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# NOTE ON THE SPLIT DOMINATION NUMBER OF THE CARTESIAN PRODUCT OF PATHS

Maciej Zwierzchowski

*Institute of Mathematics*
*Technical University of Szczecin*
*Al. Piastów 48/49, 70–310 Szczecin, Poland*

**e-mail:** mzwierz@ps.pl

**Abstract**

In this note the split domination number of the Cartesian product of two paths is considered. Our results are related to [2] where the domination number of $P_m \Box P_n$ was studied. The split domination number of $P_2 \Box P_n$ is calculated, and we give good estimates for the split domination number of $P_m \Box P_n$ expressed in terms of its domination number.

**Keywords:** domination number, split domination number, Cartesian product of graphs.

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## 1. Introduction

In this paper we consider finite undirected simple graphs. For any graph $G$ we denote $V(G)$ and $E(G)$, the vertex set of $G$ and the edge set of $G$, respectively. If $n$ is the cardinality of $V(G)$, then we say that $G$ is of *order* $n$. By $\langle X\rangle_G$ we mean a subgraph of a graph $G$ induced by a subset $X \subseteq V(G)$. A subset $D \subseteq V(G)$ is a *dominating set* of $G$, if for every $x \in V(G) - D$, there is a vertex $y \in D$ such that $xy \in E(G)$. We also say that $x$ is dominated by $D$ in $G$ or by $y$ in $G$. A dominating set $D$ of $G$ is a *split dominating set* of $G$, if the induced subgraph $\langle V(G) - D\rangle_G$ of $G$ is disconnected. The domination number, [the split domination number] of a graph $G$, denoted $\gamma(G)$, [$\gamma_s(G)$] is the cardinality of the smallest dominating [the smallest split dominating] set of $G$. A dominating set $D$ is called a $\gamma(G)$-set if $D$ realizes the domination number. Similarly we define a $\gamma_s(G)$-set. From the definition of a split dominating set it follows immediately that $\gamma(G) \leq \gamma_s(G)$. Additionally note that for a connected graph $G$ a $\gamma_s(G)$-set exists if and only if $G$ is different from a complete graph. More information about a split dominating set and the split domination number can be found in [3]. The Cartesian product of two graphs $G$ and $H$, is a graph $G \Box H$ with $V(G \Box H) = V(G) \times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G \Box H)$ if and only if ($g_1 = g_2$ and $h_1 h_2 \in E(H)$) or ($g_1 g_2 \in E(G)$ and $h_1 = h_2$).

Any other terms not defined in this paper can be found in [1].

## 2. Main Results

**Theorem 1.** *For any* $n, m \geq 2$

$$\gamma(P_m \Box P_n) \leq \gamma_s(P_m \Box P_n) \leq \gamma(P_m \Box P_n) + 1.$$

***Proof.*** Let $m, n \geq 2$ and let $D$ be the minimum dominating set of $P_m \Box P_n$. According to the definition of a split dominating set we have $\gamma(P_m \Box P_n) \leq \gamma_s(P_m \Box P_n)$. Thus to prove this theorem we will show that $\gamma_s(P_m \Box P_n) \leq \gamma(P_m \Box P_n) + 1$. Consider the graph $P_m \Box P_n$, as $m$ canonical copies of $P_n$ with vertices labelled $x_{i,j}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, and with edges $x_{i,j} x_{i+1,j}$ and $x_{i,j} x_{i,j+1}$.

If $x_{1,1} \in D$, then the subset $D' = D - \{x_{1,1}\} \cup \{x_{1,2}, x_{2,1}\}$ is also a dominating set of $P_m \Box P_n$. Moreover, since $N_{P_m \Box P_n}(x_{1,1}) = \{x_{1,2}, x_{2,1}\} \subset D'$, then $x_{1,1}$ is an isolated vertex of the induced subgraph $\langle V(P_m \Box P_n) - D' \rangle_{P_m \Box P_n}$ of a graph $P_m \Box P_n$. It means that $D'$ is a split dominating set of $P_m \Box P_n$, with $|D'| \leq \gamma(P_m \Box P_n) + 1$.

