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THE CYCLE-COMPLETE GRAPH RAMSEY NUMBER $R(C_5, K_7)$

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Abstract

The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that every graph G of order N contains a cycle C_m on m vertices or has independence number $\alpha(G) \geq n$. It has been conjectured by Erdős, Faudree, Rousseau and Schelp that $r(C_m, K_n) = (m-1)(n-1)+1$ for all $m \geq n \geq 3$ (except $r(C_3, K_3) = 6$). This conjecture holds for $3 \leq n \leq 6$. In this paper we will present a proof for $r(C_5, K_7) = 25$.

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1. Introduction

We use [3] for terminology and notation not defined here and consider finite and simple graphs only.

For two graphs G and H, the Ramsey number $r(G, H)$ is the smallest integer N such that every 2-colouring of the edges of the complete graph K_N contains a subgraph isomorphic to G in the first colour or a subgraph isomorphic to H in the second colour.

A cycle on m vertices will be denoted by C_m and the independence number of a graph by $\alpha(G)$. The cycle-complete graph Ramsey number $r(C_m, K_n)$ is the smallest integer N such that for every graph G of order

N, $C_m \subset G$ or $\alpha(G) \geq n$. The graph $(n-1)K_{m-1}$ shows that $r(C_m, K_n) \geq (m-1)(n-1)+1$ for all $m \geq n \geq 3$.

Question 1 [5]**.** With n given, what is the smallest value of m such that

$$r(C_m, K_n) = (m-1)(n-1)+1 \ ? \tag{1}$$

Conjecture 1 [5]**.** With the only exception of $r(C_3, K_3) = 6$, formula (1) holds for all $m \geq n \geq 3$.

2. Results

The following observation is easily verified.

Observation 1. Formula (1) also holds for $n = 1, 2$ and all $m \geq 3$.

Conjecture 1 was confirmed for $n = 3$ in early work on Ramsey theory ([6], [12]), and it has been proved recently for $n = 4$ [14], $n = 5$ [2] and $n = 6$ [13].

Table 1. Exact Values of $r(C_m, K_n)$.

$m \backslash n$	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	7	10	14	18	22	26	
5	9	13	17	21	25		
≥ 6	$2m-1$	$3m-2$	$4m-3$	$5m-4$			

Bondy and Erdős [1] have proved that formula (1) holds if $m \geq n^2 - 2$. This was improved by Thomason [15] to $m \geq n^2 - n - 1$ for all $n \geq 4$ and further to $m \geq n^2 - 2n$ for all $n \geq 5$ in [13]. Recently, Nikiforov succeeded to show a lower bound which is linear in n.

Theorem 1 [9]**.** $r(C_m, K_n) = (m-1)(n-1)+1$ *for all* $m \geq 4n+2$ *and all* $n \geq 4$.

Nikiforov has also posed the following challenging conjecture.

Conjecture 2. For every k there exists $n_0 = n_0(k)$ such that for $n > n_0$ and $m > n^{1/k}$,

$$r(C_m, K_n) = (m-1)(n-1)+1.$$

The known numbers for small values of m and n do not contradict this conjecture.

In [8] it has been proved that $r(C_5, K_6) = 21$. In this paper we will compute $r(C_5, K_7)$.

Theorem 2. $r(C_5, K_7) = 25$.

Moreover, the fact that $r(C_5, K_6) = 21$ and $r(C_5, K_7) = 25$, justifies the following question.

Question 2. Does Formula (1) hold for all $m \geq 5$?

3. Preliminary Results

For a vertex $u \in V(G)$ let $N_i(u) = \{v \in V(G) | d(u, v) = i\}$ and $N_i^*(u) = \{v \in V(G) | d(u, v) \geq i\}$. For given $N_i(u)$ and $N_i^*(u)$ let $G_i = G[N_i(u)]$ and $G_i^* = G[N_i^*(u)]$.

