Further, let $m(G) = \max_{F \subseteq G} |E(F)|/|V(F)|$ and define the Ramsey density $m_{inf}(H, r)$ as the infimum of $m(G)$ over all graphs G such that $G \to (H, r)$.

In the first part of this paper we show that when H is a complete graph K_k on k vertices, then $m_{inf}(H, r) = (R-1)/2$, where $R = R(k; r)$ is the classical Ramsey number. As a corollary we derive a new proof of the result credited to Chvatál that the size Ramsey number for K_k equals $\binom{R}{2}$.

We also study an on-line version of the size Ramsey number, related to the following two-person game: Painter colors on-line the edges provided by Builder, and Painter's goal is to avoid a monochromatic copy of K_k. The on-line Ramsey number $\overline{R}(k; r)$ is the smallest number of moves (edges) in which Builder can force Painter to lose if r colors are available. We show that $\overline{R}(3; 2) = 8$ and $\overline{R}(k; 2) \leq 2k\binom{2k-2}{k-1}$, but leave unanswered the question if $\overline{R}(k; 2) = o(R^2(k; 2))$.

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1. Introduction

For integers $k \geq 2$ and $r \geq 2$, let $R(k;r)$ be the *Ramsey number*, that is the smallest integer n such that every r-coloring of the edges of the complete graph K_n results in a monochromatic copy of K_k. The existence of $R(k;r)$ was proved in a seminal paper of Ramsey [21]. For $n < R(k;r)$, any coloring of the edges of K_n without a monochromatic copy of K_k will be called *proper*. For instance, $R(3;2) = 6$, and any proper coloring of K_5 consists of a Red C_5 and a Blue C_5. For more on Ramsey theory see [11] and for a dynamic update of the state-of-art of the Ramsey numbers see [20].

A related notion of size Ramsey number was introduced by Erdős, Faudree, Rousseau and Schelp in [5]. Given a graph H and an integer $r \geq 2$, let $G \to (H, r)$ denote the Ramsey property of a graph G, that is, every r-coloring of the edges of G results in a monochromatic copy of H. We then call G *a Ramsey graph* (with respect to H and r). *The size Ramsey number* of H (given r) is the smallest number of edges in a Ramsey graph G. It is shown in [5], with a proof attributed to V. Chvatál, that if $H = K_k$ then the size Ramsey number equals $\binom{R}{2}$, and, moreover, every connected Ramsey graph G with at most $\binom{R}{2}$ edges must be isomorphic to K_R, where $R = R(k;r)$.

In this paper we investigate yet another question in this direction, where instead of asking for the smallest number of vertices or edges in a Ramsey graph, we are interested in the smallest density, by which we mean the ratio of the number of edges to the number of vertices, or half of the average degree. This notion of graph density appears naturally, e.g., in the theory of random graphs (cf. [12]), or, in a modified form, in the Nash-Williams arboricity theorem (cf. [4], Theorem 3.5.4).

Similar to other applications, also in the context of Ramsey graphs, the notion of density has to be refined to yield a meaningful problem. Indeed, by adding sufficiently many isolated vertices to a Ramsey graph we obtain another Ramsey graph with density arbitrarily close to zero. To avoid this triviality, for a graph G we define its *density* as

$$m(G) = \max_{F \subseteq G} |E(F)|/|V(F)|$$

and, for a graph H and an integer $r \geq 2$, define the *Ramsey density* as

$$m_{inf}(H, r) = \inf\{m(G) : G \to (H)_r\}.$$

In particular, it is easy to verify that for the k-armed star S_k

$$m_{inf}(S_k, r) = \frac{r(k-1)+1}{r(k-1)+2}$$

and, using a nontrivial result on star-arboricity from [1] and [14], that $m_{inf}(P_3, 2) = 1$, where P_3 is a path of length three (on four vertices).

