

## DOMINATION NUMBERS IN GRAPHS WITH REMOVED EDGE OR SET OF EDGES

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### Abstract

It is known that the removal of an edge from a graph  $G$  cannot decrease a domination number  $\gamma(G)$  and can increase it by at most one. Thus we can write that  $\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1$  when an arbitrary edge  $e$  is removed. Here we present similar inequalities for the weakly connected domination number  $\gamma_w$  and the connected domination number  $\gamma_c$ , i.e., we show that  $\gamma_w(G) \leq \gamma_w(G - e) \leq \gamma_w(G) + 1$  and  $\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2$  if  $G$  and  $G - e$  are connected. Additionally we show that  $\gamma_w(G) \leq \gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$  and  $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$  if  $G$  and  $G - E_p$  are connected and  $E_p = E(H_p)$  where  $H_p$  of order  $p$  is a connected subgraph of  $G$ .

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## 1. Introduction

Let  $G = (V, E)$  be a connected undirected graph. The *neighbourhood*  $N_G(v)$  of a vertex  $v \in V$  is the set of all vertices adjacent to  $v$ . For a set  $X \subseteq V$ , the *open neighbourhood*  $N_G(X)$  is defined to be  $\bigcup_{v \in X} N_G(v)$  and the *closed neighbourhood*  $N_G[X] = N_G(X) \cup X$ . A set  $D \subseteq V$  is a *dominating set* if

$N_G[D] = V$ . Further,  $D$  is a *connected dominating set* if  $D$  is dominating and  $\langle D \rangle$ , the subgraph induced by  $D$ , is connected.

The *domination number* of  $G$ , denoted  $\gamma(G)$ , is  $\min\{|D| : D \text{ is a dominating set of } G\}$ , while the *connected domination number* of  $G$ , denoted  $\gamma_c(G)$ , is  $\min\{|D| : D \text{ is a connected dominating set of } G\}$ .

A dominating set  $D$  is a *weakly connected dominating set* if the subgraph weakly induced by  $D$ ,  $\langle D \rangle_w = (N(D), E_w)$ , is connected, where  $E_w$  is the set of all edges having at least one vertex in  $D$ . The *weakly connected domination number* of  $G$ , denoted  $\gamma_w(G)$ , is  $\min\{|D| : D \text{ is a weakly connected dominating set of } G\}$ . For unexplained terms and symbols see [1].

Let  $H_p$  be a connected subgraph of  $G$  with  $p$  vertices for  $p \geq 2$  and  $E_p = E(H_p)$  the set of edges of  $H_p$ . By  $G - e$  we denote the graph formed by removing an edge  $e$  from  $G$  and by  $G - E_p$  the graph formed by removing the set of edges  $E_p$  from  $G$ .

It is known [2] that the removal of an edge from  $G$  cannot decrease  $\gamma(G)$  and can increase it by at most one. Thus we can write that  $\gamma(G) \leq \gamma(G - e) \leq \gamma(G) + 1$  when an arbitrary edge  $e$  is removed. Here we present similar inequalities for numbers  $\gamma_c(G)$  and  $\gamma_w(G)$ , i.e., we show that  $\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2$  and  $\gamma_w(G) \leq \gamma_w(G - e) \leq \gamma_w(G) + 1$  if  $G$  and  $G - e$  are connected.

We also prove that  $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$  and  $\gamma_w(G) \leq \gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$  if  $G$  and  $G - E_p$  are connected.

## 2. Connected Domination Number

We study the behavior the connected domination number, with respect to edge or set of edges deletion. First we show that removing an edge cannot decrease the connected domination number and can increase it by at most two.

**Theorem 1.** *If  $e$  is an edge of  $G$  and if  $G$  and  $G - e$  are connected, then  $\gamma_c(G) \leq \gamma_c(G - e) \leq \gamma_c(G) + 2$ .*

**Proof.** First we show that  $\gamma_c(G) \leq \gamma_c(G - e)$ . Let  $D_0$  be a minimum connected dominating set of  $G - e$ . Certainly,  $D_0$  is a connected dominating set of  $G$ . Thus  $\gamma_c(G) \leq |D_0| = \gamma_c(G - e)$ .