Lemma 1. *Let G be a C_5-free graph. Then the graphs G_1 and G_2 are P_4-free for every vertex $u \in V(G)$.*

Proof. If $G_1 = G[N_1(u)]$ contains a P_4, then u is contained in a C_5, a contradiction. Hence, G_1 is P_4-free.

Suppose now that G_2 contains a P_4 with vertices labeled $w_1w_2w_3w_4$. If $N(u) \cap N(w_1) \cap N(w_4) \neq \emptyset$, then there is a C_5, a contradiction. Hence we may assume that there are two vertices $u_1, u_2 \in V(G_1)$ such that $u_1w_1, u_2w_4 \in E(G)$. Now consider the vertex w_2. If $w_2v \in E(G)$ for a vertex $v \in V(G_1) - \{u_1\}$, then there is a C_5, a contradiction. Hence we may assume that $w_2u_1 \in E(G)$. Now consider the vertex w_3. Then w_3 is always contained in a C_5, a contradiction. Hence, G_2 is P_4-free. ■

The following lemma is an immediate consequence of Lemma 1.

Lemma 2. *Let G be a C_5-free graph and $u \in V(G)$. Then the components of G_1 and G_2 are of the form K_1, K_2, K_3 or $K_{1,r}$ for $r \geq 2$.*

Using Lemma 2 we obtain the following lemma.

Lemma 3. *Let G be a C_5-free graph with $\alpha(G) \leq 6$. Then*

(a) $\alpha(G_2) \leq 5$ *and* $|V(G_2)| \leq 15$,

(b) $\alpha(G_3^*) \leq 6 - \alpha(G_1)$ *and* $|V(G_3^*)| \leq 24 - 4\alpha(G_1)$,

(c) *If* $W \subset V(G_2)$, *then* $\alpha(G_2[W]) \geq \lceil \frac{|W|}{3} \rceil$.

Using the assumption that G is C_5-free we obtain the following lemmas.

Lemma 4. *Let G be a C_5-free graph and $F \subset G$ with $F \cong K_4$. Then $d_F(v) \leq 1$ for all $v \in V(G) - V(F)$.*

Lemma 5. *Let G be a C_5-free graph with $|V(G)| = 25$ and $\alpha(G) \leq 6$. If $I \subset V(G)$ is independent with $|I| = k$, $1 \leq k \leq 5$, then $|N(I)| \geq 3k + 1$.*

Proof. Suppose there is an independent set $I \subset V(G)$ with $|I| = k$, $1 \leq k \leq 5$, and $|N(I)| \leq 3k$. Let $G' = G - (I \cup N(I))$. Then $|V(G')| \geq 25 - 4k = 4(7-k) - 3$. Since G is C_5-free, we conclude by Table 1 and Observation 1 that $\alpha(G') \geq 7 - k$. Let J be an independent set of size $\alpha(G') \geq 7 - k$ in G'. Then $I \cup J$ is an independent set of size at least 7 in G, a contradiction. ■

The following two lemmas are easily verified using the fact that G is C_5-free.

Lemma 6. *If F_i is a component of G_2 with $|V(F_i)| \geq 2$, then $|N(F_i) \cap N(u)| = 1$.*

Lemma 7. *Let F_1, F_2 be two components of G_1. If $|V(F_2)| \geq 2$, then $N(F_1) \cap N(F_2) \cap V(G_2) = \emptyset$.*

Lemma 8. *Let $F \cong K_2$ be a component of G_2 with $V(F) = \{w_1, w_2\}$ and $J = N(w_1) \cap N(w_2) \cap V(G_3)$. Then J is independent.*

Proof. Suppose J is not independent. Then there is an edge in $G_3[J]$, say xy. By lemma 6 there is a vertex $v \in N(w_1) \cap N(w_2) \cap N(u)$. But then $C_5 \subseteq G[\{v, w_1, w_2, x, y\}]$, a contradiction. ■

Jayawardene and Rousseau have determined all C_5-free graphs G with $\alpha(G) = 3$ and order 11 and 12.

