Discussiones Mathematicae Graph Theory 25 (2005) 129–139

# THE CYCLE-COMPLETE GRAPH RAMSEY NUMBER $R(C_5, K_7)$

INGO SCHIERMEYER

Institut für Diskrete Mathematik und Algebra Technische Universität Bergakademie Freiberg 09596 Freiberg, Germany

e-mail: schierme@tu-freiberg.de

#### Abstract

The cycle-complete graph Ramsey number  $r(C_m, K_n)$  is the smallest integer N such that every graph G of order N contains a cycle  $C_m$  on m vertices or has independence number  $\alpha(G) \ge n$ . It has been conjectured by Erdős, Faudree, Rousseau and Schelp that  $r(C_m, K_n) = (m-1)(n-1) + 1$  for all  $m \ge n \ge 3$  (except  $r(C_3, K_3) = 6$ ). This conjecture holds for  $3 \le n \le 6$ . In this paper we will present a proof for  $r(C_5, K_7) = 25$ .

Keywords: Ramsey numbers, extremal graphs.

2000 Mathematics Subject Classification: 05C55, 05C35.

### 1. Introduction

We use [3] for terminology and notation not defined here and consider finite and simple graphs only.

For two graphs G and H, the Ramsey number r(G, H) is the smallest integer N such that every 2-colouring of the edges of the complete graph  $K_N$  contains a subgraph isomorphic to G in the first colour or a subgraph isomorphic to H in the second colour.

A cycle on *m* vertices will be denoted by  $C_m$  and the independence number of a graph by  $\alpha(G)$ . The cycle-complete graph Ramsey number  $r(C_m, K_n)$  is the smallest integer *N* such that for every graph *G* of order N,  $C_m \subset G$  or  $\alpha(G) \geq n$ . The graph  $(n-1)K_{m-1}$  shows that  $r(C_m, K_n) \geq (m-1)(n-1) + 1$  for all  $m \geq n \geq 3$ .

**Question 1** [5]. With n given, what is the smallest value of m such that

(1) 
$$r(C_m, K_n) = (m-1)(n-1) + 1$$
?

**Conjecture 1** [5]. With the only exception of  $r(C_3, K_3) = 6$ , formula (1) holds for all  $m \ge n \ge 3$ .

### 2. Results

The following observation is easily verified.

**Observation 1.** Formula (1) also holds for n = 1, 2 and all  $m \ge 3$ .

Conjecture 1 was confirmed for n = 3 in early work on Ramsey theory ([6], [12]), and it has been proved recently for n = 4 [14], n = 5 [2] and n = 6 [13].

$m \backslash n$	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	7	10	14	18	22	26	
5	9	13	17	21	25		
$\geq 6$	2m-1	3m-2	4m - 3	5m - 4			

Table 1. Exact Values of  $r(C_m, K_n)$ .

Bondy and Erdős [1] have proved that formula (1) holds if  $m \ge n^2 - 2$ . This was improved by Thomason [15] to  $m \ge n^2 - n - 1$  for all  $n \ge 4$  and further to  $m \ge n^2 - 2n$  for all  $n \ge 5$  in [13]. Recently, Nikiforov succeeded to show a lower bound which is linear in n.

**Theorem 1** [9].  $r(C_m, K_n) = (m-1)(n-1) + 1$  for all  $m \ge 4n+2$  and all  $n \ge 4$ .

Nikiforov has also posed the following challenging conjecture.

**Conjecture 2.** For every k there exists  $n_0 = n_0(k)$  such that for  $n > n_0$  and  $m > n^{1/k}$ ,

$$r(C_m, K_n) = (m-1)(n-1) + 1.$$

The known numbers for small values of m and n do not contradict this conjecture.

In [8] it has been proved that  $r(C_5, K_6) = 21$ . In this paper we will compute  $r(C_5, K_7)$ .

**Theorem 2.**  $r(C_5, K_7) = 25$ .

Moreover, the fact that  $r(C_5, K_6) = 21$  and  $r(C_5, K_7) = 25$ , justifies the following question.