If $x_{1,1} \notin D$, then it must be that $x_{1,2} \in D$ or $x_{2,1} \in D$ (otherwise $x_{1,1}$ would not be dominated by $D$ in $P_m \Box P_n$). Assume that $x_{1,2} \in D$, then $D' = D \cup \{x_{1,2}\}$ is a split dominating set of $P_m \Box P_n$ and $|D'| \leq |D| + 1 = \gamma(P_m \Box P_n) + 1$, as desired.

Thus $\gamma_s(P_m \Box P_n) \leq \gamma(P_m \Box P_n) + 1$, for any $m, n \geq 2$ and the proof is complete. ■

In [2] it was obtained that $\lim_{n,m \to \infty} \frac{\gamma(P_m \Box P_n)}{mn} = \frac{1}{5}$. As a consequence from the above fact and from Theorem 1 we obtain the following

**Corollary 2.**

$$\lim_{n,m\to\infty} \frac{\gamma_s(P_m \Box P_n)}{mn} = \frac{1}{5}.$$

■

The following result was proved in [2].

**Theorem 3** [2]. *For $n \geq 2$,*

$$\gamma(P_2 \Box P_n) = \left\lceil \frac{n+1}{2} \right\rceil.$$

■

Inspired by this result we shall calculate the split domination number of $P_2 \Box P_n$, for $n \geq 2$. Before proceeding we give a few necessary results.

Let $V(P_2) = \{v_1, v_2\}$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. For convenience, in the rest of the paper we will write $x_i$ instead of $(v_1, u_i) \in V(P_2 \Box P_n)$ and $y_i$ instead of $(v_2, u_i) \in V(P_2 \Box P_n)$, for $i = 1, 2, \ldots, n$. Hence $V(P_2 \Box P_n) = \{x_i, y_i : i = 1, 2, \ldots, n\}$ and $E(P_2 \Box P_n) = \{x_i x_{i+1},\ y_i y_{i+1},\ x_i y_i,\ x_n y_n : i = 1, 2, \ldots, n-1\}$.

**Lemma 4.** *If $n \equiv 2 (\mathrm{mod}\, 4)$, $n \geq 2$, then*

$$D = \{x_i : i \equiv 1 (\mathrm{mod}\, 4)\} \cup \{y_j : j \equiv 3 (\mathrm{mod}\, 4)\} \cup \{y_n\}$$

*is the $\gamma_s(P_2 \Box P_n)$-set with $|D| = \lceil \frac{n+1}{2} \rceil$.*

***Proof.*** Let $D = \{x_i : i \equiv 1 (\mathrm{mod}\, 4)\} \cup \{y_j : j \equiv 3 (\mathrm{mod}\, 4)\} \cup \{y_n\}$ be a subset of $V(P_2 \Box P_n)$.

We show that any vertex of $P_2 \Box P_n$ is either in $D$ or it is adjacent to some vertex from $D$. Let $r$ be an integer not greater than $n$.

If $r = 4q$, $q \geq 1$, then the vertex $x_r$ is adjacent to $x_{r+1} = x_{4q+1} \in D$ and $y_r$ is adjacent to $y_{r-1} = y_{4q-1} \in D$.

If $r = 4q + 1$, $q \geq 0$, then $x_r \in D$ and $y_r$ is adjacent to $x_r$.

If $r = 4q + 2$, $q \geq 0$, then $x_r$ is adjacent to $x_{r-1} \in D$. If $r = n$, then $y_r = y_n \in D$ and if $r < n$, then $y_r$ is adjacent to $y_{r+1} \in D$.

Finally, if $r = 4q + 3$, $q \geq 0$, then $y_r \in D$ and $x_r$ is adjacent to $y_r$.
All this together gives that $D$ is a dominating set of $P_2 \Box P_n$.

Let $n = 4s + 2$, $s \geq 0$. We state that $|D| = \lceil \frac{n+1}{2} \rceil$. Indeed, partition $V(P_2 \Box P_n)$ into subsets $B_i = \{x_{4i-3}, y_{4i-3}, \ldots, x_{4i}, y_{4i}\}$, for $i = 1, 2, \ldots, s$

and $B_{s+1} = \{x_{n-1}, y_{n-1}, x_n, y_n\}$. Note that $|D \cap B_i| = 2$, for $i = 1, 2, \ldots, s+1$. Thus $|D| = 2s + 2 = \lceil \frac{n+1}{2} \rceil = \gamma(P_2 \Box P_n)$, by Theorem 3. Since $N_{P_2 \Box P_n}(x_n) = \{x_{n-1}, y_n\} \subset D$, hence $x_n$ is an isolated vertex of $\langle V(P_2 \Box P_n) - D \rangle_{P_2 \Box P_n}$. Thus this induced subgraph is disconnected. All this together gives that $D$ is a $\gamma_s(P_2 \Box P_n)$-set, since $D$ is a split dominating set of $P_2 \Box P_n$ with the minimum cardinality. Hence the result is true. ■