Now we prove that  $\gamma_c(G - e) \leq \gamma_c(G) + 2$ . Let  $D$  be a minimum connected dominating set of  $G$  and let  $e$ , say  $e = xy$ , be an edge of  $G$  such that  $G - e$  is connected. We consider three cases.

*Case 1.*  $x, y \notin D$ . It is easy to see that  $D$  is a connected dominating set of  $G - e$  and  $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$ .

*Case 2.*  $|\{x, y\} \cap D| = 1$ , say  $x \in D, y \notin D$ . If  $N_{G-e}(y) \cap D \neq \emptyset$ , then  $D$  is a connected dominating set of  $G - e$  and we have  $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$ .

If  $N_{G-e}(y) \cap D = \emptyset$ , then  $N_{G-e}(y) \cap (V - D) \neq \emptyset$  as  $G - e$  is connected. Thus, there exists a vertex  $y' \in N_{G-e}(y) \cap (V - D)$  such that  $N_{G-e}(y') \cap D \neq \emptyset$ . In this case  $D \cup \{y'\}$  is a connected dominating set of  $G - e$  and  $\gamma_c(G - e) \leq |D \cup \{y'\}| = \gamma_c(G) + 1 \leq \gamma_c(G) + 2$ .

*Case 3.*  $x, y \in D$ . Let  $\langle D \rangle_{G-e}$  be the subgraph induced by  $D$  in  $G - e$ . If  $\langle D \rangle_{G-e}$  is connected, then  $D$  is a connected dominating set of  $G - e$  and  $\gamma_c(G - e) \leq |D| = \gamma_c(G) \leq \gamma_c(G) + 2$ .

If  $\langle D \rangle_{G-e}$  is not connected, then it has exactly two components with vertex sets, say  $D_1$  and  $D_2$ . Since  $G - e$  is connected, there exists a path connecting  $D_1$  and  $D_2$ . Let  $P = (x_1, \dots, x_k)$  be a shortest path between  $D_1$  and  $D_2$ , say  $x_1 \in D_1, x_k \in D_2$ . From the choice of  $P$  it follows that  $x_2, \dots, x_{k-1}$  belong to  $V - D$  and  $3 \leq k \leq 4$  (otherwise some of vertices from a path would not be dominated).

If  $k = 3$ , then  $D \cup \{x_2\}$  is a connected dominating set of  $G - e$  and  $\gamma_c(G - e) \leq |D \cup \{x_2\}| = \gamma_c(G) + 1 \leq \gamma_c(G) + 2$ .

If  $k = 4$ , then  $D \cup \{x_2, x_3\}$  is a connected dominating set of  $G - e$  and thus  $\gamma_c(G - e) \leq |D \cup \{x_2, x_3\}| = \gamma_c(G) + 2$ . ■

Now we study the effects on the connected domination number when a graph is modified by deleting a set of edges.

**Theorem 2.** *Let  $H_p$  be a connected subgraph of order  $p$  in  $G$ , let  $E_p$  be the edge set of  $H_p$  and let  $G - E_p$  be the graph obtained from  $G$  by deleting edges of  $E_p$ . If  $G$  and  $G - E_p$  are connected, then  $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$ .*

**Proof.** Let  $D_0$  be a minimum connected dominating set of  $G - E_p$ . Then  $D_0$  is a connected dominating set of  $G$  and obviously  $\gamma_c(G) \leq |D_0| = \gamma_c(G - E_p)$ .

We now prove the inequality  $\gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$ . Let  $D$  be a minimum connected dominating set of  $G$  and let us denote  $V(H_p) \cap D$  and  $V(H_p) \cap (V - D)$  by  $S_1$  and  $S_2$ , respectively. Certainly,  $0 \leq |S_1| \leq p$  and  $0 \leq |S_2| \leq p$ . If  $H_p$  is not a tree, then let  $\{C_1, \dots, C_k\}$  be a fundamental basis of  $H_p$ . We sequentially remove edges belonging to  $E_p$  from a graph  $G$  according to the algorithm.