**Question 2.** Does Formula (1) hold for all  $m \ge 5$ ?

## 3. Preliminary Results

For a vertex  $u \in V(G)$  let  $N_i(u) = \{v \in V(G) | d(u, v) = i\}$  and  $N_i^*(u) = \{v \in V(G) | d(u, v) \ge i\}$ . For given  $N_i(u)$  and  $N_i^*(u)$  let  $G_i = G[N_i(u)]$  and  $G_i^* = G[N_i^*(u)]$ .

**Lemma 1.** Let G be a C<sub>5</sub>-free graph. Then the graphs  $G_1$  and  $G_2$  are  $P_4$ -free for every vertex  $u \in V(G)$ .

**Proof.** If  $G_1 = G[N_1(u)]$  contains a  $P_4$ , then u is contained in a  $C_5$ , a contradiction. Hence,  $G_1$  is  $P_4$ -free.

Suppose now that  $G_2$  contains a  $P_4$  with vertices labeled  $w_1w_2w_3w_4$ . If  $N(u) \cap N(w_1) \cap N(w_4) \neq \emptyset$ , then there is a  $C_5$ , a contradiction. Hence we may assume that there are two vertices  $u_1, u_2 \in V(G_1)$  such that  $u_1w_1, u_2w_4 \in E(G)$ . Now consider the vertex  $w_2$ . If  $w_2v \in E(G)$  for a vertex  $v \in V(G_1) - \{u_1\}$ , then there is a  $C_5$ , a contradiction. Hence we may assume that  $w_2u_1 \in E(G)$ . Now consider the vertex  $w_3$ . Then  $w_3$  is always contained in a  $C_5$ , a contradiction. Hence,  $G_2$  is  $P_4$ -free.

The following lemma is an immediate consequence of Lemma 1.

**Lemma 2.** Let G be a  $C_5$ -free graph and  $u \in V(G)$ . Then the components of  $G_1$  and  $G_2$  are of the form  $K_1, K_2, K_3$  or  $K_{1,r}$  for  $r \ge 2$ .

Using Lemma 2 we obtain the following lemma.

**Lemma 3.** Let G be a C<sub>5</sub>-free graph with  $\alpha(G) \leq 6$ . Then

- (a)  $\alpha(G_2) \le 5 \text{ and } |V(G_2)| \le 15$ ,
- (b)  $\alpha(G_3^*) \leq 6 \alpha(G_1)$  and  $|V(G_3^*)| \leq 24 4\alpha(G_1)$ ,
- (c) If  $W \subset V(G_2)$ , then  $\alpha(G_2[W]) \geq \lceil \frac{|W|}{3} \rceil$ .

Using the assumption that G is  $C_5$ -free we obtain the following lemmas.

**Lemma 4.** Let G be a C<sub>5</sub>-free graph and  $F \subset G$  with  $F \cong K_4$ . Then  $d_F(v) \leq 1$  for all  $v \in V(G) - V(F)$ .

**Lemma 5.** Let G be a C<sub>5</sub>-free graph with |V(G)| = 25 and  $\alpha(G) \leq 6$ . If  $I \subset V(G)$  is independent with |I| = k,  $1 \leq k \leq 5$ , then  $|N(I)| \geq 3k + 1$ .

**Proof.** Suppose there is an independent set  $I \subset V(G)$  with  $|I| = k, 1 \leq k \leq 5$ , and  $|N(I)| \leq 3k$ . Let  $G' = G - (I \cup N(I))$ . Then  $|V(G')| \geq 25 - 4k = 4(7-k) - 3$ . Since G is  $C_5$ -free, we conclude by Table 1 and Observation 1 that  $\alpha(G') \geq 7 - k$ . Let J be an independent set of size  $\alpha(G') \geq 7 - k$  in G'. Then  $I \cup J$  is an independent set of size at least 7 in G, a contradiction.

The following two lemmas are easily verified using the fact that G is  $C_5$ -free.

**Lemma 6.** If  $F_i$  is a component of  $G_2$  with  $|V(F_i)| \ge 2$ , then  $|N(F_i) \cap N(u)| = 1$ .

**Lemma 7.** Let  $F_1, F_2$  be two components of  $G_1$ . If  $|V(F_2)| \ge 2$ , then  $N(F_1) \cap N(F_2) \cap V(G_2) = \emptyset$ .