Lemma 9 [8]**.** *Let G be a graph with $C_5 \not\subseteq G$ and $\alpha(G) = 3$.*

(a) *If* $|V(G)| = 12$, *then* $3K_4 \subset G$.

(b) *If* $|V(G)| = 11$, *then* $2K_4 \cup K_3 \subset G$.

For a vertex $u \in V(G)$, an independent set $I \subset V(G)$ of type $(n_0, n_1, \dots, n_{k-1}, n_k^*)$ is an independent set of size $\sum_{i=0}^{k} n_i$, which contains n_i vertices from G_i, $1 \le i \le k-1$, and n_k^* vertices from G_k^*. Furthermore, $n_0 = 1$ (0), if u is (not) contained in I.

Lemma 10. *Let G be a graph with $C_5 \not\subset G$. Suppose G_2 has five components $F_1, F_2, \dots, F_5$ with $|V(F_i)| = 1$, $1 \le i \le p, |V(F_i)| = 2$, $p+1 \le i \le q$, $|V(F_i)| = 3$, $q+1 \le i \le 5$. Further there are vertices $u_i \in V(G_1)$ such that $G_2[N(u_i)] = F_i$ for $p+1 \le i \le q$ and $u_u u_j \in E(G)$ for $p+1 \le i \le q$. Suppose $q > p$ and $|V(G_3^*) - (\cup_{i=1}^{p} N(F_i))| \ge q-p+1$. Then there exists an independent set of type $(1,0,5,1)$ or $(1,0,4,2)$.*

Proof. Suppose there is no independent set of type $(1,0,5,1)$. Since $|V(G_3^*) - (\cup_{i=1}^{p} N(F_i))| \ge q-p+1$ there exists i with $p+1 \le i \le q$, say $i = p+1$, and two vertices $v_1, v_2 \in V(G_3)$ with $v_1 w_i, v_2 w_i \in E(G)$ for $i = 1, 2$, where $V(F_{p+1}) = \{w_1, w_2\}$. By Lemma 8, $v_1 v_2 \notin E(G)$. Since $G_1[\{u_{p+1}, \dots, u_q\}]$ is complete and G is C_5-free, we have $N(v_i) \cap V(F_j) = \emptyset$ for $i = 1, 2$ and $p+2 \le j \le q$. But then v_1, v_2 are contained in an independent set I containing F_i for $1 \le i \le p$ and a vertex from each F_i for $p+2 \le i \le 5$. Hence I is an independent set of type $(1,0,4,2)$, a contradiction. ∎

Lemma 11 [8]**.** *Let G be a graph with $\delta(G) \ge 4$ and $C_5 \not\subset G$. Then $\alpha(G) \ge \Delta(G)$.*

4. Proof of Theorem 2

Let $|V(G)| = 25$. By Lemma 5 and Lemma 11 we may assume that $4 \le \delta(G) \le \Delta(G) \le 6$. We distinguish these three cases.

1. $\Delta(G) = 4$

Then G is 4-regular. Moreover, by Lemma 5, if $d(u, v) = 2$ for two vertices $u, v \in V(G)$, then

$$|N(u) \cap N(v)| = 1. \tag{2}$$

Hence G contains no induced $K_4 - e$ and no induced C_4. For the neighbourhood of a vertex u we distinguish the following cases.

Case 1. $\alpha(G_1) = 4$
By (2) we conclude that $|V(G_2)| = 3 \cdot 4 = 12$. Since $\alpha(G_2) \leq 6, F_i = G[N_{G_2}(u_i)] \cong K_3$ for some i with $1 \leq i \leq 4$. But then $\alpha(G[N_{G_3}(F_i)]) = 3$. Hence there are three independent vertices in $N_{G_3}(F_i)$ which are contained together with $\{u_1, u_2, u_3, u_4\}$ in an independent set of type $(0, 4, 0, 3)$, a contradiction.

