

ON DOMINATION IN GRAPHS

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Abstract

For a finite undirected graph G on n vertices two continuous optimization problems taken over the n -dimensional cube are presented and it is proved that their optimum values equal the domination number γ of G . An efficient approximation method is developed and known upper bounds on γ are slightly improved.

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1. Introduction and Results

For terminology and notation not defined here we refer to [3]. Let $V = V(G) = \{1, \dots, n\}$ be the vertex set of an undirected graph G , and for $i \in V$, $N(i)$ be the neighbourhood of i in G , $N_2(i) = \{k \in V \mid k \in \bigcup_{j \in N(i)} N(j) \setminus (\{i\} \cup N(i))\}$, $d_i = |N(i)|$, $t_i = |N_2(i)|$, $\delta = \min_{i \in V} d_i$, and $\Delta = \max_{i \in V} d_i$.

A set $D \subseteq V(G)$ is a *dominating set* of G if $(\{i\} \cup N(i)) \cap D \neq \emptyset$ for every $i \in V$. The minimum cardinality of a dominating set of G is the *domination*

number γ of G . In [7] $\gamma = \min_{x_1, \dots, x_n \in [0,1]} \sum_{i \in V} (x_i + (1-x_i) \prod_{j \in N(i)} (1-x_j))$ was proved. With $x_1 = \dots = x_n = x$ we have $\gamma \leq (x + (1-x)^{\delta+1})n \leq (x + e^{-(\delta+1)x})n$ for every $x \in [0, 1]$. Minimizing $x + (1-x)^{\delta+1}$ and $x + e^{-(\delta+1)x}$, the well-known inequalities $\gamma \leq (1 - \frac{1}{(\delta+1)^{\frac{1}{\delta}}} + \frac{1}{(\delta+1)^{\frac{\delta+1}{\delta}}})n \leq \frac{1+\ln(\delta+1)}{\delta+1}n$ (see [4, 8]) follow. Obviously, it is easily checked whether $\gamma = 1$ or not. Thus, we will assume $G \in \Gamma$ in the sequel, where Γ is the set of graphs G such that each component of G has domination number greater than 1. Without mentioning in each case, we will use $d_i, t_i \geq 1$ for $i = 1, \dots, n$ if $G \in \Gamma$. For $x_1, \dots, x_n \in [0, 1]$ let

$$\begin{aligned} f_G(x_1, \dots, x_n) &= \sum_{i \in V} \left(x_i \left(1 - \left(\prod_{j \in N(i)} x_j \right) \left(1 - \prod_{k \in N_2(i)} x_k \right) \right) + (1-x_i) \prod_{j \in N(i)} (1-x_j) \right) \\ g_G(x_1, \dots, x_n) &= f_G(x_1, \dots, x_n) \\ &\quad - \sum_{i \in V} \left(\frac{1}{1+d_i} (1-x_i) \left(\prod_{j \in N(i)} (1-x_j) \right) \left(\prod_{k \in N_2(i)} (1-x_k) \right) \right). \end{aligned}$$

Theorem 1. *If $G \in \Gamma$ then*

$$\begin{aligned} \gamma &= \min_{x_1, \dots, x_n \in [0,1]} f_G(x_1, \dots, x_n) = \min_{x_1, \dots, x_n \in [0,1]} g_G(x_1, \dots, x_n) \\ &\leq \min_{x \in [0,1]} \sum_{i \in V} \left(x \left(1 - x^{d_i} (1 - x^{t_i}) \right) + (1-x)^{d_i+1} \left(1 - \frac{1}{1+d_i} (1-x)^{t_i} \right) \right) \\ &\leq \min_{x \in [0,1]} \left(x \left(1 - x^\Delta (1-x) \right) + (1-x)^{\delta+1} \left(1 - \frac{1}{1+\Delta} (1-x)^{\Delta(\Delta-1)} \right) \right) n. \end{aligned}$$

Since DOMINATING SET is an NP-complete decision problem ([5]), it is difficult to solve the continuous optimization problem \mathcal{P} :

$$\min_{x_1, \dots, x_n \in [0,1]} g_G(x_1, \dots, x_n).$$

However, if (x_1, \dots, x_n) is the solution of any approximation method for \mathcal{P} , then (see Theorem 2) we can easily find a dominating set of G of cardinality at most $g_G(x_1, \dots, x_n)$.