**Lemma 8.** Let  $F \cong K_2$  be a component of  $G_2$  with  $V(F) = \{w_1, w_2\}$  and  $J = N(w_1) \cap N(w_2) \cap V(G_3)$ . Then J is independent.

**Proof.** Suppose J is not independent. Then there is an edge in  $G_3[J]$ , say xy. By lemma 6 there is a vertex  $v \in N(w_1) \cap N(w_2) \cap N(u)$ . But then  $C_5 \subseteq G[\{v, w_1, w_2, x, y\}]$ , a contradiction.

Jayawardene and Rousseau have determined all  $C_5$ -free graphs G with  $\alpha(G) = 3$  and order 11 and 12.

**Lemma 9** [8]. Let G be a graph with  $C_5 \not\subset G$  and  $\alpha(G) = 3$ . (a) If |V(G)| = 12, then  $3K_4 \subset G$ . (b) If |V(G)| = 11, then  $2K_4 \cup K_3 \subset G$ .

132

For a vertex  $u \in V(G)$ , an independent set  $I \subset V(G)$  of type  $(n_0, n_1, \ldots, n_{k-1}, n_k^*)$  is an independent set of size  $\sum_{i=0}^k n_i$ , which contains  $n_i$  vertices from  $G_i$ ,  $1 \leq i \leq k-1$ , and  $n_k^*$  vertices from  $G_k^*$ . Furthermore,  $n_0 = 1$  (0), if u is (not) contained in I.

**Lemma 10.** Let G be a graph with  $C_5 \not\subset G$ . Suppose  $G_2$  has five components  $F_1, F_2, \ldots, F_5$  with  $|V(F_i)| = 1$ ,  $1 \leq i \leq p$ ,  $|V(F_i)| = 2$ ,  $p+1 \leq i \leq q$ ,  $|V(F_i)| = 3$ ,  $q+1 \leq i \leq 5$ . Further there are vertices  $u_i \in V(G_1)$  such that  $G_2[N(u_i)] = F_i$  for  $p+1 \leq i \leq q$  and  $u_u u_j \in E(G)$  for  $p+1 \leq i \leq q$ . Suppose q > p and  $|V(G_3^*) - (\bigcup_{i=1}^p N(F_i))| \geq q-p+1$ . Then there exists an independent set of type (1, 0, 5, 1) or (1, 0, 4, 2).

**Proof.** Suppose there is no independent set of type (1, 0, 5, 1). Since  $|V(G_3^*) - (\bigcup_{i=1}^p N(F_i))| \ge q - p + 1$  there exists i with  $p + 1 \le i \le q$ , say i = p + 1, and two vertices  $v_1, v_2 \in V(G_3)$  with  $v_1 w_i, v_2 w_i \in E(G)$  for i = 1, 2, where  $V(F_{p+1}) = \{w_1, w_2\}$ . By Lemma 8,  $v_1 v_2 \notin E(G)$ . Since  $G_1[\{u_{p+1}, \ldots, u_q\}]$  is complete and G is  $C_5$ -free, we have  $N(v_i) \cap V(F_j) = \emptyset$  for i = 1, 2 and  $p+2 \le j \le q$ . But then  $v_1, v_2$  are contained in an independent set I containing  $F_i$  for  $1 \le i \le p$  and a vertex from each  $F_i$  for  $p+2 \le i \le 5$ . Hence I is an independent set of type (1, 0, 4, 2), a contradiction.

**Lemma 11** [8]. Let G be a graph with  $\delta(G) \ge 4$  and  $C_5 \not\subset G$ . Then  $\alpha(G) \ge \Delta(G)$ .

## 4. Proof of Theorem 2

Let |V(G)| = 25. By Lemma 5 and Lemma 11 we may assume that  $4 \leq \delta(G) \leq \Delta(G) \leq 6$ . We distinguish these three cases.

1.  $\Delta(G) = 4$ 

Then G is 4-regular. Moreover, by Lemma 5, if d(u, v) = 2 for two vertices  $u, v \in V(G)$ , then

$$(2) \qquad \qquad |N(u) \cap N(v)| = 1.$$

Hence G contains no induced  $K_4 - e$  and no induced  $C_4$ . For the neighbourhood of a vertex u we distinguish the following cases.