Case 2. $\alpha(G_1) = 3$
Let $E(G_1) = \{u_1u_2\}$ and $F_i = G[N_{G_2}(u_i)]$ with $V(F_i) = \{w_{i1}, w_{i2}\}$ for $i = 1, 2$. Suppose $F_i = G[N_{G_2}(u_i)] \cong K_3$ for some i with $3 \leq i \leq 4$, say $i = 3$. Then $|N_{G_3}(F_3)| = 3$ and $\alpha(G[N_{G_3}(F_3)]) = 3$. By (2) $d_{F_1 \cup F_2}(v) \leq 1$ for all vertices $v \in N_{G_3}(F_3)$. Hence we may assume that $N_{G_3}(w_{11}) \cap N_{G_3}(F_3) = \emptyset$. But then $\{u_2, u_3, u_4, w_{11}\} \cup N_{G_3}(F_3)$ is an independent set of type $(0, 3, 1, 3)$, a contradiction.

Suppose now $\alpha(G[N_{G_2}(u_i)]) \geq 2$ for $3 \leq i \leq 4$. Since $\alpha(G_2) \leq 6$, we conclude $w_{11}w_{12}, w_{21}w_{22} \in E(G)$. Let $N_{G_3}(w_{ij}) = \{x_{ij1}x_{ij2}\}$ for $1 \leq i, j \leq 2$. Then there are three independent vertices in $N_{G_3}(w_{ij})$ for $ij = 12, 21, 22$. These three vertices are contained together with w_{11} and u_2, u_3, u_4 in an independent set of type $(0, 3, 1, 3)$, a contradiction.

For the remaining part we may assume that $|E(G[N(v)])| \geq 2$ for every vertex $v \in V(G)$.

Case 3. $\alpha(G_1) = 2$
Let $E(G_1) = \{u_1u_2, u_3u_4\}$. Then $N_{G_2}(u_i) = V(F_i) = \{w_{i1}, w_{i2}\}$ with $w_{i1}w_{i2} \in E(G)$ for $1 \leq i \leq 4$. As above we conclude that there are three independent vertices in $N_{G_3}(w_{ij})$ for $ij = 32, 41, 42$ which are contained together with u_2, u_4, w_{11} and w_{31} in an independent set of type $(0, 2, 2, 3)$, a contradiction.

Case 4. $\alpha(G_1) = 2$
Let $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$. We may assume that $G[N(v)] \cong K_3 \cup K_1$ for every vertex $v \in V(G)$. Choose an edge uw with $N(u) = \{w, u_1, u_2, u_3\}$ and $N(w) = \{u, w_1, w_2, w_3\}$ such that $G[\{u_1, u_2, u_3\}] \cong K_3 \cong G[\{w_1, w_2, w_3\}]$. Then there exist vertices x_i and y_i for $1 \leq i \leq 3$ such that $u_ix_i, w_iy_i \in E(G)$. Let $V(G_1) = \{u_1, u_2, u_3, w_1, w_2, w_3\}$ and $V(G_2) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. Hence $\alpha(G_2) \leq 5$. If $\alpha(G_2) = 5$, then there is an independent set of type $(1, 1, 5)$, a contradiction. Hence we may assume $\alpha(G_2) \leq 4$. Since $E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})$ contains only independent edges, we may assume that $|E(\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\})| = 2$ (else consider u_1, x_1 instead of u, w).

We may assume that x_2, y_2 and x_3, y_3 are contained in a K_4. Hence $|V(G_3)| = 2 \cdot 2 + 2 \cdot 3 = 10$. Therefore, $V(G_4^*) \neq \emptyset$ and so there is an independent set of type $(1, 1, 4, 0, 1)$ (with respect to the edge uw), a contradiction.