**Lemma 5.** *If $n \equiv 0 (\mathrm{mod}\, 4)$, $n \geq 2$, then*

$$D = \{x_i : i \equiv 1 (\mathrm{mod}\, 4)\} \cup \{y_j : j \equiv 3 (\mathrm{mod}\, 4)\} \cup \{x_n\}$$

*is the $\gamma_s(P_2 \Box P_n)$-set with $|D| = \lceil \frac{n+1}{2} \rceil$.*

***Proof.*** Let $D$ be as in the statement of the theorem. Arguing similarly as in the proof of above lemma, it follows that $D$ is a dominating set of $P_2 \Box P_n$. Now, we show that $|D| = \lceil \frac{n+1}{2} \rceil$. Put $n = 4s$ and partition $V(P_2 \Box P_n)$ into the subsets $B_i = \{x_{4i-3}, y_{4i-3}, \ldots, x_{4i}, y_{4i}\}$, for $i = 1, 2, \ldots, s$. It is easy to observe that $|D \cap B_i| = 2$, for $i = 1, 2, \ldots, s-1$ and $|D \cap B_s| = 3$. Hence $|D| = 2(s-1)+3 = 2s+1 = \lceil \frac{n+1}{2} \rceil = \gamma(P_2 \Box P_n)$, as desired. Finally, observe that $y_n$ is an isolated vertex of $\langle V(P_2 \Box P_n) - D \rangle_{P_2 \Box P_n}$. This means that the last subgraph is disconnected and as a consequence $D$ is a split dominating set of $P_2 \Box P_n$. Since $D$ is also a $\gamma(P_2 \Box P_n)$-set, it is a $\gamma_s(P_2 \Box P_n)$-set, as required. ■

**Lemma 6.** *Let $n \geq 5$ be odd and let $D$ be a $\gamma(P_2 \Box P_n)$-set. Then exactly one of $x_1$ and $y_1$ belong to $D$.*

***Proof.*** Let $n = 2k + 1$ with $k \geq 2$ and let $D$ be a $\gamma(P_2 \Box P_n)$-set. Assume that $x_1, y_1 \notin D$, then it must be that $x_2, y_2 \in D$ (otherwise $x_1$ or $y_1$ would not be dominated by $D$). Since $n \geq 5$ is odd, then $\{x_3, y_3\} \subset V(P_2 \Box P_n)$. Moreover $x_3, y_3 \notin D$. Indeed, without loss of generality, suppose that $x_3 \in D$. Then $D \cup \{y_1\} - \{x_2, y_2\}$ is a dominating set of $P_2 \Box P_n$, having the cardinality $|D| - 1$. This contradicts the fact that $D$ is the minimum dominating set of $P_2 \Box P_n$.

So, we have $x_1, y_1, x_3, y_3 \notin D$ and $x_2, y_2 \in D$. Consider two induced subgraphs of $P_2 \Box P_n$ :

$$X_1 = \langle \{x_1, y_1, x_2, y_2, x_3, y_3\} \rangle_{P_2 \Box P_n} \quad \text{and}$$
$$X_2 = \langle \{x_4, y_4, \ldots, x_n, y_n\} \rangle_{P_2 \Box P_n} .$$

Since $X_2 \cong P_2 \square P_{n-3}$, then by Theorem 3 we have $\gamma(X_2) = \lceil \frac{n-2}{2} \rceil = \lceil \frac{2k-1}{2} \rceil = k$. Further $|D| = \gamma(X_1) + \gamma(X_2) = 2 + k = \lceil \frac{n+3}{2} \rceil > \lceil \frac{n+1}{2} \rceil = \gamma(P_2 \square P_n)$, — a contradiction, since $D$ is a $\gamma(P_2 \square P_n)$-set.

