ON DOUBLE DOMINATION IN GRAPHS

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Abstract

In a graph G, a vertex dominates itself and its neighbors. A subset $S \subseteq V(G)$ is a double dominating set of G if S dominates every vertex of G at least twice. The minimum cardinality of a double dominating set of G is the double domination number $\gamma_{\times 2}(G)$. A function $f(\mathbf{p})$ is defined, and it is shown that $\gamma_{\times 2}(G) = \min f(\mathbf{p})$, where the minimum is taken over the n-dimensional cube $C^n = \{\mathbf{p} = (p_1, \ldots, p_n) \mid p_i \in \mathbb{R}, 0 \leq p_i \leq 1, i = 1, \ldots, n\}$. Using this result, it is then shown that if G has order n with minimum degree δ and average degree d, then $\gamma_{\times 2}(G) \leq ((\ln(1 + d) + \ln \delta + 1)/\delta)n$.

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1.   Introduction

In this paper we continue the study of double domination in graphs started by Harary and Haynes [5] and studied further in [1, 2, 3, 8, 9, 10] and elsewhere.

Domination in graphs is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [6, 7]. For a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a *dominating set* if each vertex in $V - S$ is adjacent to at least one vertex of S. Equivalently, S is a dominating set of G if for every vertex $v \in V$, $|N[v] \cap S| \geq 1$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set.

In [5] Harary and Haynes defined a generalization of domination as follows: a subset S of V is a *k-tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq k$, that is, v is in S and has at least $k-1$ neighbors in S or v is in $V - S$ and has at least k neighbors in S. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G. Clearly, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$, while $\gamma_t(G) \leq \gamma_{\times 2}(G)$ where $\gamma_t(G)$ denotes the total domination number of G (see [6, 7]). For a graph to have a k-tuple dominating set, its minimum degree is at least $k - 1$. Hence for trees, $k \leq 2$. A k-tuple dominating set where $k = 2$ is called a *double dominating set* (DDS). A DDS of cardinality $\gamma_{\times 2}(G)$ we call a $\gamma_{\times 2}(G)$-set. The redundancy involved in k-tuple domination makes it useful in many applications.

For notation and graph theory terminology we in general follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. The degree of a vertex v in G is denoted by $d(v)$. The minimum degree among the vertices of G is denoted by $\delta(G)$, while the average degree of G is denoted by $d(G) = \frac{1}{n} \sum_{v \in V} d(v)$.

2.   Main Result

Let $I\!R$ be the set of real numbers and let $f \colon C^n = \{\mathbf{p} = (p_1, \ldots, p_n) \mid p_i \in I\!R, 0 \leq p_i \leq 1, i = 1, \ldots, n\} \to I\!R$ be the function defined by

$$f(\mathbf{p}) = \sum_{i=1}^{n} p_i + \left(\sum_{i=1}^{n} p_i \cdot \prod_{j \in N(i)} (1 - p_j) \right) + \left(2 \sum_{i=1}^{n} (1 - p_i) \cdot \prod_{j \in N(i)} (1 - p_j) \right)$$
$$+ \sum_{i=1}^{n} (1 - p_i) \left(\sum_{j \in N(i)} p_j \prod_{k \in N(i) \backslash \{j\}} (1 - p_k) \right).$$

Using similar techniques to those employed in [4], we shall show:

Lemma 1. *If G is a graph of order n, then $\gamma_{\times 2}(G) = \min_{\mathbf{p} \in C^n} f(\mathbf{p})$.*

Proof. Let $G = (V, E)$ where $V = \{1, 2, \ldots, n\}$. For $i = 1, \ldots, n$, let $d_i = d(i)$ and let i' denote a neighbor of vertex i, i.e., $i' \in N(i)$. We form a set $X \subseteq V$ by random and independent choice of $i \in V$, where $P(i \in X) = p_i$ with $0 \leq p_i \leq 1$ denotes the probability that the vertex i belongs to X. Let X_0, Y_0 and Y_1 be the sets defined by

$$X_0 = \{i \in X : |N(i) \cap X| = 0\},$$
$$Y_0 = \{i \notin X : |N(i) \cap X| = 0\}, \text{ and}$$
$$Y_1 = \{i \notin X : |N(i) \cap X| = 1\},$$

and let

$$X_0' = \bigcup_{i \in X_0} \{i'\} \quad \text{and} \quad Y_0' = \bigcup_{i \in Y_0} \{i'\}.$$