INPUT: a graph  $G$ , a subgraph  $H_p$   
 OUTPUT: a spanning tree  $H'_p$  of  $H_p$   
 $H'_p := H_p$  ;  
**for**  $i = 1$  **to**  $k$  **do**  
 Let  $\{C_1, \dots, C_{k-i+1}\}$  be a fundamental basis of  $H'_p$  ;  
   **if**  $V(C_i) \subset S_1$  or  $V(C_i) \subset S_2$  **then** remove from  $H'_p$  any edge  $e$  of  $C_i$ ;  
   **else if** there exists an edge  $e$  of  $C_i$  joining two vertices of  $S_2$  **then**  
     remove  $e$  from  $H'_p$ ;  
   **else** there exists a vertex  $v$  belonging to  $V(C_i) \cap S_2$  such that its  
     neighbours on  $C_i$ , say  $x$  and  $y$ , belong to  $S_1$ , then we remove  
     from  $H'_p$  either the edge  $vx$  or  $vy$   
   **fi**;  
**fi**;  
**od**;

Let  $E_s$  be the set of edges removed according to the above algorithm. Since  $\{C_1, \dots, C_k\}$  is a fundamental basis, the graph  $H'_p = H_p - E_s$  is a spanning tree of  $H_p$ , so  $|H_p - E_s| = p - 1$  and  $|E_s| \leq \binom{p}{2} - p + 1$ . Certainly, the set  $D$  is a minimum connected dominating set of the graph  $G_0 = G - E_s$  and  $\gamma_c(G_0) = \gamma_c(G - E_s) = \gamma_c(G)$ .

Let  $e_1, \dots, e_{p-1}$  be the edges of  $H_p - E_s$  and let  $G_i = G_{i-1} - e_i = G_0 - \{e_1, \dots, e_i\}$  for  $i = 1, \dots, p - 1$ .

As  $\gamma_c(G - E_p) = \gamma_c(G_{p-1})$ , by Theorem 1 we have

$$\begin{aligned} \gamma_c(G - E_p) &= \gamma_c(G_{p-1}) \leq \gamma_c(G_{p-2}) + 2 \leq \gamma_c(G_{p-3}) + 4 \\ &\leq \dots \leq \\ \gamma_c(G_1) + 2p - 4 &\leq \gamma_c(G_0) + 2p - 2. \end{aligned}$$

Thus  $\gamma_c(G - E_p) \leq \gamma_c(G) + 2p - 2$  since  $\gamma_c(G_0) = \gamma_c(G)$ . ■

Following theorem is an obvious generalisation of obtained results.

**Theorem 3.** *If  $G$  and  $G - E_p$  are connected and  $H_p$  has  $k$  components, then  $\gamma_c(G) \leq \gamma_c(G - E_p) \leq \gamma_c(G) + 2(p - k)$ .*

### 3. Weakly Connected Domination Number

In this part we study the behavior the weakly connected domination number with respect to edge or set of edges deletion from a graph.

**Theorem 4.** *If  $e$  is an edge of a graph  $G$  and if  $G$  and  $G - e$  are connected, then  $\gamma_w(G) \leq \gamma_w(G - e) \leq \gamma_w(G) + 1$ .*

**Proof.** Let  $D_0$  be a minimum weakly connected dominating set of  $G - e$ . Certainly,  $D_0$  is also a weakly connected dominating set of  $G$  and  $\gamma_w(G) \leq |D_0| = \gamma_w(G - e)$ .

To prove the inequality  $\gamma_w(G - e) \leq \gamma_w(G) + 1$ , let  $D$  be a minimum weakly connected dominating set of  $G$ , and let  $e$ , say  $e = xy$ , be an edge of  $G$  such that  $G - e$  is connected. We consider three cases.

*Case 1.* If  $x, y \in V - D$ , then  $D$  is a weakly connected dominating set of  $G - e$  and  $\gamma_w(G - e) \leq |D| = \gamma_w(G) \leq \gamma_w(G) + 1$ .

*Case 2.*  $x, y \in D$ . Let  $F$  be the subgraph weakly induced by  $D$  in  $G - e$ . If  $F$  is connected, then  $D$  is a weakly connected dominating set of  $G - e$  and  $\gamma_w(G - e) \leq |D| = \gamma_w(G) \leq \gamma_w(G) + 1$ . If  $F$  is not connected, then it has exactly two components with vertex sets, say  $D_1, D_2$ , and suppose  $x \in D_1, y \in D_2$ .

