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DOMINATING BIPARTITE SUBGRAPHS IN GRAPHS

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Abstract

A graph G is hereditarily dominated by a class $\mathcal{D}$ of connected graphs if each connected induced subgraph of G contains a dominating induced subgraph belonging to $\mathcal{D}$. In this paper we characterize graphs hereditarily dominated by classes of complete bipartite graphs, stars, connected bipartite graphs, and complete k-partite graphs.

Keywords: dominating set, dominating subgraph, forbidden induced subgraph, bipartite graph, k-partite graph.

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1. Introduction

The general problem of structural domination can be considered as a subfield of the theory of domination in graphs and also of the theory of induced hereditary properties. It is formulated in [1, 3] in the following way:

Basic problem
Given a (finite or infinite) class $\mathcal{D}$ of connected graphs, characterize the class of those graphs in which every connected induced subgraph contains a dominating induced subgraph isomorphic to some $D \in \mathcal{D}$.

Several papers have been published in which this problem is the focus of study. Researchers have considered various families of connected graphs.

The first result of this type can be found in Wolk's paper [7] where $\mathcal{D} = \{K_1\}$ and the class of hereditarily one-vertex-dominated graphs was characterized in terms of the forbidden induced subgraphs P_4, C_4. The next result (as regards characterization) was given by G. Bacsó and Zs. Tuza [1], and independently by M.B. Cozzens and L.L. Kelleher [5], for dominating cliques. In this case the family of forbidden subgraphs consists of P_5 and C_5. The hereditarily dominated graphs have been characterized for some further families of graphs, too, e.g. for $\mathcal{D} = \{G : diam(G) \leq t\}$ for every given $t \geq 2\}$ in [2]. References to various related results, also including sufficient conditions, can be found in [3].

An interesting direction, not really explored so far, was initiated by J. Liu and H. Zhou [6] who characterized the graphs hereditarily dominated by the family of complete bipartite graphs *within the class of K_3-free graphs.* This work is the main motivation of our present paper; our Theorem 1 extends its characterization for all graphs, hence dropping the condition of triangle-freeness.

2. Preliminaries

We consider only finite, simple graphs. As usual, by $V(G)$ and $E(G)$ we denote the vertex set and the edge set, respectively. Also, K_n, P_n, C_n, and $K_{1,n}$ denotes the complete graph, the path, and the cycle with n vertices, and the star with n edges, respectively. Moreover, the *paw* denoted by PW is the graph on vertex set $\{a, b, c, d\}$ with edge set $\{ab, ac, ad, bc\}$.

A set $D \subseteq V(G)$ is called a *dominating set* if for each $v \in V(G) - D$ there exists a neighbor $w \in D$ of v; in the other words, $N(v) \cap D \neq \emptyset$, where

$N(v)$ is the set of all vertices in G adjacent to v. A subgraph induced by a dominating set is called a *dominating subgraph.*

Let $\mathcal{D}$ be a nonempty class of connected graphs. A graph G is called *hereditarily dominated by* $\mathcal{D}$ if in each of its connected induced subgraphs there exists a dominating induced subgraph belonging to $\mathcal{D}$.

A class $\mathcal{D}$ of connected graphs is called a *compact class* if it is closed under taking connected induced subgraphs. Each compact class $\mathcal{D}$ is uniquely characterized by a set of graphs minimal not in $\mathcal{D}$, that we shall denote by $L(\mathcal{D}) = \{H \notin \mathcal{D} :$ *each connected, induced, proper subgraph of* H *is in* $\mathcal{D}\}$.

In the theory of dominating sets, an important role is played by private neighbors, or in other words, private dominated vertices. Let $G = (V, E)$, and D be a dominating induced subgraph of G, with $u \in V(D)$. We say that u *has a private neighbor* if there exists a $u' \in V(G) \setminus D$ such that $N(u') \cap V(D) = \{u\}$.

Obviously, if u has a private neighbor, then $D \setminus \{u\}$ is not a dominating subgraph in G.