Then, $|X_0'| \leq |X_0|$ and $|Y_0'| \leq |Y_0|$. Further the set

$$D = X \cup X_0' \cup Y_0 \cup Y_0' \cup Y_1$$

is a double dominating set of G. By the linearity of expectation,

$$E(|D|) \leq E(|X| + |X_0| + 2|Y_0| + |Y_1|) = E(|X|) + E(|X_0|) + 2E(|Y_0|) + E(|Y_1|).$$

Hence using the well-known fact that for a random subset M of a given finite set N,

$$E(|M|) = \sum_{n \in N} P(n \in M),$$

we have

$$
\begin{aligned}
E(|D|) &\leq \sum_{i \in V} P(i \in X) + \sum_{i \in V} P(i \in X_0) + 2\sum_{i \in V} P(i \in Y_0) + \sum_{i \in V} P(i \in Y_1) \\
&= \sum_{i=1}^{n} p_i + \left(\sum_{i=1}^{n} p_i \cdot \prod_{j \in N(i)} (1 - p_j) \right) + \left(2\sum_{i=1}^{n} (1 - p_i) \cdot \prod_{j \in N(i)} (1 - p_j) \right) \\
&\quad + \sum_{i=1}^{n} (1 - p_i) \left(\sum_{j \in N(i)} p_j \prod_{k \in N(i)\setminus\{j\}} (1 - p_k) \right) \\
&= f(\mathbf{p}).
\end{aligned}
$$

The expectation being an average value, there is consequently a double dominating set of G of cardinality at most $E(|D|)$. Hence,

$$
\gamma_{\times 2}(G) \leq \min_{\mathbf{p} \in C^n} f(\mathbf{p}).
$$

Now let D^* be a double dominating set of G of minimum cardinality $\gamma_{\times 2}(G)$. Then for $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$ where $p_i^* = 1$ if $i \in D^*$ and $p_i^* = 0$ otherwise,

$$
f(\mathbf{p}^*) = \sum_{i=1}^{n} p_i = |D^*| = \gamma_{\times 2}(G),
$$

whence $\gamma_{\times 2}(G) = \min_{\mathbf{p} \in C^n} f(\mathbf{p})$. ■

As a consequence of Lemma 1, we have our main result.

Theorem 2. *If G is a graph of order n with $\delta = \delta(G) \geq 1$ and $d = d(G)$, then*

$$
\gamma_{\times 2}(G) \leq \left(\frac{\ln(1 + d) + \ln \delta + 1}{\delta} \right) n.
$$

Proof. Following the notation introduced in the proof of Lemma 1, we let $\mathbf{p} = (p_1, \ldots, p_n)$ and we set $p_i = p$ for all $i = 1, \ldots, n$, where $0 \leq p \leq 1$. Let $m = |E(G)|$. Then,

$$f(\mathbf{p}) \leq np + \sum_{i=1}^{n} p \cdot (1-p)^{d_i} + 2\sum_{i=1}^{n}(1-p)^{d_i+1} + \sum_{i=1}^{n} d_i \cdot p \cdot (1-p)^{d_i}$$

$$\leq np + np(1-p)^{\delta} + 2n(1-p)^{\delta+1} + p(1-p)^{\delta} \cdot 2m$$

$$\leq np + np\,e^{-\delta p} + 2n(1-p)\,e^{-\delta p} + p \cdot 2m\,e^{-\delta p}$$

$$\text{(since for } x \in I\!\!R,\ 1-x \leq e^{-x})$$

$$= np + e^{-\delta p}\left(np + 2n(1-p) + p \cdot 2m\right)$$

$$= np + e^{-\delta p}\left(2n + p(2m - n)\right)$$

$$\leq np + e^{-\delta p}(n + 2m) \qquad \text{(since } p \leq 1).$$

The function $g(p) = np + e^{-\delta p}(n + 2m)$ is minimized when $p = p^*$ where

$$e^{-\delta p^*} = \frac{n}{\delta(n + 2m)} = \frac{1}{\delta(1 + d)},$$

i.e., where $p^* = (\ln(1 + d) + \ln \delta)/\delta$. If $p^* > 1$, then $\gamma_{\times 2}(G) \leq n < p^* n$ and the desired upper bound (although meaningless in this case) follows. Hence we may assume $p^* \leq 1$. Thus, by Lemma 1,

$$\gamma_{\times 2}(G) \leq g(p^*) = np^* + \frac{n}{\delta} = \left(\frac{\ln(1 + d) + \ln \delta + 1}{\delta}\right)n,$$

which is the desired upper bound. ∎

We close with a few remarks. As with most bounds established using the probabilistic method, the upper bound in Theorem 2 is only interesting for large minimum degree. Further, for fixed minimum degree the upper bound becomes uninteresting for large average degree. We have yet to establish whether Theorem 2 is sharp.

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