Now, assume that $x_1$ and $y_1 \in D$, then $x_2, y_2, x_3, y_3 \notin D$ (otherwise there would exist a dominating set of $P_2 \square P_n$ with order strictly less than the cardinality of $D$). Arguing as above, for $X_1 = \langle \{x_1, y_1, x_2, y_2\} \rangle_{P_2 \square P_n}$ and $X_2 = \langle \{x_3, y_3, \ldots, x_n, y_n\} \rangle_{P_2 \square P_n}$, we also come to a contradiction. Hence the proof is complete. ■

In [2] the following was proved

**Lemma 7** [2]**.** *If* $n \geq 5$ *and* $n$ *is odd, then*

$$D = \{x_i : i \equiv 1 (\mathrm{mod}\, 4)\} \cup \{y_j : j \equiv 3 (\mathrm{mod}\, 4)\}$$

*is the* $\gamma(P_2 \square P_n)$*-set with* $|D| = \lceil \frac{n+1}{2} \rceil$. ■

**Lemma 8.** *Let* $n \geq 5$ *be odd and let* $D$ *be a* $\gamma(P_2 \square P_n)$*-set. Then*

$$|D \cap \{x_i, y_i, x_{i+1}, y_{i+1}\}| = 1,$$

*for* $i = 1, 2, \ldots, n-1$.

***Proof.*** We prove this lemma by induction. First consider the base case, when $n = 5$. By Lemma 6, either $x_1 \in D$ or $y_1 \in D$ and $x_5 \in D$ or $y_5 \in D$. Since $\gamma(P_2 \square P_5) = 3$, then

$$|D \cap \{x_2, y_2, x_3, y_3, x_4, y_4\}| = 1.$$

If $x_3, y_3 \notin D$, then $x_3$ or $y_3$ is not dominated by $D$ in $P_2 \square P_5$. So it must be that either $x_3 \in D$ or $y_3 \in D$. Thus the result holds for $n = 5$.

Assume that the result holds for $n = 2k+1$ and consider $n = 2k+3$. By Lemma 6, either $x_1 \in D$ or $y_1 \in D$. If $x_2, y_2 \notin D$, then by the assumption

$$|D \cap \{x_i, y_i, x_{i+1}, y_{i+1}\}| = 1,$$

for $i = 3, 4, \ldots, n-1$. Moreover,

$$|D \cap \{x_1, y_1, x_2, y_2\}| = 1 \text{ and}$$
$$|D \cap \{x_2, y_2, x_3, y_3\}| = 1.$$

Thus the result holds for $n = 2k + 3$.

If $x_2 \in D$ or $y_2 \in D$, then $D_1 = D \cap \{x_i, y_i : i = 4, \ldots, n\}$ is a $\gamma(P_2 \Box P_{2k})$-set and $|D_1| = \lceil \frac{2k+1}{2} \rceil = k+1$, by Theorem 3. Thus

$$|D| \geq |D_1| + 2 = k + 3 > \left\lceil \frac{2k+3}{2} \right\rceil = \gamma(P_2 \Box P_{2k+3})$$

but this is impossible, since $D$ is a $\gamma(P_2 \Box P_{2k+3})$-set.

Hence the result is true for all odd $n \geq 5$. ■

**Theorem 9.** *For $n \geq 2$,*

$$\gamma_s(P_2 \Box P_n) = \begin{cases} \lceil \frac{n+1}{2} \rceil, & \text{if } n \text{ is even or } n = 3, \\ \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n \geq 5 \text{ is odd.} \end{cases}$$

***Proof.*** Let $n \geq 2$ be even. According to Lemma 4 and Lemma 5 the result is true.

If $n = 3$, then the set $\{x_2, y_2\}$ is the minimum split dominating set of $P_2 \Box P_3$, with the required cardinality.

Next, suppose that $n \geq 5$ is odd. Then $n = 2k + 1$, $(k \geq 2)$. According to Lemma 8 we have that the set $D$ of Lemma 7 is unique (modulo the automorphism that exchanges paths $P_n$). Moreover, observe that $D$ is not a split dominating set of $P_2 \Box P_n$. Thus $\gamma(P_2 \Box P_n) < \gamma_s(P_2 \Box P_n)$ and by Theorem 1 we obtain that $\gamma_s(P_2 \Box P_n) = \gamma(P_2 \Box P_n) + 1$. ■

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