In the proofs below, the following concept will occur frequently. Suppose that the graph G is connected, and let H be a subgraph of G. We say that H is *d-minimal* if it is a *connected, dominating induced subgraph*, and moreover it is *minimal under inclusion* with respect to these properties; that is, each of its connected induced subgraphs is non-dominating in G. Being d-minimal implies, in particular, that each non-cutting vertex of H has a private neighbor.

To attach a leaf to a given vertex v of G means to take a new vertex v' and the edge vv'. The *leaf-graph* of a graph G, denoted $F(G)$, is the graph obtained from G by attaching a leaf to each of its non-cutting vertices.

In this paper we consider bipartite graphs and complete k-partite graphs as dominating subgraphs, and obtain characterizations of hereditarily dominated graphs in terms of forbidden induced subgraphs. In each case, the necessity of conditions can be derived immediately from a general method developed in [3], that characterizes the minimally non-$\mathcal{D}$-dominated, *non*-2-*connected* graphs in the following way:

Cutpoint Lemma [3]**.** *Let* $\mathcal{D}$ *be a compact class. A graph* G *with at least one cutpoint is minimal non-*$\mathcal{D}$*-dominated if and only if it is isomorphic to a leaf-graph* $F(L)$ *where* L *is a graph minimal not in* $\mathcal{D}$*.*

3. Dominating Complete Bipartite Subgraphs

We begin with considering the class of complete bipartite graphs. Let $\mathcal{D}_1 = \{K_{p,q} : p, q \geq 1\} \cup \{K_1\}$. It is easily checked that the family of graphs minimal not in $\mathcal{D}_1$ is $L(\mathcal{D}_1) = \{C_3, P_4\}$.

Theorem 1. *A graph G is hereditarily dominated by $\mathcal{D}_1$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_1)$.*

Proof. It is easy to see that none of graphs $F(C_3)$, $P_6 = F(P_4)$, or C_6 can contain any connected dominating subgraph which is a member of D_1. This proves the "only if" part. Conversely, let us suppose for a contradiction that there exists a minimal non-$\mathcal{D}_1$-dominated graph G with no induced subgraph $F(C_3)$, P_6, and C_6. Since G is P_6-free and C_6-free, each of its d-minimal subgraphs has to be P_4-free. (This follows, e.g., from the results of [2].)

Suppose first that every dominating, connected, induced subgraph of G contains at least one triangle. Among those subgraphs H, we choose one with the minimum possible number of triangles, and furthermore with as many leaves attached to some triangle $T \subseteq H$ as possible. Let T have t leaves attached inside H. Our goal is to prove $t = 3$, which is equivalent to saying that $F(C_3) \subseteq H$. This would lead to the contradiction that G is not $F(C_3)$-free, thus some H should be triangle-free.

Let x be any vertex of T. If x is a cut-vertex of H, then it is adjacent to some $x' \in V(H)$ which belongs to a connected component of $H - x$ not containing $T - x$. This x' is then a leaf for T, inside H, attached to x. On the other hand, if x is a non-cutting vertex of H, then the *connected* subgraph $H - x$ contains fewer triangles than H, thus cannot dominate G. We obtain that x has a private neighbor, say x'. Then the subgraph H' induced by $V(H) \cup \{x'\}$ is connected, dominating, and has precisely the same number of triangles as H does. Inside H', however, the number of leaves attached to T is greater than that in H, i.e., we should have chosen H' instead of H. This contradiction proves that $t = 3$ would indeed hold if H were not triangle-free.

Thus, G is dominated by some triangle-free, connected, induced subgraph H. We choose a subgraph $H' \subseteq H$ which is d-minimal. This H' remains triangle-free and, by our earlier observations, also P_4-free. Consequently, H' is complete bipartite, contradicting the assumption that G is not $\mathcal{D}_1$-dominated. ∎

Using Theorem 1 we can characterize hereditarily dominated graphs for the family of complete bipartite graphs with a bounded number of vertices, too. For $n \geq 3$, let $\mathcal{D}_2 = \mathcal{D}_2(n) = \{K_{p,q} : 1 \leq p, q \leq n\} \cup \{K_1\}$. The family of minimal graphs not in $\mathcal{D}_2$ is $L(\mathcal{D}_2) = \{C_3, P_4, K_{1,n+1}\}$.