In [16] an analogous parameter was studied in the case of vertex coloring (see also [18, 23, 24]. Except for stars, edge coloring seems to be harder. Some initial, general bounds were obtained in [22] in the context of Ramsey properties of random graphs. In this paper we make a next step and prove a surprising result that for $H = K_k$ the sparsest in the above sense Ramsey graph is the complete graph with $R = R(k; r)$ vertices. For example, it follows that no graph G with density $m(G) < 2.5$ can be Ramsey with respect to $H = K_3$ and $r = 2$. From our result we can quickly derive the above mentioned fact that the size Ramsey number for $H = K_k$ equals $\binom{R}{2}$. All of this is presented in Section 2.

In Section 3 we study an on-line version of the size Ramsey number, related to a two-person, avoidance type game.

2. Ramsey Density for Complete Graphs

The result presented in this section has first appeared in [15].

Theorem 1. *For all $k \geq 3$ and $r \geq 2$,*

$$m_{inf}(K_k, r) = \frac{R(k; r) - 1}{2}.$$

Proof. Set $R = R(k; r)$ and note that $m(K_R) = (R-1)/2$. Thus, it remains to prove that

$$m(G) < (R-1)/2 \quad \Longrightarrow \quad G \not\to (K_k)_r. \tag{1}$$

The proof of (1) consists of two claims. Let $\chi(G)$ be the chromatic number of a graph G.

Claim 1. *If $\chi(G) \leq R - 1$, then $G \not\to (K_k)_r$.*

Indeed, partition $V(G)$ into $R-1$ independent sets $V_1, \ldots, V_{R-1}$ and color all edges between V_i and V_j by the color used to color the edge ij in a proper r-coloring of K_{R-1}.

Claim 2. *If $m(G) < (n-1)/2$, then $\chi(G) \le n-1$.*

Indeed, if $m(G) < (n-1)/2$, then every subgraph F of G contains a vertex of degree at most $n-2$. Hence, the so called coloring number (see, e.g., [4]), which is an upper bound for the chromatic number, is at most $n-1$. ∎

It is straightforward to deduce Chvatál's result from Theorem 1.

Corollary 1 [5]**.** *Fix $k \ge 3$ and $r \ge 2$ and set $R = R(k;r)$. The size Ramsey number of the complete graph K_k with respect to r colors equals $\binom{R}{2}$. Moreover, if a Ramsey graph has $\binom{R}{2}$ edges, then it consists of a copy of K_R and, possibly, a number of isolated vertices.*

Proof. The only way a graph G satisfies both $|E(G)| \le \binom{R}{2}$ and $m(G) \ge (R-1)/2$ is when it contains a subgraph isomorphic to K_R. ∎

Another simple consequence of Theorem 1 is the following property of Ramsey graphs. Let $\Delta(G)$ stand for the maximum degree in a graph G.

Corollary 2. *If $G \to (K_k)_r$ then $\Delta(G) \ge R-1$.*

Theorem 1 serves as a lemma in the proof of a sharp threshold for Ramsey properties of random graphs in [6], while Corollary 2 has been utilized in [8].

Remark 1. For a graph G, let us set $d(G) = |E(G)|/|V(G)|$ and call G *balanced* if $m(G) = d(G)$. As proved in [10], and independently in [19], for every graph G there exists a balanced supergraph $F \supseteq G$ with $d(F) = m(G)$. Hence, in the infimum defining $m_{inf}(H,r)$ one can restrict oneself to balanced graphs only. Then, in the case of $H = K_k$, Theorem 1 says that no balanced graph with n vertices and less than $(R-1)n/2$ edges is Ramsey.

Remark 2. All the above results can be routinely generalized to the off-diagonal case, when, given integers $k_1,\dots,k_r \ge 2$, one requires a copy of K_{k_i} in the ith color, for some $i \in \{1,2,\dots,r\}$.

3. A Ramsey Game

In this section we present yet another variant of the size Ramsey number, related to the following two-person, avoidance game $\mathcal{G}$, introduced by Beck in [3]. Let integers $k, r \geq 2$ be given. There are two players, Builder and Painter, who move on the originally empty graph with an unbounded number of vertices. In each move, Builder draws a new edge which is immediately colored by Painter. The goal of Builder is to force Painter to complete a monochromatic copy of K_k; the goal of Painter is opposite: to avoid it. The payoff to Painter is the number of edges colored until this happens. The higher the payoff, the better for Painter.

