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# ON DOMINATION IN GRAPHS

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**Abstract**

For a finite undirected graph $G$ on $n$ vertices two continuous optimization problems taken over the $n$-dimensional cube are presented and it is proved that their optimum values equal the domination number $\gamma$ of $G$. An efficient approximation method is developed and known upper bounds on $\gamma$ are slightly improved.

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## 1. Introduction and Results

For terminology and notation not defined here we refer to [3]. Let $V = V(G) = \{1, \ldots, n\}$ be the vertex set of an undirected graph $G$, and for $i \in V$, $N(i)$ be the neighbourhood of $i$ in $G$, $N_2(i) = \{k \in V \mid k \in \bigcup_{j \in N(i)} N(j) \setminus (\{i\} \cup N(i))\}$, $d_i = |N(i)|$, $t_i = |N_2(i)|$, $\delta = \min_{i \in V} d_i$, and $\Delta = \max_{i \in V} d_i$.

A set $D \subseteq V(G)$ is a *dominating set* of $G$ if $(\{i\} \cup N(i)) \cap D \neq \emptyset$ for every $i \in V$. The minimum cardinality of a dominating set of $G$ is the *domination* *number* $\gamma$ of $G$. In [7] $\gamma = \min_{x_1,\ldots,x_n \in [0,1]} \sum_{i\in V}(x_i + (1-x_i)\prod_{j\in N(i)}(1-x_j))$ was proved. With $x_1 = \ldots = x_n = x$ we have $\gamma \le (x + (1-x)^{\delta+1})n \le (x + e^{-(\delta+1)x})n$ for every $x \in [0,1]$. Minimizing $x + (1-x)^{\delta+1}$ and $x + e^{-(\delta+1)x}$, the well-known inequalities $\gamma \le (1 - \frac{1}{(\delta+1)^{\frac{1}{\delta}}} + \frac{1}{(\delta+1)^{\frac{\delta+1}{\delta}}})n \le \frac{1+\ln(\delta+1)}{\delta+1}n$ (see [4, 8]) follow. Obviously, it is easily checked whether $\gamma = 1$ or not. Thus, we will assume $G \in \Gamma$ in the sequel, where $\Gamma$ is the set of graphs $G$ such that each component of $G$ has domination number greater than 1. Without mentioning in each case, we will use $d_i, t_i \ge 1$ for $i = 1, \ldots, n$ if $G \in \Gamma$. For $x_1, \ldots, x_n \in [0,1]$ let

$$f_G(x_1,\ldots,x_n) = \sum_{i\in V}\left(x_i\Big(1-\Big(\prod_{j\in N(i)} x_j\Big)\Big(1-\prod_{k\in N_2(i)} x_k\Big)\Big) + (1-x_i)\prod_{j\in N(i)}(1-x_j)\right)$$

$$g_G(x_1,\ldots,x_n) = f_G(x_1,\ldots,x_n) - \sum_{i\in V}\left(\frac{1}{1+d_i}(1-x_i)\Big(\prod_{j\in N(i)}(1-x_j)\Big)\Big(\prod_{k\in N_2(i)}(1-x_k)\Big)\right).$$

**Theorem 1.** *If* $G \in \Gamma$ *then*

$$\begin{aligned}\gamma &= \min_{x_1,\ldots,x_n\in[0,1]} f_G(x_1,\ldots,x_n) = \min_{x_1,\ldots,x_n\in[0,1]} g_G(x_1,\ldots,x_n)\\ &\le \min_{x\in[0,1]}\sum_{i\in V}\left(x\Big(1-x^{d_i}(1-x^{t_i})\Big) + (1-x)^{d_i+1}\Big(1-\frac{1}{1+d_i}(1-x)^{t_i}\Big)\right)\\ &\le \min_{x\in[0,1]}\left(x\Big(1-x^{\Delta}(1-x)\Big) + (1-x)^{\delta+1}\Big(1-\frac{1}{1+\Delta}(1-x)^{\Delta(\Delta-1)}\Big)\right)n.\end{aligned}$$

Since DOMINATING SET is an NP-complete decision problem ([5]), it is difficult to solve the continuous optimization problem $\mathcal{P}$ :

$$\min_{x_1,\ldots,x_n\in[0,1]} g_G(x_1,\ldots,x_n).$$

However, if $(x_1,\ldots,x_n)$ is the solution of any approximation method for $\mathcal{P}$, then (see Theorem 2) we can easily find a dominating set of $G$ of cardinality at most $g_G(x_1,\ldots,x_n)$.