#### *Case* 1. $\alpha(G_1) = 4$

By (2) we conclude that  $|V(G_2)| = 3 \cdot 4 = 12$ . Since  $\alpha(G_2) \leq 6, F_i = G[N_{G_2}(u_i)] \cong K_3$  for some *i* with  $1 \leq i \leq 4$ . But then  $\alpha(G[N_{G_3}(F_i)]) = 3$ . Hence there are three independent vertices in  $N_{G_3}(F_i)$  which are contained together with  $\{u_1, u_2, u_3, u_4\}$  in an independent set of type (0, 4, 0, 3), a contradiction.

*Case* 2.  $\alpha(G_1) = 3$ 

Let  $E(G_1) = \{u_1u_2\}$  and  $F_i = G[N_{G_2}(u_i)]$  with  $V(F_i) = \{w_{i1}, w_{i2}\}$  for i = 1, 2. Suppose  $F_i = G[N_{G_2}(u_i)] \cong K_3$  for some i with  $3 \le i \le 4$ , say i = 3. Then  $|N_{G_3}(F_3)| = 3$  and  $\alpha(G[N_{G_3}(F_3)]) = 3$ . By (2)  $d_{F_1 \cup F_2}(v) \le 1$  for all vertices  $v \in N_{G_3}(F_3)$ . Hence we may assume that  $N_{G_3}(w_{11}) \cap N_{G_3}(F_3) = \emptyset$ . But then  $\{u_2, u_3, u_4, w_{11}\} \cup N_{G_3}(F_3)$  is an independent set of type (0, 3, 1, 3), a contradiction.

Suppose now  $\alpha(G[N_{G_2}(u_i)]) \geq 2$  for  $3 \leq i \leq 4$ . Since  $\alpha(G_2) \leq 6$ , we conclude  $w_{11}w_{12}, w_{21}w_{22} \in E(G)$ . Let  $N_{G_3}(w_{ij}) = \{x_{ij1}x_{ij2}\}$  for  $1 \leq i, j \leq 2$ . Then there are three independent vertices in  $N_{G_3}(w_{ij})$  for ij = 12, 21, 22. These three vertices are contained together with  $w_{11}$  and  $u_2, u_3, u_4$  in an independent set of type (0, 3, 1, 3), a contradiction.

For the remaining part we may assume that  $|E(G[N(v)])| \ge 2$  for every vertex  $v \in V(G)$ .

*Case* 3.  $\alpha(G_1) = 2$ 

Let  $E(G_1) = \{u_1u_2, u_3u_4\}$ . Then  $N_{G_2}(u_i) = V(F_i) = \{w_{i1}, w_{i2}\}$  with  $w_{i1}w_{i2} \in E(G)$  for  $1 \leq i \leq 4$ . As above we conclude that there are three independent vertices in  $N_{G_3}(w_{ij})$  for ij = 32, 41, 42 which are contained together with  $u_2, u_4, w_{11}$  and  $w_{31}$  in an independent set of type (0, 2, 2, 3), a contradiction.

Case 4.  $\alpha(G_1) = 2$ 

Let  $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$ . We may assume that  $G[N(v)] \cong K_3 \cup K_1$  for every vertex  $v \in V(G)$ . Choose an edge uw with  $N(u) = \{w, u_1, u_2, u_3\}$  and  $N(w) = \{u, w_1, w_2, w_3\}$  such that  $G[\{u_1, u_2, u_3\}] \cong K_3 \cong G[\{w_1, w_2, w_3\}]$ . Then there exist vertices  $x_i$  and  $y_i$  for  $1 \le i \le 3$  such that  $u_ix_i, w_iy_i \in E(G)$ . Let  $V(G_1) = \{u_1, u_2, u_3, w_1, w_2, w_3\}$  and  $V(G_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ . Hence  $\alpha(G_2) \le 5$ . If  $\alpha(G_2) = 5$ , then there is an independent set of type (1, 1, 5), a contradiction. Hence we may assume  $\alpha(G_2) \le 4$ . Since  $E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$  contains only independent edges, we may assume that  $|E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})| = 2$  (else consider  $u_1, x_1$  instead of u, w). We may assume that  $x_2, y_2$  and  $x_3, y_3$  are contained in a  $K_4$ . Hence  $|V(G_3)| = 2 \cdot 2 + 2 \cdot 3 = 10$ . Therefore,  $V(G_4^*) \neq \emptyset$  and so there is an independent set of type (1, 1, 4, 0, 1) (with respect to the edge uw), a contradiction.