Theorem 2. *A graph G is hereditarily dominated by $\mathcal{D}_2(n)$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_2)$.*

Proof. If G contains an induced subgraph $F(L)$ for $L \in L(\mathcal{D}_2)$, or an induced C_6, then this subgraph is not dominated by any member of $\mathcal{D}_2$.

For the other direction, let us suppose that there exists a minimal non-$\mathcal{D}_2$-dominated graph G with no induced subgraph $F(L)$, $L \in L(\mathcal{D}_2)$ — i.e., $F(C_3)$, $P_6 = F(P_4)$, and $F(K_{1,n+1})$ — and with no induced C_6.

Since G does not contain $F(C_3)$, P_6 and C_6, applying Theorem 1 we obtain that G is hereditarily dominated by the class of complete bipartite graphs. Hence each induced subgraph of G has a complete bipartite dominating subgraph H. If G is non-$\mathcal{D}_2$-dominated, then each dominating complete bipartite subgraph $H = (V_1, V_2)$ of G has $\max(|V_1|, |V_2|) > n$. Say, $|V_2| \geq n + 1$.

We choose the dominating $H = K_{p,q}$ so that p is smallest, and with this p the value of q is also smallest. By assumption, we have $p \geq 1$ and $q \geq n + 1 \geq 3$. Let us choose a vertex $a \in V_1$ and denote by $b_1, \ldots, b_q$ the vertices in the larger class V_2 of H. By the minimality of q, those vertices have private neighbors, say $b'_1, \ldots, b'_q$.

We consider the possible positions of edges in the subgraph H' induced by $b'_1, \ldots, b'_q$. If H' contains a triangle, e.g. b'_1, b'_2, b'_3 are mutually adjacent, then G contains an $F(C_3)$ induced by $b_1, b_2, b_3, b'_1, b'_2, b'_3$. If H' contains a (triangle-free) component with more than two vertices, say $b'_1 b'_2 b'_3$ is an induced P_3, then it forms an induced C_6 together with the vertices b_1, a, b_3. If $b'_1 b'_2$ is an isolated edge of H', then these three vertices with b_2, a_1, b_3, b'_3 is an induced P_6. Excluding all these possibilities we obtain that the b'_i are mutually nonadjacent. Thus, the set $\{a, b_1, \ldots, b_q, b'_1, \ldots, b'_q\}$ induces $F(K_{1,n+1})$, contradicting the assumption that G is $F(K_{1,n+1})$-free. ■

4. Dominating Connected Bipartite Subgraphs

Next we consider connected bipartite dominating subgraphs. Let $\mathcal{D}_3 = \{G : G \textit{ is connected and bipartite}\}$. (This also includes K_1.) It is clear that

$L(\mathcal{D}_3) = \{C_3, C_5, \ldots\}$, the set of odd cycles.

Theorem 3. *A graph G is hereditarily dominated by $\mathcal{D}_3$ if and only if G is $F(L)$-free for all $L \in L(\mathcal{D}_3)$.*

Proof. The leaf graph $F(C_{2k+1})$, $k \geq 1$, does not contain any connected dominating bipartite subgraph, i.e., the condition is necessary.

Let us suppose that there exists a minimal non-$\mathcal{D}_3$-dominated graph G with no induced subgraph $F(L)$ for $L \in L(\mathcal{D}_3)$.

If G has some cut vertex, then by the Cutpoint Lemma the graph G is isomorphic to $F(L)$ for $L \in L(\mathcal{D}_3)$ and the proof is done. Otherwise $G - x$ is connected and there exists a dominating subgraph $B = B(x)$ of $G - x$ which is a connected bipartite graph, for each vertex $x \in V(G)$. Adding a neighbor w of x to B, we obtain a dominating subgraph which is almost bipartite.

We now apply the method used in [4] for the characterization of graphs hereditarily dominated by paths. We choose the dominating bipartite subgraph B of $G - x$ and the neighbor w of x in such a way that the number of induced cycles of odd lengths in $H = B \cup \{w\}$ is minimum.