2. $\Delta(G) = 5$

Case 1. $\alpha(G_1) = 5$
Since $\alpha(G_1) = 5$ we conclude that $\alpha(G_3^*) \leq 1$ and thus $|V(G_2)| \geq 25 - (1 + 5 + 4) = 15$. By Lemma 3 we have $|V(G_2)| \leq 15$. Therefore, $G_2 \cong 5K_3$ and $G_3 \cong K_4$. Hence by Lemma 4 every vertex of G_3 is contained in an independent set of type $(1, 0, 5, 1)$, a contradiction.

Case 2. $\alpha(G_1) = 4$
Then $|V(G_3^*)| \leq 8$ by Lemma 3 and thus $|V(G_2)| \geq 11$.

Case 2.1. $E(G_1) = \{u_1u_2\}$
Let $U_1 = \{u_1, u_2\}$ and $U_2 = \{u_3, u_4, u_5\}$. By Lemma 5 we conclude that $|N_{G_2}(U_2)| \geq 9$. Since $\alpha(G_2[N(U_1)]) \geq 2$, we get $\alpha(G_2[N(U_1)]) = 2$ and $\alpha(G_2[N(U_2)]) = 3$ by Lemma 3 and Lemma 7. Moreover, $G_2[N(U_2)] \cong 3K_3$ by Lemma 6.

Let $J = \{u_3, u_4, u_5\}$ and $G' = G - (J \cup N(J))$. Then $|V(G')| = 12$ and $\alpha(G') \geq 3$ by Table 1. Since $I \cup J$ is an independent set in G with $|I \cup J| \leq \alpha(G) \leq 6$ for every independent set I of G', we conclude $\alpha(G') = 3$. Hence $3K_4 \subset G'$ by Lemma 9. Therefore, $G_2[N(u_1)] = \{F_1, F_2\} \cong \{K_3, K_3\}$ and we follow the arguments of Case 1 above.

Case 2.2. $E(G_1) = \{u_1u_2, u_1u_3\}$
Let $U_1 = \{u_1, u_2, u_3\}$ and $U_2 = \{u_4, u_5\}$. Similarily as in the previous case we conclude that $\alpha(G_2[N(U_1)]) = 3$, $\alpha(G_2[N(U_2)]) = 2$ and $G_2[N(U_2)] \cong 2K_3$. Let F_1, F_2, F_3 be the three components of $G_2[N(U_1)]$ with $F_i = G_2[N(u_i)]$ for $i = 1, 2, 3$. Let $J = \{u_1, u_4, u_5\}$ and $G' = G - (J \cup N(J))$. Then $11 \leq |V(G')| \leq 12$ and thus $3K_4 \subset G'$ or $2K_4 \cup K_3 \subset G'$ by Lemma 9. Since $N_{G_3}(F_1)$ and F_2, F_3 are independent, $N_{G_3}(F_i) \cong K_3$ for $i = 2$ or 3. But then $H_i = N_{G_3}(F_i) \cong K_4$ is contained in a $K_4 \subset G'$ for $i = 2$ or 3, a contradiction.

Case 2.3. $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4\}$

Case 2.4. $E(G_1) = \{u_1u_2, u_1u_3, u_1u_4, u_1u_5\}$
For both cases let $J = \{u_2, u_3, u_4\}$ and $G' = G - (J \cup N(J))$. By Lemma 5 we need $|J \cup N(J)| \geq 13$. Since $4 \leq \alpha(G_2) \leq 5$ we conclude that

$G[N_{G_2}(u_i)] \cong K_3$ for some $i, 2 \leq i \leq 4$. Now we can follow the proof of Case 4 (by considering u_i with $d(u_i) = 5$ instead of u).