The on-line Ramsey number $\overline{R}(k;r)$ is the smallest payoff over all possible strategies of Builder, assuming Painter uses an optimal strategy. In the game theoretic terms, $\overline{R}(k;r)$ is the value of $\mathcal{G}$, that is the saddle point of the payoff matrix achieved at its equilibrium pair. As this is a full information, zero-sum game, an equilibrium pair does exists. Equivalently, $\overline{R}(k;r)$ is the smallest t for which Builder has a winning strategy in the related game $\mathcal{G}_t$, which is won by Builder if the payoff to Painter in $\mathcal{G}$ is at most t and lost otherwise. Again, $\mathcal{G}_t$ is a two-person, full information game with no ties, so one of the players must have a winning strategy. For basic notions of the Game Theory cf. [17].

Similar to the classical Ramsey numbers, it is hard to compute exact values of $\overline{R}(k;r)$. Besides the trivial $\overline{R}(2;2) = 1$, we have determined only one more instance, $\overline{R}(3;2)$, for which an obvious upper bound is $\binom{R(3;2)}{2} = 15$.

Proposition 1. $\overline{R}(3;2) = 8$.

Proof. By mimicking the proof of the bound $R(3;2) \leq 6$, in five moves Builder can force Painter to color by the same color three edges with the same endpoint. Then the next three edges force Painter to complete a monochromatic triangle.

Now we turn to the proof of the lower bound. We will describe a strategy of Painter to be followed for the first six moves which guarantees that the seventh edge is safe. Note that Painter is forced to complete a monochromatic triangle if and only if the current edge to be colored is a diagonal of a cycle C_4 whose two consecutive edges are Blue and the other two are Red, and each of these two pairs of monochromatic edges forms a triangle with this diagonal. Such colored copy of C_4 will be called *fatal*. An uncolored edge is called *dangerous* if it belongs to a copy of C_4 whose two consecutive edges are of the same color and the remaining edge is of the other color. For as long as she can, Painter will try to avoid creating a fatal C_4 as well as a vertex incident to three edges of the same color.

As long as at most five vertices are introduced to the game by Builder (that is, at most five vertices have positive degree in the current graph), the strategy of Painter is to follow a fixed proper coloring of K_5, which is a union of two pentagons, one Red, the other Blue. (Note that such a union does not contain a fatal C_4.) Otherwise, if the current edge e is not dangerous, then color it Red unless this would raise the maximum degree in Red to three or complete a Red triangle, in which cases color e Blue. If e is dangerous, then color it so as to avoid a fatal C_4, unless this would create a monochromatic triangle.

Now, let us see that this is indeed a successful strategy. If after six moves only at most five vertices are introduced then, clearly, the seventh edge is safe, no matter where it is. On the other hand, six edges cannot create two copies of C_4 on more than five vertices. Thus, it remains to consider the case when the graph has six edges, at least six (non-isolated) vertices and at most one copy of C_4. We claim that Painter can avoid making this C_4 fatal.

Suppose to the contrary, and consider the very moment when the fourth edge of the 4-cycle, call it $e = uv$, was chosen by Builder. Let us denote the other two vertices of that C_4 by w, z and assume that the edges uz and zw are of the same color, while vw is of the other color. According to her strategy, Painter colors e by the majority color, i.e., the color of uz and zw. It remains to explain that this move does not create a monochromatic triangle.

Indeed, e cannot form a triangle with a vertex other than w or z, because then we would have six edges on five vertices — a contradiction with our assumption. So, the only other possibility is the triangle uvz (uvw cannot be monochromatic), but only if all three edges zu, zv and zw are of the same color. According to the chosen strategy of Painter, this is impossible in Red before a first dangerous edge is colored.