Let C be an odd cycle in H. We consider the vertices v of $C - w$ one by one. If v is a non-cutting vertex of H, then its private neighbor v' surely exists, otherwise $B - v$ would be a dominating set with fewer odd cycles than B. In this case we insert v' into B, hence keeping it bipartite, induced and dominating, with the same number of induced odd cycles. On the other hand, if v is a cut vertex of H, then it has a neighbor v' in a component of $H - v$ other than the component containing $C - v$. At the end, having found v' for each $v \in V(C)$ and defining $w' = x$ as the private neighbor of w, we obtain the contradiction that G contains $F(C)$ as an induced subgraph. (This $F(C)$ is indeed an induced subgraph, for otherwise the extended bipartite graph B with the vertices v' would dominate the entire G.) ∎

5. Complete k-Partite Dominating Subgraphs

Let $\mathcal{D}_4 = \{K_{n_1,n_2,\ldots,n_k} : k \geq 2,\ n_1, \ldots n_k \geq 1\} \cup \{K_1\}$. The minimal graphs not in $\mathcal{D}_4$ are P_4 and PW, i.e., $L(\mathcal{D}_4) = \{P_4, PW\}$.

Theorem 4. *A graph G is hereditarily dominated by $\mathcal{D}_4$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_4)$.*

Proof. It is obvious that the graphs $F(P_4) = P_6$, $F(PW)$, and C_6 are not dominated by any complete k-partite subgraph. To prove the converse, suppose for a contradiction that G is a minimal counterexample, i.e., G is a minimal non-$\mathcal{D}_4$-dominated graph that does not contain P_6, C_6, and $F(PW)$ as induced subgraphs.

If G has some cut vertex, then by the Cutpoint Lemma the graph G is isomorphic to $F(PW)$ or P_6, and the proof is done. Otherwise $G - x$ is a connected graph and there exists a dominating subgraph of $G - x$ which is a complete k-partite graph, for each $x \in V(G)$. Since G is P_6-free and C_6-free, the d-minimal subgraphs in G contain no induced path P_4.

We are going to prove that G is dominated by some PW-free induced subgraph. Suppose not. Let H be a dominating induced subgraph of G, containing an unavoidable paw with vertex set $\{a, b, c, d\}$ and edge set $\{ab, ac, ad, bc\}$, and suppose that the number of triangles inside H is as small as possible. By "unavoidable" we mean that the leaf vertex d cannot be removed. If d is a non-cutting vertex of H, this assumption means that d has a private neighbor d'; and in the opposite case it is adjacent to some vertex d' in a connected component of $H - d$ that does not contain a, b, c. Applying now the argument from the proof of Theorem 1, we would obtain leaves b' and c' for b and c, too. Thus, the contradiction would follow that G is not $F(PW)$-free.

Hence, let H be a PW-free, dominating, connected, induced subgraph of G. Take an induced subgraph of H which is d-minimal in G. By what has been said above, this H is also P_4-free, so that a complete multipartite dominating subgraph is found. ■

For complete multipartite graphs of bounded-size parts, let $n \geq 2$ and $\mathcal{D}_5 = \mathcal{D}_5(n) = \{K_{n_1,\ldots,n_k} : k \geq 2,\ 1 \leq n_1, \ldots, n_k \leq n\} \cup \{K_1\}$. Here the number k of vertex classes is not fixed. The class of graphs minimal not in $\mathcal{D}_5$ is $L(\mathcal{D}_5) = \{P_4, PW, K_{1,n+1}\}$.

Theorem 5. *A graph G is hereditarily dominated by $\mathcal{D}_5(n)$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_5)$.*

The proof of Theorem 5 is omitted because it is very similar to that of Theorem 3. We note that an analogous result can be proved also for complete multipartite graphs with a bounded number k of vertex classes. Then K_{k+1} occurs as a further graph minimal not in $\mathcal{D}$, and hence $F(K_{k+1})$ as a minimal forbidden induced subgraph.

6. Hereditary Domination by Induced Stars

We close this paper with the case of induced stars, which form probably the most interesting subfamily of the complete bipartite graphs. The stars with a restricted number of leaves will also be considered.