Theorem 2. *Given a graph $G \in \Gamma$ on $V = \{1, \dots, n\}$ with maximum degree Δ , $x_1, \dots, x_n \in [0, 1]$, there is an $O(\Delta^4 n)$ -algorithm finding a dominating set D of G with $|D| \leq g_G(x_1, \dots, x_n)$.*

2. Proofs

Proof of Theorem 1. For events A and B and for a random variable Z of an arbitrary random space, $P(A)$, $P(A|B)$, and $E(Z)$ denote the probability of A , the conditional probability of A given B , and the expectation of Z , respectively. Let \bar{A} be the complementary event of A . We will use the well-known facts that $P(B)P(A|B) = P(A \cap B) = P(B) - P(\bar{A} \cap B) = P(B)(1 - P(\bar{A}|B))$ and $E(|S'|) = \sum_{s \in S} P(s \in S')$ for a random subset S' of a given finite set S . $I \subset V$ is an *independent set* if $N(i) \cap I = \emptyset$ for all $i \in I$. Consider fixed $x_1, \dots, x_n \in [0, 1]$. $X \subseteq V$ is formed by random and independent choice of $i \in V$, where $P(i \in X) = x_i$. Let $X' = \{i \in X \mid N(i) \subseteq X\}$, $X'' = \{i \in X' \mid N(i) \cap (X \setminus X') \neq \emptyset\}$, $Y = \{i \in V \mid i \notin X \wedge N(i) \cap X = \emptyset\}$, $Y' = \{i \in Y \mid N(i) \cap Y \neq \emptyset\}$, and I be a maximum independent set of the subgraph of G induced by Y' .

Lemma 3. $(X \setminus X'') \cup (Y \setminus I)$ is a dominating set of G .

Proof. Obviously, $X'' \subseteq X' \subseteq X$ and $(X \setminus X') \subseteq (X \setminus X'')$. If $i \in V \setminus (X \cup Y)$ then $N(i) \cap (X \setminus X') \neq \emptyset$, if $i \in X''$ then again $N(i) \cap (X \setminus X') \neq \emptyset$, and if $i \in I$ then $N(i) \cap (Y \setminus I) \neq \emptyset$. ■

Lemma 4. $\gamma \leq E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$.

Proof. Let D be a random dominating set of G . Because of the property of the expectation to be an average value we have $\gamma \leq E(|D|)$. With Lemma 3 and linearity of the expectation, $\gamma \leq E(|(X \setminus X'') \cup (Y \setminus I)|) = E(|X| - |X''| + |Y| - |I|) = E(|X|) - E(|X''|) + E(|Y|) - E(|I|)$ since $(X \setminus X'') \cap (Y \setminus I) = \emptyset$. ■

Lemma 5. $E(|X|) = \sum_{i \in V} x_i$, $E(|X''|) = \sum_{i \in V} x_i \left(\prod_{j \in N(i)} x_j \right) \left(1 - \prod_{k \in N_2(i)} x_k \right)$,

$$E(|Y|) = \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j), \text{ and}$$

$$E(|I|) \geq \sum_{i \in V} \frac{1}{1 + d_i} (1 - x_i) \left(\prod_{j \in N(i)} (1 - x_j) \right) \left(\prod_{k \in N_2(i)} (1 - x_k) \right).$$

Proof. $E(|X|) = \sum_{i \in V} P(i \in X) = \sum_{i \in V} x_i.$

$$\begin{aligned}
E(|X''|) &= \sum_{i \in V} P(i \in X'') = \sum_{i \in V} P(i \in X \wedge N(i) \subseteq X \wedge N(i) \cap (X \setminus X') \neq \emptyset) \\
&= \sum_{i \in V} P(i \in X) P(N(i) \subseteq X) P(N(i) \cap (X \setminus X') \neq \emptyset \mid i \in X \wedge N(i) \subseteq X) \\
&= \sum_{i \in V} x_i \left(\prod_{j \in N(i)} x_j \right) (1 - P(N(i) \subseteq X' \mid i \in X \wedge N(i) \subseteq X)) \\
&= \sum_{i \in V} x_i \left(\prod_{j \in N(i)} x_j \right) (1 - P(N_2(i) \subseteq X)) = \sum_{i \in V} x_i \left(\prod_{j \in N(i)} x_j \right) \left(1 - \prod_{k \in N_2(i)} x_k \right). \\
E(|Y|) &= \sum_{i \in V} P(i \in Y) = \sum_{i \in V} P(i \notin X) P(N(i) \cap X = \emptyset) \\
&= \sum_{i \in V} (1 - x_i) \prod_{j \in N(i)} (1 - x_j).
\end{aligned}$$