Since  $F$  is disconnected and  $G - e$  is connected, there are adjacent vertices  $a, b \in V - D$  such that  $a \in D_1, b \in D_2$ .

Thus  $D \cup \{a\}$  (and  $D \cup \{b\}$ ) is a weakly connected dominating set of  $G - e$  and  $\gamma_w(G - e) \leq |D \cup \{a\}| = \gamma_w(G) + 1$ .

*Case 3.*  $|\{x, y\} \cap D| = 1$ , say  $x \in D, y \in V - D$ . As in Case 2, let  $F$  be the subgraph weakly induced by  $D$  in  $G - e$ . If  $F$  is connected and  $N_{G-e}(y) \cap D \neq \emptyset$ , then  $D$  is a weakly connected dominating set in  $G - e$  and we have desired inequality.

If  $F$  is connected and  $N_{G-e}(y) \cap D = \emptyset$ , then, since  $G - e$  is connected,  $N_{G-e}(y) \cap (V - D) \neq \emptyset$ . Thus, there is a vertex  $y' \in N_{G-e}(y) \cap (V - D)$  such that  $N_{G-e}(y') \cap D \neq \emptyset$ . In this case  $D \cup \{y\}$  is a weakly connected dominating set of  $G - e$  and  $\gamma_w(G - e) \leq |D \cup \{y\}| = \gamma_w(G) + 1$ .

If  $F$  is not connected, then it has exactly two components with vertex sets, say  $D_1, D_2$  and assume that  $x \in D_1, y \in D_2$ . Then it is no problem to observe that  $N_{G-e}(y) \cap D \neq \emptyset$ , i.e.,  $y$  has a neighbour in  $D$  in  $G - e$ . This implies, that  $D$  is a dominating set of  $G - e$ .

Since  $F$  is disconnected and  $G - e$  is connected, there are adjacent vertices  $a, b \in V - D$  such that  $a \in D_1, b \in D_2$ .

Thus  $D \cup \{a\}$  (and  $D \cup \{b\}$ ) is a weakly connected dominating set of  $G - e$  and  $\gamma_w(G - e) \leq |D \cup \{a\}| = \gamma_w(G) + 1$ . ■

**Theorem 5.** *Let  $H_p$  be a connected subgraph of order  $p$  in  $G$ , let  $E_p$  be the edge set of  $H_p$  and let  $G - E_p$  be the graph formed by removing edges  $E_p$  from  $G$ . If  $G$  and  $G - E_p$  are connected, then  $\gamma_w(G) \leq \gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$ .*

**Proof.** Let  $D_0$  be a minimum weakly connected dominating set of  $G - E_p$ . It is no problem to observe that  $D_0$  is a weakly connected dominating set of  $G$ , so  $\gamma_w(G) \leq |D_0| = \gamma_w(G - E_p)$ .

Now we prove that  $\gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$ . Let  $D$  be a minimum weakly connected dominating set of  $G$ . As in the proof of Theorem 2, let  $E_s$  be a subset of  $E_p$  such that  $H_p - E_s$  is a spanning tree of  $H_p$ , let  $e_1, \dots, e_{p-1}$  be the edges of  $H_p - E_s$  and  $G_i = G_{i-1} - e_i = G_0 - \{e_1, \dots, e_i\}$  for  $i = 1, \dots, p - 1$ . The set  $D$  is a minimum weakly connected dominating set of a graph  $G_0 = G - E_s$ . Thus  $\gamma_w(G_0) = \gamma_w(G - E_s) = \gamma_w(G)$ .

As  $\gamma_w(G - E_p) = \gamma_w(G_{p-1})$ , by Theorem 4 we have

$$\begin{aligned} \gamma_w(G - E_p) &= \gamma_w(G_{p-1}) \leq \gamma_w(G_{p-2}) + 1 \leq \gamma_c(G_{p-3}) + 2 \\ &\leq \dots \leq \\ &\gamma_c(G_1) + p - 2 \leq \gamma_c(G_0) + p - 1. \end{aligned}$$

Thus  $\gamma_w(G - E_p) \leq \gamma_w(G) + p - 1$  as  $\gamma_w(G_0) = \gamma_w(G)$ . ■

## References

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