135

2.  $\Delta(G) = 5$ 

*Case* 1.  $\alpha(G_1) = 5$ 

Since  $\alpha(G_1) = 5$  we conclude that  $\alpha(G_3^*) \leq 1$  and thus  $|V(G_2)| \geq 25 - (1 + 5 + 4) = 15$ . By Lemma 3 we have  $|V(G_2)| \leq 15$ . Therefore,  $G_2 \cong 5K_3$  and  $G_3 \cong K_4$ . Hence by Lemma 4 every vertex of  $G_3$  is contained in an independent set of type (1, 0, 5, 1), a contradiction.

Case 2.  $\alpha(G_1) = 4$ Then  $|V(G_3^*)| \le 8$  by Lemma 3 and thus  $|V(G_2)| \ge 11$ .

Case 2.1.  $E(G_1) = \{u_1u_2\}$ 

Let  $U_1 = \{u_1, u_2\}$  and  $U_2 = \{u_3, u_4, u_5\}$ . By Lemma 5 we conclude that  $|N_{G_2}(U_2)| \geq 9$ . Since  $\alpha(G_2[N(U_1)]) \geq 2$ , we get  $\alpha(G_2[N(U_1)]) = 2$  and  $\alpha(G_2[N(U_2)]) = 3$  by Lemma 3 and Lemma 7. Moreover,  $G_2[N(U_2)] \cong 3K_3$  by Lemma 6.

Let  $J = \{u_3, u_4, u_5\}$  and  $G' = G - (J \cup N(J))$ . Then |V(G')| = 12and  $\alpha(G') \geq 3$  by Table 1. Since  $I \cup J$  is an independent set in G with  $|I \cup J| \leq \alpha(G) \leq 6$  for every independent set I of G', we conclude  $\alpha(G') = 3$ . Hence  $3K_4 \subset G'$  by Lemma 9. Therefore,  $G_2[N(u_1)] = \{F_1, F_2\} \cong \{K_3, K_3\}$ and we follow the arguments of Case 1 above.

Case 2.2.  $E(G_1) = \{u_1u_2, u_1u_3\}$ 

Let  $U_1 = \{u_1, u_2, u_3\}$  and  $U_2 = \{u_4, u_5\}$ . Similarly as in the previous case we conclude that  $\alpha(G_2[N(U_1)]) = 3$ ,  $\alpha(G_2[N(U_2)]) = 2$  and  $G_2[N(U_2)] \cong 2K_3$ . Let  $F_1, F_2, F_3$  be the three components of  $G_2[N(U_1)]$  with  $F_i = G_2[N(u_i)]$  for i = 1, 2, 3. Let  $J = \{u_1, u_4, u_5\}$  and  $G' = G - (J \cup N(J))$ . Then  $11 \leq |V(G')| \leq 12$  and thus  $3K_4 \subset G'$  or  $2K_4 \cup K_3 \subset G'$  by Lemma 9. Since  $N_{G_3}(F_1)$  and  $F_2, F_3$  are independent,  $N_{G_3}(F_i) \cong K_3$  for i = 2 or 3. But then  $H_i = N_{G_3}(F_i) \cong K_4$  is contained in a  $K_4 \subset G'$  for i = 2 or 3, a contradiction.

Case 2.3.  $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4\}$ 

Case 2.4.  $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5\}$ 

For both cases let  $J = \{u_2, u_3, u_4\}$  and  $G' = G - (J \cup N(J))$ . By Lemma 5 we need  $|J \cup N(J)| \ge 13$ . Since  $4 \le \alpha(G_2) \le 5$  we conclude that

 $G[N_{G_2}(u_i)] \cong K_3$  for some  $i, 2 \leq i \leq 4$ . Now we can follow the proof of Case 4 (by considering  $u_i$  with  $d(u_i) = 5$  instead of u).

