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MULTICOLOR RAMSEY NUMBERS FOR PATHS AND CYCLES

Tomasz Dzido

Department of Computer Science
University of Gdańsk
Wita Stwosza 57, 80–952 Gdańsk, Poland

e-mail: tdz@math.univ.gda.pl

Abstract

For given graphs $G_1, G_2, \ldots, G_k, k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph on n vertices with k colors, then it is always a monochromatic copy of some G_i, for $1 \leq i \leq k$. We give a lower bound for k-color Ramsey number $R(C_m, C_m, \ldots, C_m)$, where $m \geq 8$ is even and C_m is the cycle on m vertices. In addition, we provide exact values for Ramsey numbers $R(P_3, C_m, C_p)$, where P_3 is the path on 3 vertices, and several values for $R(P_l, P_m, C_p)$, where $l, m, p \geq 2$. In this paper we present new results in this field as well as some interesting conjectures.

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1. Introduction

In this paper all graphs considered are undirected, finite and contain neither loops nor multiple edges. Let G be such a graph. The vertex set of G will be denoted by $V(G)$ and the edge set of G by $E(G)$. C_m denotes the cycle of length m, P_m — the path on m vertices. In this article we consider only edge colorings of graphs. For given graphs $G_1, G_2, \ldots, G_k,\ k \geq 2$, the *multicolor Ramsey number* $R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that if we arbitrarily color the edges of the complete graph of order n with

k colors, then it always contains a monochromatic copy of some G_i, for $1 \le i \le k$. We often consider only 3-color Ramsey number $R(G_1, G_2, G_3)$ (i.e., we color the edges of the complete graph K_n with color red, blue and green). A 3-coloring of K_n is called a $(G_1, G_2, G_3; n)$-coloring if it contains neither a red G_1 nor a blue G_2 nor a green G_3. $(G_1, G_2, \ldots, G_k; n)$-coloring is defined analogously. A $(G_1, G_2, \ldots, G_k; n)$-coloring is said to be *critical* if $n = R(G_1, G_2, \ldots, G_k) - 1$. For $v \in V(G)$, we define functions $r(v)$, $b(v)$ and $g(v)$ to be the numbers of red, blue and green neighbours of v. Very little is known about multicolor Ramsey numbers. We refer the reader to [9] and [1] for a survey.

2. The Ramsey Numbers for Even Cycles

Up to now, there have been known a few exact values for 3-color Ramsey numbers for cycles. The following numbers are proved: $R(C_3, C_3, C_3) = 17$ [7] and $R(C_4, C_4, C_4) = 11$ [4]. By using computer support, the two following numbers have been determined: $R(C_5, C_5, C_5) = 17$ [10] and $R(C_6, C_6, C_6) = 12$ [11]. Recently, Faudree, Schelten and Schiermeyer have shown (without using a computer) that

Theorem 1 [6]**.**

$$R(C_7, C_7, C_7) = 25.$$

In 1973, Bondy and Erdös formulated the following conjecture.

Conjecture 2 [2]**.** *For all odd integers* $m \ge 5$*,*

$$R(C_m, C_m, C_m) = 4m - 3.$$

Łuczak has proved that this bound holds asymptotically.

Theorem 3 [8]**.** *For all integers* $m \ge 5$*,*

$$R(C_m, C_m, C_m) = (4 + o(1))m.$$

Now, we shall give formula for a lower bound for multicolor Ramsey numbers.

Theorem 4. *For all even integers $m \geq 8$ and an integer $k \geq 2$,*

$$R\underbrace{(C_m, C_m, \ldots, C_m)}_{k\ times} > m + (k-1)\frac{m}{2} - (k-1) - 1.$$

Proof. All we need is to provide any k-coloring of graph G on $m+(k-1)\frac{m}{2}-(k-1)-1$ vertices which is a good $\underbrace{(C_m, C_m, \ldots, C_m)}_{k\ times}$-coloring. Consider a partition of $V(G)$ into subsets $V_1, V_2, \ldots, V_{k+1}$, where $|V_1| = \frac{m}{2}$ and $|V_i| = \frac{m}{2} - 1$, $i \in \{2, \ldots, k+1\}$. Let $e = \{x, y\}$. Then this coloring is as follows:

(i) $c(e) = 1$ if $x, y \in V_i$, where $i \in \{1, 2, \ldots, k+1\}$,

(ii) $c(e) = j - 1$ if $x \in V_i$, $y \in V_j$, where $i \in \{1, 2\}$, $j \in \{2, 3, \ldots, k+1\}$ and $i < j$,

(iii) $c(e) = i - 1$ if $x \in V_i$, $y \in V_j$, where $i \in \{3, 4, \ldots, k+1\}$, $j \in \{4, 5 \ldots, k+1\}$ and $i < j$.