Case 3. $\alpha(G_1) = 3$

Case 3.1. $E(G_1) = \{u_1u_2, u_3u_4\}$
As in previous cases we conclude that $\alpha(G_2) = 5$ and $G_2[N(u_5)] \cong K_3$. Suppose G_2 is isomorphic to one of $\{K_{n_1}, K_{n_2}, K_{n_3}, K_{n_4}, K_{n_5}\}$ with $2 \leq n_1 \leq n_2 \leq 3, 2 \leq n_3 \leq n_4 \leq 3, n_5 = 3$. If $d(w_1) = 4 = d(w_3)$, let $J = \{u, w_1, w_3\}$ and $G' = G-(J \cup N(J))$. Then $|V(G')| = 11$ and thus $2K_4 \cup K_3 \subset G'$ by Lemma 9. Since F_2, F_4 and F_5 are independent and $|V(F_i)| \geq 2$ for $i = 2, 4, 5$, there exist $F_i, i = 2, 4$ or 5, such that F_i is contained in a $K_4 \subset G' - \{u_i\}$, a contradiction. Suppose G_2 is isomorphic to one of $\{K_1, K_1, K_{n_3}, K_{n_4}, K_{n_5}\}$ with $2 \leq n_3 \leq n_4 \leq 3, n_5 = 3$. Then $|V(G_3)| \geq 25 - 6 - (5 + n_3 + n_4) = 14 - n_3 - n_4$. Hence $|V(G_3) - (N(F_1) \cup N(F_2))| \geq 14 - n_3 - n_4 - 6 = 8 - n_3 - n_4$. Now by Lemma 10 there is an independent set of type $(1, 0, 5, 1)$ or $(1, 0, 4, 2)$, a contradiction.

Finally suppose that G_2 is isomorphic to $\{K_1, K_1, K_1, K_1, K_3\}$. Let w_1, $w_2, w_3, w_4 \in V(G_2)$ be four independent vertices with $N_{G_2}(u_1) = N_{G_2}(u_2) = \{w_1, w_2\}$ and $N_{G_2}(u_3) = N_{G_2}(u_4) = \{w_3, w_4\}$. If there is a vertex $v \in V(G_3^*)$ with $v \notin N(w_i)$ for $1 \leq i \leq 4$, then v is contained in an independent set of type $(1, 0, 5, 1)$ by Lemma 4, a contradiction.

Hence we may assume that $V(G_3^*) = V(G_3) \subset N(w_1) \cup N(w_2) \cup N(w_3) \cup N(w_4)$. Furthermore, $d_{G_3}(w_i) = 3$ for $1 \leq i \leq 4$. Let $H_i = G[N_{G_3}(w_i)]$ for $1 \leq i \leq 4$. Since $|V(G_3)| = 12$ we have $V(H_i) \cap V(H_j) = \emptyset$ for $1 \leq i < j \leq 4$. Moreover, there are no edges between $V(H_i)$ and $V(H_{i+1})$ for $i = 1, 3$, since G contains no C_5. Suppose $\alpha(G_2[H_i \cup H_{i+1}]) \geq 3$, then there are three independent vertices in $V(H_i) \cup V(H_{i+1})$, which are contained together with $u_i, u_5, w_{4-i}, w_{5-i}$ in an independent set of type $(0, 2, 2, 3)$, a contradiction.

Hence we may assume that $G_2[H_i \cup H_{i+1}] \cong 2K_3$ for $i = 1, 3$. Now any two vertices $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$ are contained in an independent set of size four in G_3 by Lemma 4. Hence $4 \leq \alpha(G_3) \leq 3$ by Lemma 3, a contradiction.

Case 3.2. $E(G_1) = \{u_1u_2, u_1u_3, u_4u_5\}$
See Case 4.

Case 3.3. $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3\}$

We first conclude that $\alpha(G_2) = 5$. Hence by Lemma 5 we get $G_2[N(u_i)] \cong K_3$ for $i = 4, 5$. We have $1 \le d_{G_2}(u_1) \le d_{G_2}(u_2) \le d_{G_2}(u_3) \le 2$. Let $F_i = G_2[N(u_i)]$ for $1 \le i \le 3$ and $J = \{u_1, u_4, u_5\}$. If $d_{G_2}(u_1) = 1$, then $G' = G[V(G) - (J \cup N(J))]$ has $|V(G')| = 12$. So $3K_4 \subset G'$ by Lemma 9. Since $N_{G_3}(F_1), V(F_2)$ and $V(F_3)$ are independent, $N_{G_3}(F_1)$ is contained in a $K_4 \subset G - F_1$. Hence there is a C_5, a contradiction.