Let $\mathcal{D}_6 = \{K_{1,j} : 0 \le j\}$ and $\mathcal{D}_7 = \mathcal{D}_7(n) = \{K_{1,j} : 0 \le j \le n\}$ for $n \ge 3$ (where $K_{1,0} = K_1$). It is clear that $L(\mathcal{D}_6) = \{C_3, C_4, P_4\}$ and $L(\mathcal{D}_7) = \{C_3, C_4, P_4, K_{1,n+1}\}$.

Theorem 6. *A graph G is hereditarily dominated by $\mathcal{D}_6$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_6)$.*

Proof. It is easy to see that an induced C_6 or an induced subgraph $F(L)$ of G, for some $L \in L(\mathcal{D}_6)$, does not contain dominating induced stars. Conversely, suppose that there exists a minimal non-$\mathcal{D}_6$-dominated graph G with no induced subgraph $F(C_3)$, $P_6 = F(P_4)$ and $F(C_4)$, and with no induced C_6. Since G does not contain $F(C_3)$, P_6 and C_6, applying Theorem 1 we obtain that G is dominated by some complete bipartite induced subgraphs. We choose a minimal one, say D, with vertex classes A and B. If G is not star-dominated, then $|A|, |B| \ge 2$ and each $v \in A \cup B$ is a non-cutting vertex of D with at least one private neighbor. We next construct a dominating induced subgraph H, starting from D itself, by sequentially considering the vertices $v \in A \cup B$. If v does not have a private neighbor with respect to the subgraph found so far, then we delete it from H unless all elements of A or B would be deleted. And if v still has a private neighbor v', then we insert v' into H.

At the end of this procedure, subsets $A' = \{a_1, \ldots, a_p\} \subseteq A$ and $B' = \{b_1, \ldots, b_q\} \subseteq B$ remain in H; and if $p > 1$ and/or $q > 1$, then every vertex of A' and/or B' has a private neighbor, mutually nonadjacent. If both $p, q \ge 2$, then the contradiction $F(C_4) \subset G$ is obtained. If $p = q = 1$, then G is dominated by some path of length $\ell \le 3$. For $\ell \le 2$, a dominating star with at most two leaves is found, contrary to our assumptions; and otherwise for $\ell = 3$ the endpoints of this induced dominating path must have private neighbors, hence an induced P_6 or C_6 occurs in G.

Finally, assume $p = 1$ and $q \ge 2$, and let $a_2 \in A \setminus A'$ with private neighbor a_2' with respect to D. Since a_2 has been removed from H, a_2' is dominated by some other private neighbor. If it is b_1', then the vertices $a_2', b_1', b_1, a_1, b_2, b_2'$ induce P_6 or C_6. On the other hand, if a_2' is dominated by a'_1, we check whether H remains dominating after the removal of all the b'_j. If so, then a dominating star centered at a_1 has been found. And if it isn't, then some b'_j of H, say b'_1 has a private neighbor b''_1. Thus, we obtain the final contradiction that $a'_2, a'_1, a_1, b_1, b'_1, b''_1$ induce P_6 or C_6. ■

Theorem 7. *A graph G is hereditarily dominated by $\mathcal{D}_7(n)$ if and only if G is C_6-free and $F(L)$-free for all $L \in L(\mathcal{D}_7)$.*

Proof. As before, neither the graphs $F(L)$ ($L \in L(\mathcal{D}_7)$) nor C_6 contain dominating induced stars on at most n end vertices.

Let us suppose that there exists a non-$\mathcal{D}_7$-dominated graph G with no induced subgraph C_6 and no $F(L)$ for $L \in L(\mathcal{D}_7)$; that is, $F(C_3)$, P_6, $F(C_4)$, $F(K_{1,n+1})$. Since G does not contain $F(C_3)$, $F(C_4)$, P_6, and C_6, applying the previous theorem we obtain that G has a dominating induced star H. If G is non-$\mathcal{D}_7$-dominated, then each minimal dominating star $H = K_{1,t}$ in G has $t \geq n+1$. We notice that, by the minimality of H, each non-cutting vertex of H has a private neighbor. Using the same method as in the proof of Theorem 2, we can easily find $F(K_{n+1})$, and this contradiction completes the proof. ■

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