By the definition of our coloring, it is easy to see that there is no cycle C_m in color 1. Furthermore, as each of the monochromatic sets of edges colored with one of colors $i = 2, \ldots, k+1$, induce a bipartite subgraph G^i with the smaller partition of the cardinality $\frac{m}{2} - 1$, there are no monochromatic cycle C_m in G^i, and thus in the whole graph as well. ■

The following three corollaries are straightforward.

Corollary 5. *For all even integers $m \geq 8$,*

$$R(C_m, C_m, C_m) > 2m - 3.$$

Corollary 6. *For all even integers $m \geq 8$,*

$$R(C_m, C_m, C_m, C_m) > \frac{5}{2}m - 4.$$

In particular, we have that $R(C_8, C_8, C_8) > 13$, $R(C_{10}, C_{10}, C_{10}) > 17$, and $R(C_8, C_8, C_8, C_8) > 16$.

3. The Ramsey Numbers $R(P_l, P_m, C_p)$

This section is devoted to the study of the 3-color Ramsey numbers for two paths and one cycle which have different length.

3..1 The Ramsey numbers $R(P_3, P_3, C_m)$

Arste *et. al* [1] gave the first two Ramsey numbers of this type: $R(P_3, P_3, C_3) = 5$ and $R(P_3, P_3, C_4) = 6$. It is easy to prove the following theorem.

Theorem 7. $R(P_3, P_3, C_5) = 6$.

Proof. We will only show the inequality $R(P_3, P_3, C_5) > 5$ by presenting a coloring of K_5 which contains neither a red path P_3 nor a blue path P_3 nor a green cycle C_5. Let the vertices of K_5 be labeled $0, 1, 2, 3, 4$. Let vertex 0 be joined by green edges to the vertices $1, 2, 3, 4$. Vertex 1 is joined by a green edge to 2, a red edge to 3 and a blue edge to 4, vertex 2 is joined by a blue edge to 3 and by a red edge to 4. This enforces the green edge {3,4}. ■

Theorem 8. *For all integers $m \geq 6$,*

$$R(P_3, P_3, C_m) = m.$$

Proof. Avoiding a red and a blue path P_3 we obtain that $r(v) \leq 1$ and $b(v) \leq 1$ for all vertices v in any 3-coloring of graph K_m. By Dirac's Theorem, we immediately obtain a cycle C_m, which completes the proof. ■

3..2 The Ramsey numbers $R(P_3, P_4, C_m)$

In [1] we can find two results for Ramsey numbers of such type:
$R(P_3, P_4, C_3) = 7$ and $R(P_3, P_4, C_4) = 7$. We will prove the analogous result for $R(P_3, P_4, C_m)$.

First, we need the following definition.

Definition 1. *Turán number* $T(m, G)$ is the maximum number of edges in any m-vertex graph which does not contain a subgraph isomorphic to G.

Let us recall the well-known Turán numbers.

Lemma 9 [12]**.** *For all integers* $n \geq 4$*,*

$$T(m, P_3) = \left\lfloor \frac{m}{2} \right\rfloor,$$

$$T(m, P_4) = \begin{cases} m & \text{if } m \equiv 0 \mod 3, \\ m{-}1 & \text{otherwise,} \end{cases}$$

$$T(m, C_m) = \binom{m-1}{2} + 1,$$

$$T(m+1, C_m) = \binom{m-1}{2} + 3.$$

Theorem 10. $R(P_3, P_4, C_5) = 7$.

Proof. The proof of $R(P_3, P_4, C_5) \leq 7$ is simple, so it is left to the reader. We will only show the inequality $R(P_3, P_4, C_5) > 6$ by presenting a critical coloring $(P_3, P_4, C_5; 6)$. Let the vertices of K_6 be labeled $0, 1, \ldots, 5$, and let us color the edges $\{0, 2\}$, $\{0, 5\}$, $\{2, 5\}$, $\{1, 3\}$, $\{1, 4\}$, $\{3, 4\}$ with blue, and the edges $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$ with red; the remaining edges are green. It is easy to check that this coloring forces a green C_6 but not C_5. ■