*Case* 3.  $\alpha(G_1) = 3$ 

Case 3.1.  $E(G_1) = \{u_1u_2, u_3u_4\}$ 

As in previous cases we conclude that  $\alpha(G_2) = 5$  and  $G_2[N(u_5)] \cong K_3$ . Suppose  $G_2$  is isomorphic to one of  $\{K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}, K_{n_5}\}$  with  $2 \leq n_1 \leq n_2 \leq 3, 2 \leq n_3 \leq n_4 \leq 3, n_5 = 3$ . If  $d(w_1) = 4 = d(w_3)$ , let  $J = \{u, w_1, w_3\}$  and  $G' = G - (J \cup N(J))$ . Then |V(G')| = 11 and thus  $2K_4 \cup K_3 \subset G'$  by Lemma 9. Since  $F_2, F_4$  and  $F_5$  are independent and  $|V(F_i)| \geq 2$  for i = 2, 4, 5, there exist  $F_i, i = 2, 4$  or 5, such that  $F_i$  is contained in a  $K_4 \subset G' - \{u_i\}$ , a contradiction. Suppose  $G_2$  is isomorphic to one of  $\{K_1, K_1, K_{n_3}, K_{n_4}, K_{n_5}\}$  with  $2 \leq n_3 \leq n_4 \leq 3, n_5 = 3$ . Then  $|V(G_3)| \geq 25 - 6 - (5 + n_3 + n_4) = 14 - n_3 - n_4$ . Hence  $|V(G_3) - (N(F_1) \cup N(F_2))| \geq 14 - n_3 - n_4 - 6 = 8 - n_3 - n_4$ . Now by Lemma 10 there is an independent set of type (1, 0, 5, 1) or (1, 0, 4, 2), a contradiction.

Finally suppose that  $G_2$  is isomorphic to  $\{K_1, K_1, K_1, K_1, K_3\}$ . Let  $w_1$ ,  $w_2, w_3, w_4 \in V(G_2)$  be four independent vertices with  $N_{G_2}(u_1) = N_{G_2}(u_2) = \{w_1, w_2\}$  and  $N_{G_2}(u_3) = N_{G_2}(u_4) = \{w_3, w_4\}$ . If there is a vertex  $v \in V(G_3^*)$  with  $v \notin N(w_i)$  for  $1 \le i \le 4$ , then v is contained in an independent set of type (1, 0, 5, 1) by Lemma 4, a contradiction.

Hence we may assume that  $V(G_3^*) = V(G_3) \subset N(w_1) \cup N(w_2) \cup N(w_3) \cup N(w_4)$ . Furthermore,  $d_{G_3}(w_i) = 3$  for  $1 \le i \le 4$ . Let  $H_i = G[N_{G_3}(w_i)]$  for  $1 \le i \le 4$ . Since  $|V(G_3)| = 12$  we have  $V(H_i) \cap V(H_j) = \emptyset$  for  $1 \le i < j \le 4$ . Moreover, there are no edges between  $V(H_i)$  and  $V(H_{i+1})$  for i = 1, 3, since G contains no  $C_5$ . Suppose  $\alpha(G_2[H_i \cup H_{i+1}]) \ge 3$ , then there are three independent vertices in  $V(H_i) \cup V(H_{i+1})$ , which are contained together with  $u_i, u_5, w_{4-i}, w_{5-i}$  in an independent set of type (0, 2, 2, 3), a contradiction.

Hence we may assume that  $G_2[H_i \cup H_{i+1}] \cong 2K_3$  for i = 1, 3. Now any two vertices  $v_1 \in V(H_1)$  and  $v_2 \in V(H_2)$  are contained in an independent set of size four in  $G_3$  by Lemma 4. Hence  $4 \leq \alpha(G_3) \leq 3$  by Lemma 3, a contradiction.

Case 3.2.  $E(G_1) = \{u_1u_2, u_1u_3, u_4u_5\}$ See Case 4.