If $d_{G_2}(u_i) = 2$ for $1 \le i \le 3$, then $|V(G')| = 11$. So $2K_4 \cup K_3 \subset G'$ by Lemma 9. Thus $F_i \cong K_2$ is contained in a $K_4 \subset G - u_i$ for some $i, 2 \le i \le 3$. Hence there is a C_5, a contradiction.

Case 4. $\alpha(G_1) = 2$

Then $G_1 = K_3 \cup K_2$. Let $E(G_1) = \{u_1u_2, u_1u_3, u_2u_3, u_4u_5\}$.

Suppose first $N_{G_2}(u_4) = N_{G_2}(u_5) = \{w_4, w_5\}$ for two vertices $w_4, w_5 \in V(G_2)$. Let $F_i = G_2[N(u_i)]$ for $1 \le i \le 3$ and set $F_i = \{w_i\}$ for $i = 4, 5$. Now let $H_i = G_3[N(F_i)]$ for $1 \le i \le 5$, $J = \{u, w_4, w_5\}$ and $G' = G - (J \cup N(J))$. Then $11 \le |V(G')| \le 12$ by Lemma 5. Suppose $|V(G')| = 12$. Then $3K_4 \subset G'$ by Lemma 9. Thus $G[F_i \cup H_i] \cong K_4$ for $1 \le i \le 3$. Since there is no C_5, we have $|F_i| = 1$ and $|H_i| = 3$ for $1 \le i \le 3$. We may assume $|V(H_4)| = 2$ and $|V(H_5)| = 3$. Thus $H_i \cong K_3$ for $i = 1, 2, 3$ and 5 and $H_4 \cong K_2$. Since $E(H_4, H_5) = \emptyset$, there is always an independent set with four vertices, one from H_2, H_3, H_4 and H_5. Together with w_1, u_2 and u_4 this gives an independent set of type $(0, 2, 1, 4)$, a contradiction. Suppose now $|V(G')| = 11$. Then $2K_4 \cup K_3 \subset G'$ by Lemma 9. We can follow the arguments above and may assume that $|V(F_3) \cup V(H_3)| = K_3$. Again we can find an independent set of type $(0, 2, 1, 4)$ as above, a contradiction.

Suppose next $F_i = G[N_{G_2}(u_i)]$ for $i = 4, 5$ with $|V(F_i)| \ge 2$ for two independent components F_4 and F_5. Furthermore, $F_i = G[N_{G_2}(u_i)]$ for $i = 1, 2, 3$, since $\alpha(G_2) = 5$. We have $1 \le |V(F_1)| \le |V(F_2)| \le |V(F_3)| \le 2$. If $|V(F_i)| = 1$ (i.e., $F_i = \{w_i\}$) for some i with $1 \le i \le 3$, then $d_{G_3}(w_i) = 3$, else we would be in a previous case.

Suppose there are two vertices $w_1 \in V(F_1)$ and $w_2 \in V(H_2)$ with $d(w_i) = 4$, $1 \le i \le 2$. Let $J = \{u, w_1, w_2\}$ and $G' = G - (J \cup N(J))$. Then $|V(G')| = 11$ and $2K_4 \cup K_3 \subset G'$ by Lemma 9. Thus F_i is contained in a $K_4 \subset G' - \{u_i\}$ for some $i, 4 \le i \le 5$. But then there is a C_5, a contradiction. Hence we may assume that $V(F_i) = \{w_{i1}, w_{i2}\}$ for $i = 2, 3$ and $d_{G_3}(w_{ij}) = 3$ for $i = 2, 3$ and $1 \le j \le 2$. But then $|V(G)| \ge 1 + 5 + (1 + 2 \cdot 2 + 2 \cdot 2) + 4 \cdot 3 = 27 > 25$, a contradiction.