Theorem 11. *For all integers* $m \geq 6$*,*

$$R(P_3, P_4, C_m) = m + 1.$$

Proof. We can easily obtain the result for $m = 6$. Consider any 3-colored K_7. By Lemma 9, in order to avoid a red P_3 and a blue P_4, there must be at most 3 red, 6 blue and remaining 12 green edges. If $g(v) \leq 2$ for some vertex $v \in V(K_7)$, then the graph $K_7 - v$ contains a green cycle C_5 and we immediately have a green C_6 or a red P_3, or a blue P_4. If $g(v) = 3$ for some vertex v, then $K_7 - v$ contains a green C_5 and also a green C_6 or $K_7 - v$ does not contain a green C_5 and C_6 but contains a vertex w such that $g(w) \leq 2$. It is easy to check that the graph $K_7 - \{v, w\}$ contains a green cycle C_4, and if there is no a green C_5, then we quickly have a blue P_4 or a red P_3. The proof of the case of $m \in \{7, 8, 9\}$ is similar, so it is left to the reader.

In general case, the proof is by contradiction. Suppose, contrary to our claim, that we have a 3-coloring of the complete graph K_{m+1}. By Lemma 9, in order to avoid a red P_3, a blue P_4 and a green C_m, graph

K_{m+1} can have at most $T(m+1, C_m) + T(m+1, P_4) + T(m+1, P_3) \leq \frac{1}{2}m^2 - \frac{3}{2}m + 3 + m + 1 + \frac{1}{2}m + \frac{1}{2} = \frac{1}{2}m^2 + 4\frac{1}{2} < \frac{1}{2}m^2 + \frac{1}{2}m$ edges, for all $m > 9$, a contradiction. ■

3..3 The Ramsey numbers $R(P_4, P_4, C_m)$

In [1] we can find two values of Ramsey numbers of this type: $R(P_4, P_4, C_3) = 9$ and $R(P_4, P_4, C_4) = 7$.

Theorem 12. $R(P_4, P_4, C_5) = 9$.

Proof. First we shall present a critical coloring $(P_4, P_4, C_5; 8)$, thus getting $R(P_4, P_4, C_5) > 8$. Let the vertices of K_8 be labeled $1, 2, \ldots, 8$. We can assume $i, i+1, i+2, i+3$, $i \in \{1, 5\}$ to be the vertices of two K_4 which are colored as follows: the edges $\{i, i+1\}, \{i, i+2\}, \{i+1, i+2\}$ are red, vertex $i+3$ is joined by a blue edge to $i, i+1, i+2$, and the remaining edges of the graph K_8 are green.

Since $R(P_4, P_4, C_4) = 7$, we can assume $1, 2, 3, 4$ to be the vertices of green C_4. Avoiding a green cycle C_5 we know that the number of red and blue edges from vertices $5, \ldots, 9$ to green cycle is at least two. This forces a red (or a blue) path P_3: (i, x, j), where $x \in C_4$ and $i, j \in \{5, \ldots, 9\}$. Without loss of generality, we assume $x = 1$, $i = 5$, $j = 6$. We have to consider three cases.

Case 1. *The edges* $\{3, 5\}$ *and* $\{3, 6\}$ *are green.* This forces: $\{2, 5\}$, $\{4, 5\}$, $\{2, 6\}$, $\{4, 6\}$ to be blue, and we immediately have a blue P_4.

Case 2. *The edges* $\{3, 5\}$ *and* $\{3, 6\}$ *are blue.* Then it is forced: the vertices 5 and 6 is joined by green edges to each of the vertices: 2, 4, 7, 8, 9. Then $\{2, 4, 7, 8, 9\}$ is the set of vertices of the complete graph K_5. Avoiding a green cycle in K_9, there is no green edges in K_5. Since $R(P_4, P_4) = 5$, we can easily obtain a red or a blue path P_4.

Case 3. *Without loss of generality:* $\{3, 5\}$ *is blue and* $\{3, 6\}$ *is green.* This forces: $\{2, 6\}$, $\{4, 6\}$ blue, $\{2, 5\}$, $\{4, 5\}$ green, and one of the following two subcases must occur:

Case 3.1. $\{1, 3\}$ *is blue.* It forces $\{5, 7\}$, $\{5, 8\}$, $\{5, 9\}$ green, $\{4, 7\}$, $\{4, 8\}$, $\{4, 9\}$ red, $\{2, 7\}$ green, $\{4, 5\}$ blue and we have a blue path P_4.