Hence, suppose that all three, zu, zv and zw, are Blue, and denote by f this one of them which was colored last. How come f has been colored Blue? Only if coloring it Red would create a Red triangle or if vertex z had already degree two in Red. But either option yields seven edges altogether — a contradiction. ■

So, unlike the size Ramsey number, its on-line counterpart, $\overline{R}(k;r)$, can be smaller than the obvious $\binom{R(k;r)}{2}$. An interesting open question, suggested by V. Rödl, is the following. For clarity, let us restrict to just two colors.

Question. Is it true that $\overline{R}(k;2) = o(R^2(k;2))$ as $k \to \infty$?

At the moment we are unable to answer this question. Instead, we only give some weaker bounds and answer in positive the same question for the off-diagonal on-line Ramsey numbers with one parameter fixed and the other growing to ∞. Let $R(k,l)$ and $\overline{R}(k,l)$ be defined as $R(k;2)$ and $\overline{R}(k;2)$, but with the objective of either a Red copy of K_k or a Blue copy of K_l. (Note that, unless $k=2$, $\overline{R}(k;2) = \overline{R}(k,k)$ is not the same as $\overline{R}(k,2)$.)

Proposition 2. *For all integers $k, l \geq 2$ we have*

$$\overline{R}(k,l) \leq (k+l)\binom{k+l-2}{l-1}.$$

In particular, (i) $\overline{R}(k,k) \leq 2k\binom{2k-2}{k-1}$ *and* (ii) $\overline{R}(k,l) = o(R^2(k,l))$ *as* $k \to \infty$ *while* $l \geq 3$ *is fixed.*

Proof. We use induction on $k+l$. Note that $\overline{R}(k,2) = \binom{k}{2} \leq (k+2)k$ and suppose the inequality is true for all k' and l' such that $k'+l' < k+l$. Consider the following strategy of Builder: first draw

$$\binom{k+l-2}{l-1} - 1 = \binom{k+l-3}{l-1} + \binom{k+l-3}{l-2} - 1$$

edges from one vertex v; then stick to the Red color if Red was used by Painter on at least $\binom{k+l-3}{l-1}$ of these edges, and stick to Blue otherwise. Repeat with the respective parameter, k or l, decreased by 1, on the subgraph induced by the endpoints of the edges with the selected color incident to v. This strategy yields a recursive estimate

$$\begin{aligned} \overline{R}(k,l) &\leq \binom{k+l-2}{l-1} + \max\{\overline{R}(k-1,l), \overline{R}(k,l-1)\} \\ &\leq \binom{k+l-2}{l-1} + \overline{R}(k-1,l) + \overline{R}(k,l-1), \end{aligned}$$

which, by the induction assumption, completes the proof. Part (ii) follows, since $R(k,l) \geq c(k/logk)^{(l+1)/2}$ for some positive constant c (cf. [13]). ∎

Note that the obtained bound is almost the same as the bound on the Ramsey number $R(k,l)$ rising from the inductive proof of the Ramsey theorem. For $k=l$, this seemingly good upper bound on $\overline{R}(k;2)$ does not answer the Question, because of a large gap between this upper bound and the best known lower bound on $R(k;2)$. (Speaking of lower bounds, Beck [3] showed that $\overline{R}(k;2) \geq R(k;2)/2$ and no better lower bound is known.)

Note also that we can repeat the above proof mutatis mutandis for an arbitrary number of colors r, obtaining the bound

$$\overline{R}(k_1,\dots,k_r) \leq \left(\sum_{i=1}^{r} k_i\right) \binom{\sum_{i=1}^{r} k_i - r}{k_1-1,\dots,k_r-1}.$$

Alon and Rödl [2] have proved that $R(k,3,3) = \Omega(k^3/\log^c n)$ for some $c>0$, which, combined with the above, yields $\overline{R}(k,3,3) = o(R^2(k,3,3))$.

Remark 3. A randomized version of $\mathcal{G}$, where Builder generates the edges randomly, was studied in [7]. For a (deterministic) generalization, in which the Builder is restricted to build a graph belonging to a specific family of graphs, see [9].

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