**Theorem 2.** *Given a graph $G \in \Gamma$ on $V = \{1, \dots, n\}$ with maximum degree $\Delta$, $x_1, \dots, x_n \in [0, 1]$, there is an $O(\Delta^4 n)$-algorithm finding a dominating set $D$ of $G$ with $|D| \le g_G(x_1, \dots, x_n)$.*

## 2. Proofs

***Proof of Theorem* 1.** For events $A$ and $B$ and for a random variable $Z$ of an arbitrary random space, $P(A)$, $P(A|B)$, and $E(Z)$ denote the probability of $A$, the conditional probability of $A$ given $B$, and the expectation of $Z$, respectively. Let $\overline{A}$ be the complementary event of $A$. We will use the well-known facts that $P(B)P(A|B) = P(A \cap B) = P(B) - P(\overline{A} \cap B) = P(B)(1 - P(\overline{A}|B))$ and $E(|S'|) = \sum_{s \in S} P(s \in S')$ for a random subset $S'$ of a given finite set $S$. $I \subset V$ is an *independent set* if $N(i) \cap I = \emptyset$ for all $i \in I$. Consider fixed $x_1, \dots, x_n \in [0, 1]$. $X \subseteq V$ is formed by random and independent choice of $i \in V$, where $P(i \in X) = x_i$. Let $X' = \{i \in X \mid N(i) \subseteq X\}$, $X'' = \{i \in X' \mid N(i) \cap (X \setminus X') \ne \emptyset\}$, $Y = \{i \in V \mid i \notin X \wedge N(i) \cap X = \emptyset\}$, $Y' = \{i \in Y \mid N(i) \cap Y \ne \emptyset\}$, and $I$ be a maximum independent set of the subgraph of $G$ induced by $Y'$.

**Lemma 3.** $(X \setminus X'') \cup (Y \setminus I)$ *is a dominating set of* $G$.

***Proof.*** Obviously, $X'' \subseteq X' \subseteq X$ and $(X \setminus X') \subseteq (X \setminus X'')$. If $i \in V \setminus (X \cup Y)$ then $N(i) \cap (X \setminus X') \ne \emptyset$, if $i \in X''$ then again $N(i) \cap (X \setminus X') \ne \emptyset$, and if $i \in I$ then $N(i) \cap (Y \setminus I) \ne \emptyset$. ■

**Lemma 4.** $\gamma \le E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$.

***Proof.*** Let $D$ be a random dominating set of $G$. Because of the property of the expectation to be an average value we have $\gamma \le E(|D|)$. With Lemma 3 and linearity of the expectation, $\gamma \le E(|(X \setminus X'') \cup (Y \setminus I)|) = E(|X| - |X''| + |Y| - |I|) = E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$ since $(X \setminus X'') \cap (Y \setminus I) = \emptyset$. ■

**Lemma 5.** $E(|X|) = \sum_{i \in V} x_i$, $E(|X''|) = \sum_{i \in V} x_i \Big( \prod_{j \in N(i)} x_j \Big) \Big( 1 - \prod_{k \in N_2(i)} x_k \Big)$,

$$E(|Y|) = \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j), \text{ and}$$

$$E(|I|) \ge \sum_{i \in V} \frac{1}{1 + d_i} (1 - x_i) \Big( \prod_{j \in N(i)} (1 - x_j) \Big) \Big( \prod_{k \in N_2(i)} (1 - x_k) \Big).$$

***Proof.*** $E(|X|) = \sum_{i \in V} P(i \in X) = \sum_{i \in V} x_i$.

$$E(|X''|) = \sum_{i \in V} P(i \in X'') = \sum_{i \in V} P(i \in X \wedge N(i) \subseteq X \wedge N(i) \cap (X \setminus X') \neq \emptyset)$$

$$= \sum_{i \in V} P(i \in X) P(N(i) \subseteq X) P(N(i) \cap (X \setminus X') \neq \emptyset \mid i \in X \wedge N(i) \subseteq X)$$

$$= \sum_{i \in V} x_i \Big( \prod_{j \in N(i)} x_j \Big) (1 - P(N(i) \subseteq X' \mid i \in X \wedge N(i) \subseteq X))$$

$$= \sum_{i \in V} x_i \Big( \prod_{j \in N(i)} x_j \Big) (1 - P(N_2(i) \subseteq X)) = \sum_{i \in V} x_i \Big( \prod_{j \in N(i)} x_j \Big) \Big( 1 - \prod_{k \in N_2(i)} x_k \Big).$$