Case 3.2. $(1,3)$ *is red.* We have two situations. In the first one, blue $\{3,7\}$ forces $\{6,7\}$ green, $\{5,7\}$ blue, $\{5,7\}$, $\{5,9\}$, $\{3,8\}$, $\{3,9\}$ green, $\{4,8\}$, $\{4,9\}$, $\{8,9\}$ red. To avoid a green cycle $(5,2,3,6,9,5)$ we have $\{6,9\}$ blue and $\{6,8\}$ is blue as well. But this forces $\{2,9\}$ green, and we have a green cycle C_5. In the latter case, green $\{3,7\}$ forces $\{2,7\}$, $\{4,7\}$ red, $\{4,8\}$, $\{4,9\}$, green, $\{5,8\}$, $\{5,9\}$ blue, $\{6,8\}$, $\{6,9\}$, $\{3,8\}$ green, and we have a green C_5 $(3,6,9,4,8,3)$. ■

In the way similar to that used in the proof of Theorem 12 we can prove

Theorem 13. $R(P_4,P_4,C_6)=8$.

This result leads us to the following conjecture.

Conjecture 14. *For all integers* $m\geq 6$, $R(P_4,P_4,C_m)=m+2$.

4. The Ramsey Numbers $R(P_3,C_m,C_p)$

In [6] Faudree, Schelten and Schiermeyer proved (without using a computer) that $R(C_7,C_7,C_7)=25$. The next definitions come from that paper.

Definition 2. By K_{12}^* we denote any graph of order 12 missing at most four edges.

Definition 3. By $ext(H,n)$ we denote the maximal number of edges a graph of order n may contain, if it does not contain a subgraph isomorphic to H.

Definition 4. $ext(C_7,n)' := \max\{|E(G)| : |V(G)|=n,\ C_7\nsubseteq G,\ B_{7,7}\nsubseteq G,\ \overline{K_{12}^*}\nsubseteq G,\ G \text{ is not bipartite}\}$.

Let us note that the graph $B_{7,7}$ is a special kind of graph, however it will not be used in our considerations.

Theorem 15. $R(P_3,C_7,C_7)=13$.

Proof. We can assume that the complete graph K_{13} is 3-colored with colors red, blue and green. Avoiding a red P_3, there are at most six red edges. Suppose that K_{13} contains only these six red edges and does not contain a blue or green C_7. Because $\binom{13}{2}=78$, either the blue or green color classes contains at least 36 edges. We have to consider two situations.

1. One of the induced color classes (blue or green) is bipartite. Without loss of generality assume this is the blue color class. One of the partitions sets has at least 7 vertices. Since $R(P_3, C_7) = 7$, the complete graph K_{13} has to contain a red P_3 or a green C_7.
2. If $K_{12}^* \subseteq K_{13}$, then one can observe that we may consider only one case: K_{12} missing at most four edges, which is colored with red and blue (green). The missing edges are colored with green (blue). It is easy to check that we quickly have a red P_3 or a blue (green) C_7.

As Faudree *et al.* [6] proved that $ext'(C_7, 13) = 33$, either we get a contradiction or a blue or green cycle C_7. ∎

Our last small Ramsey number is the following.

Theorem 16. $R(P_3, C_5, C_5) = 9$.

Proof. Consider any 3-coloring of the complete graph K_9. Avoiding a red P_3, there are at most four red edges. Suppose that K_9 contains only four red edges and does not contain a blue or a green C_5. It is obvious that there is a vertex $v \in V(G)$ such that $r(v) = 0$. Next, one of the two cases must hold.

1. the vertex v has at least five blue (green) edges,
2. the vertex v is joined by exactly four blue and four green edges to the remaining vertices of K_9.

Both of the above cases can be solved by using simple combinatoric properties and because lack of space we skip the rest of the proof (see [5] for details). ∎

These last two results lead us to the following conjecture.

Conjecture 17. *For all odd integers $m \geq 5$,*

$$R(P_3, C_m, C_m) = 2m - 1 = R(C_m, C_m).$$

With the aid of a computer we have obtained six new values for 3-color Ramsey numbers $R(P_3, C_m, C_p)$. Old and new results are summarized in the following table (the last two columns contain new values).

P_3	C_3	C_4	C_5	C_6
C_3	**11** [3]	**8** [1]	**9** [5]	**11** [5]
C_4		**8** [1]	**8** [5]	**8** [5]
C_5			**9** Thm. 16	**11** [5]
C_6				**9** [5]

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