Case 3.3.  $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$ 

We first conclude that  $\alpha(G_2) = 5$ . Hence by Lemma 5 we get  $G_2[N(u_i)] \cong K_3$  for i = 4, 5. We have  $1 \leq d_{G_2}(u_1) \leq d_{G_2}(u_2) \leq d_{G_2}(u_3) \leq 2$ . Let  $F_i = G_2[N(u_i)]$  for  $1 \leq i \leq 3$  and  $J = \{u_1, u_4, u_5\}$ . If  $d_{G_2}(u_1) = 1$ , then  $G' = G[V(G) - (J \cup N(J))]$  has |V(G')| = 12. So  $3K_4 \subset G'$  by Lemma 9. Since  $N_{G_3}(F_1), V(F_2)$  and  $V(F_3)$  are independent,  $N_{G_3}(F_1)$  is contained in a  $K_4 \subset G - F_1$ . Hence there is a  $C_5$ , a contradiction.

137

If  $d_{G_2}(u_i) = 2$  for  $1 \leq i \leq 3$ , then |V(G')| = 11. So  $2K_4 \cup K_3 \subset G'$  by Lemma 9. Thus  $F_i \cong K_2$  is contained in a  $K_4 \subset G - u_i$  for some  $i, 2 \leq i \leq 3$ . Hence there is a  $C_5$ , a contradiction.

*Case* 4.  $\alpha(G_1) = 2$ 

Then  $G_1 = K_3 \cup K_2$ . Let  $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3, u_4u_5\}.$ 

Suppose first  $N_{G_2}(u_4) = N_{G_2}(u_5) = \{w_4, w_5\}$  for two vertices  $w_4, w_5 \in V(G_2)$ . Let  $F_i = G_2[N(u_i)]$  for  $1 \leq i \leq 3$  and set  $F_i = \{w_i\}$  for i = 4, 5. Now let  $H_i = G_3[N(F_i)]$  for  $1 \leq i \leq 5, J = \{u, w_4, w_5\}$  and  $G' = G - (J \cup N(J))$ . Then  $11 \leq |V(G')| \leq 12$  by Lemma 5. Suppose |V(G')| = 12. Then  $3K_4 \subset G'$  by Lemma 9. Thus  $G[F_i \cup H_i] \cong K_4$  for  $1 \leq i \leq 3$ . Since there is no  $C_5$ , we have  $|F_i| = 1$  and  $|H_i| = 3$  for  $1 \leq i \leq 3$ . We may assume  $|V(H_4)| = 2$  and  $|V(H_5)| = 3$ . Thus  $H_i \cong K_3$  for i = 1, 2, 3 and 5 and  $H_4 \cong K_2$ . Since  $E(H_4, H_5) = \emptyset$ , there is always an independent set with four vertices, one from  $H_2, H_3, H_4$  and  $H_5$ . Together with  $w_1, u_2$  and  $u_4$  this gives an independent set of type (0, 2, 1, 4), a contradiction. Suppose now |V(G')| = 11. Then  $2K_4 \cup K_3 \subset G'$  by Lemma 9. We can follow the arguments above and may assume that  $|V(F_3) \cup V(H_3)| = K_3$ . Again we can find an independent set of type (0, 2, 1, 4) as above, a contradiction.

Suppose next  $F_i = G[N_{G_2}(u_i)]$  for i = 4, 5 with  $|V(F_i)| \ge 2$  for two independent components  $F_4$  and  $F_5$ . Furthermore,  $F_i = G[N_{G_2}(u_i)]$  for i =1, 2, 3, since  $\alpha(G_2) = 5$ . We have  $1 \le |V(F_1)| \le |V(F_2)| \le |V(F_3)| \le 2$ . If  $|V(F_i)| = 1$  (i.e.,  $F_i = \{w_i\}$ ) for some i with  $1 \le i \le 3$ , then  $d_{G_3}(w_i) = 3$ , else we would be in a previous case.

Suppose there are two vertices  $w_1 \in V(F_1)$  and  $w_2 \in V(H_2)$  with  $d(w_i) = 4, 1 \leq i \leq 2$ . Let  $J = \{u, w_1, w_2\}$  and  $G' = G - (J \cup N(J))$ . Then |V(G')| = 11 and  $2K_4 \cup K_3 \subset G'$  by Lemma 9. Thus  $F_i$  is contained in a  $K_4 \subset G' - \{u_i\}$  for some  $i, 4 \leq i \leq 5$ . But then there is a  $C_5$ , a contradiction. Hence we may assume that  $V(F_i) = \{w_{i1}, w_{i2}\}$  for i = 2, 3 and  $d_{G_3}(w_{ij}) = 3$  for i = 2, 3 and  $1 \leq j \leq 2$ . But then  $|V(G)| \geq 1 + 5 + (1 + 2 \cdot 2 + 2 \cdot 2) + 4 \cdot 3 = 27 > 25$ , a contradiction.