A lower bound on $|I|$ (see [1, 9, 2, 6]) is given by the following inequality $|I| \geq \sum_{i \in Y'} \frac{1}{1+d_i}$. For $i \in V(G)$ define the random variable Z_i with $Z_i = \frac{1}{1+d_i}$ if $i \in Y'$ and $Z_i = 0$ if $i \notin Y'$. Hence,

$$\begin{aligned}
E(|I|) &\geq E\left(\sum_{i \in V} Z_i\right) = \sum_{i \in V} E(Z_i) = \sum_{i \in V} \frac{1}{1+d_i} P(i \in Y') \\
&= \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X \wedge N(i) \cap X = \emptyset \wedge N(i) \cap Y \neq \emptyset).
\end{aligned}$$

Because $d_i \geq 1$, $N_2(i) \cap X = \emptyset$ implies $N(i) \cap Y \neq \emptyset$. Hence,

$$\begin{aligned}
E(|I|) &\geq \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X \wedge N(i) \cap X = \emptyset \wedge N_2(i) \cap X = \emptyset) \\
&= \sum_{i \in V} \frac{1}{1+d_i} P(i \notin X) P(N(i) \cap X = \emptyset) P(N_2(i) \cap X = \emptyset) \\
&= \sum_{i \in V} \frac{1}{1+d_i} (1 - x_i) \left(\prod_{j \in N(i)} (1 - x_j) \right) \left(\prod_{k \in N_2(i)} (1 - x_k) \right).
\end{aligned}$$

■

From Lemma 4 and Lemma 5 we have $\gamma \leq g_G(x_1, \dots, x_n) \leq f_G(x_1, \dots, x_n)$. Let D^* be a minimum dominating set of G and let $y_i = 1$ if $i \in D^*$ and $y_i = 0$ if $i \notin D^*$. Then $y_i \prod_{j \in N(i)} y_j = 0$ and $(1 - y_i) \prod_{j \in N(i)} (1 - y_j) = 0$ for every $i \in V$, $\gamma = |D^*| = \sum_{i \in V} y_i = g_G(y_1, \dots, y_n) = f_G(y_1, \dots, y_n)$, and the proof of Theorem 1 is complete. ■

Proof of Theorem 2. Given a graph H on n_H vertices with m_H edges, there is an $O(n_H + m_H)$ -algorithm \mathcal{A} finding an independent set of H with cardinality at least $\sum_{y \in V(H)} \frac{1}{1 + d_H(y)}$, where $d_H(y)$ is the degree of $y \in V(H)$ in H (see [2]).

First we present an algorithm that constructs a set $D \subseteq V$.

Algorithm

INPUT: a graph $G \in \Gamma$ on $V = \{1, \dots, n\}$, $x_1, \dots, x_n \in [0, 1]$

OUTPUT: D

- (1) For $l = 1, \dots, n$ do if $\frac{\partial g_G(x_1, \dots, x_n)}{\partial x_l} \geq 0$ then $x_l := 0$ else $x_l := 1$.
- (2) $X := \{l \in \{1, \dots, n\} \mid x_l = 1\}$. Calculate X'', Y, Y' , and I using \mathcal{A} .
- (3) $D := (X \setminus X'') \cup (Y \setminus I)$.

END

Let $g^* = g_G(x_1, \dots, x_n)$, where (x_1, \dots, x_n) is the input vector. Note that the function g_G is linear in each variable. Thus, in step (1), for fixed $x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n$ we always choose x_l in such a way that the value of $g_G(x_1, \dots, x_n)$ is not increased. Hence, $x_l \in \{0, 1\}$ for $l = 1, \dots, n$ and $g_G(x_1, \dots, x_n) \leq g^*$ after step (1) of the algorithm. With Lemma 3, D is a dominating set, and with $|S| = E(|S|)$ for a deterministic set S and Lemma 5, $|D| \leq g^*$. It is easy to see that $\frac{\partial g_G(x_1, \dots, x_n)}{\partial x_l}$ can be calculated in $O(\Delta^4)$ time. Since G has $O(\Delta n)$ edges, the algorithm runs in $O(\Delta^4 n)$ time. ■

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