3. $\Delta(G) = 6$

Case 1. $\alpha(G_1) = 6$
Since $\alpha(G_1) = 6$ we conclude that $V(G_3^*) = \emptyset$ and thus $15 \geq |V(G_2)| = 25 - 7 = 18$ by Lemma 3, a contradiction.

Case 2. $\alpha(G_1) = 5$
Then $E(G_1) = \{u_1u_2, u_1u_3, \ldots, u_1u_r\}, 2 \leq r \leq 6$. Since $\alpha(G_1) = 5$ we conclude by Lemma 3 (b) that $|V(G_3^*)| \leq 4$ and thus $|V(G_2)| \geq 25 - 7 - 4 = 14$. Then $\alpha(G_2) \geq 5$ by Lemma 3 (c). Thus $\alpha(G_2^*) = 5$ and $G_2 \cong 5K_3$ or $G_2 \cong 4K_3 \cup K_2$. By Lemma 6 we conclude $|V(G_1)| \leq 5 < 6$, a contradiction.

Case 3. $\alpha(G_1) \leq 4$
Using Lemma 2 and Lemma 7 we can show that $\alpha(G_2) \geq 6$ and thus there is an independent set of type $(1, 0, 6)$, a contradiction. ∎

References

[1] J.A. Bondy and P. Erdős, *Ramsey numbers for cycles in graphs*, J. Combin. Theory (B) **14** (1973) 46–54.

[2] B. Bollobás, C.J. Jayawardene, Z.K. Min, C.C. Rousseau, H.Y. Ru and J. Yang, *On a conjecture involving cycle-complete graph Ramsey numbers*, Australas. J. Combin. **22** (2000) 63–72.

[3] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).

[4] V. Chvátal and P. Erdős, *A note on hamiltonian circuits*, Discrete Math. **2** (1972) 111–113.

[5] P. Erdős, R.J. Faudree, C.C. Rousseau and R.H. Schelp, *On cycle-complete graph Ramsey numbers*, J. Graph Theory **2** (1978) 53–64.

[6] R.J. Faudree and R.H. Schelp, *All Ramsey numbers for cycles in graphs*, Discrete Math. **8** (1974) 313–329.

[7] C.J. Jayawardene and C.C. Rousseau, *The Ramsey number for a qudrilateral versus a complete graph on six vertices*, Congr. Numer. **123** (1997) 97–108.

[8] C.J. Jayawardene and C.C. Rousseau, *The Ramsey Number for a Cycle of Length Five vs. a Complete Graph of Order Six*, J. Graph Theory **35** (2000) 99–108.

[9] V. Nikiforov, *The cycle-complete graph Ramsey numbers*, preprint 2003, Univ. of Memphis.

[10] S.P. Radziszowski, *Small Ramsey numbers*, Elec. J. Combin. **1** (1994) DS1.

[11] S.P. Radziszowski and K.-K. Tse, *A Computational Approach for the Ramsey Numbers $R(C_4, K_n)$*, J. Comb. Math. Comb. Comput. **42** (2002) 195–207.

[12] V. Rosta, *On a Ramsey Type Problem of J.A. Bondy and P. Erdős*, I & II, J. Combin. Theory (B) **15** (1973) 94–120.

[13] I. Schiermeyer, *All Cycle-Complete Graph Ramsey Numbers $r(C_m, K_6)$*, J. Graph Theory **44** (2003) 251–260.

[14] Y.J. Sheng, H.Y. Ru and Z.K. Min, *The value of the Ramsey number $R(C_n, K_4)$ is* $3(n-1)+1$ $(n \geq 4)$, Australas. J. Combin. **20** (1999) 205–206.

[15] A. Thomason, private communication.

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