$$E(|Y|) = \sum_{i \in V} P(i \in Y) = \sum_{i \in V} P(i \notin X) P(N(i) \cap X = \emptyset)$$

$$= \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j).$$

A lower bound on $|I|$ (see [1, 9, 2, 6]) is given by the following inequality $|I| \geq \sum_{i \in Y'} \frac{1}{1+d_i}$. For $i \in V(G)$ define the random variable $Z_i$ with $Z_i = \frac{1}{1+d_i}$ if $i \in Y'$ and $Z_i = 0$ if $i \notin Y'$. Hence,

$$E(|I|) \geq E\Big( \sum_{i \in V} Z_i \Big) = \sum_{i \in V} E(Z_i) = \sum_{i \in V} \frac{1}{1+d_i} P(i \in Y')$$

$$= \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X \wedge N(i) \cap X = \emptyset \wedge N(i) \cap Y \neq \emptyset).$$

Because $d_i \geq 1$, $N_2(i) \cap X = \emptyset$ implies $N(i) \cap Y \neq \emptyset$. Hence,

$$E(|I|) \geq \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X \wedge N(i) \cap X = \emptyset \wedge N_2(i) \cap X = \emptyset)$$

$$= \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X) P(N(i) \cap X = \emptyset) P(N_2(i) \cap X = \emptyset)$$

$$= \sum_{i \in V} \frac{1}{1+d_i} (1 - x_i) \Big( \prod_{j \in N(i)} (1 - x_j) \Big) \Big( \prod_{k \in N_2(i)} (1 - x_k) \Big).$$

■

From Lemma 4 and Lemma 5 we have $\gamma \leq g_G(x_1,\ldots,x_n) \leq f_G(x_1,\ldots,x_n)$. Let $D^*$ be a minimum dominating set of $G$ and let $y_i = 1$ if $i \in D^*$ and $y_i = 0$ if $i \notin D^*$. Then $y_i \prod_{j\in N(i)} y_j = 0$ and $(1-y_i)\prod_{j\in N(i)}(1-y_j) = 0$ for every $i \in V$, $\gamma = |D^*| = \sum_{i\in V} y_i = g_G(y_1,\ldots,y_n) = f_G(y_1,\ldots,y_n)$, and the proof of Theorem 1 is complete. ■

***Proof of Theorem* 2.** Given a graph $H$ on $n_H$ vertices with $m_H$ edges, there is an $O(n_H + m_H)$-algorithm $\mathcal{A}$ finding an independent set of $H$ with cardinality at least $\sum_{y\in V(H)} \frac{1}{1+d_H(y)}$, where $d_H(y)$ is the degree of $y \in V(H)$ in $H$ (see [2]).

First we present an algorithm that constructs a set $D \subseteq V$.

**Algorithm**
INPUT: a graph $G \in \Gamma$ on $V = \{1,\ldots,n\}$, $x_1,\ldots,x_n \in [0,1]$
OUTPUT: $D$
(1) For $l = 1,\ldots,n$ do if $\frac{\partial g_G(x_1,\ldots,x_n)}{\partial x_l} \geq 0$ then $x_l := 0$ else $x_l := 1$.
(2) $X := \{l \in \{1,\ldots,n\} \mid x_l = 1\}$. Calculate $X''$,$Y$,$Y'$, and $I$ using $\mathcal{A}$.
(3) $D := (X \setminus X'') \cup (Y \setminus I)$.
END

Let $g^* = g_G(x_1,\ldots,x_n)$, where $(x_1,\ldots,x_n)$ is the input vector. Note that the function $g_G$ is linear in each variable. Thus, in step (1), for fixed $x_1,\ldots,x_{l-1},x_{l+1},\ldots,x_n$ we always choose $x_l$ in such a way that the value of $g_G(x_1,\ldots,x_n)$ is not increased. Hence, $x_l \in \{0,1\}$ for $l = 1,\ldots,n$ and $g_G(x_1,\ldots,x_n) \leq g^*$ after step (1) of the algorithm. With Lemma 3, $D$ is a dominating set, and with $|S| = E(|S|)$ for a deterministic set $S$ and Lemma 5, $|D| \leq g^*$. It is easy to see that $\frac{\partial g_G(x_1,\ldots,x_n)}{\partial x_l}$ can be calculated in $O(\Delta^4)$ time. Since $G$ has $O(\Delta n)$ edges, the algorithm runs in $O(\Delta^4 n)$ time. ■

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