#### 3. $\Delta(G) = 6$

*Case* 1.  $\alpha(G_1) = 6$ 

Since  $\alpha(G_1) = 6$  we conclude that  $V(G_3^*) = \emptyset$  and thus  $15 \ge |V(G_2)| = 25 - 7 = 18$  by Lemma 3, a contradiction.

*Case* 2.  $\alpha(G_1) = 5$ 

Then  $E(G_1) = \{u_1u_2, u_1u_3, \ldots, u_1u_r\}, 2 \leq r \leq 6$ . Since  $\alpha(G_1) = 5$  we conclude by Lemma 3 (b) that  $|V(G_3^*)| \leq 4$  and thus  $|V(G_2)| \geq 25 - 7 - 4 = 14$ . Then  $\alpha(G_2) \geq 5$  by Lemma 3 (c). Thus  $\alpha(G_2^*) = 5$  and  $G_2 \cong 5K_3$  or  $G_2 \cong 4K_3 \cup K_2$ . By Lemma 6 we conclude  $|V(G_1)| \leq 5 < 6$ , a contradiction.

Case 3.  $\alpha(G_1) \leq 4$ 

Using Lemma 2 and Lemma 7 we can show that  $\alpha(G_2) \ge 6$  and thus there is an independent set of type (1, 0, 6), a contradiction.

## References

- J.A. Bondy and P. Erdős, Ramsey numbers for cycles in graphs, J. Combin. Theory (B) 14 (1973) 46–54.
- [2] B. Bollobás, C.J. Jayawardene, Z.K. Min, C.C. Rousseau, H.Y. Ru and J. Yang, On a conjecture involving cycle-complete graph Ramsey numbers, Australas. J. Combin. 22 (2000) 63–72.
- [3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).
- [4] V. Chvátal and P. Erdős, A note on hamiltonian circuits, Discrete Math. 2 (1972) 111–113.
- [5] P. Erdős, R.J. Faudree, C.C. Rousseau and R.H. Schelp, On cycle-complete graph Ramsey numbers, J. Graph Theory 2 (1978) 53–64.
- [6] R.J. Faudree and R.H. Schelp, All Ramsey numbers for cycles in graphs, Discrete Math. 8 (1974) 313–329.
- [7] C.J. Jayawardene and C.C. Rousseau, The Ramsey number for a quarilateral versus a complete graph on six vertices, Congr. Numer. 123 (1997) 97–108.
- [8] C.J. Jayawardene and C.C. Rousseau, The Ramsey Number for a Cycle of Length Five vs. a Complete Graph of Order Six, J. Graph Theory 35 (2000) 99–108.

- [9] V. Nikiforov, *The cycle-complete graph Ramsey numbers*, preprint 2003, Univ. of Memphis.
- [10] S.P. Radziszowski, Small Ramsey numbers, Elec. J. Combin. 1 (1994) DS1.
- [11] S.P. Radziszowski and K.-K. Tse, A Computational Approach for the Ramsey Numbers  $R(C_4, K_n)$ , J. Comb. Math. Comb. Comput. **42** (2002) 195–207.
- [12] V. Rosta, On a Ramsey Type Problem of J.A. Bondy and P. Erdős, I & II, J. Combin. Theory (B) 15 (1973) 94–120.
- [13] I. Schiermeyer, All Cycle-Complete Graph Ramsey Numbers  $r(C_m, K_6)$ , J. Graph Theory 44 (2003) 251–260.
- [14] Y.J. Sheng, H.Y. Ru and Z.K. Min, The value of the Ramsey number  $R(C_n, K_4)$  is 3(n-1)+1  $(n \ge 4)$ , Australas. J. Combin. **20** (1999) 205–206.
- [15] A. Thomason, private communication.

Received 6 November 2003 Revised 16 February 2005