# A GALLAI-TYPE EQUALITY FOR THE TOTAL DOMINATION NUMBER OF A GRAPH

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#### Abstract

We prove the following Gallai-type equality

$$\gamma_t(G) + \varepsilon_t(G) = p$$

for any graph G with no isolated vertex, where p is the number of vertices of G,  $\gamma_t(G)$  is the total domination number of G, and  $\varepsilon_t(G)$  is the maximum integer s such that there exists a spanning forest F with s the number of pendant edges of F minus the number of star components of F.

 $\mathbf{Keywords:}\;$  domination number; total domination number; Gallai equality.

2000 Mathematics Subject Classification: 05C69.

### 1. Introduction

Let G = (V(G), E(G)) be a graph with p = |V(G)| vertices. Let  $\alpha(G), \beta(G), \alpha'(G)$  and  $\beta'(G)$  be the vertex covering number, the vertex independence number, the edge covering number and the edge independence number of G, respectively. In [3], Gallai established his now classic equalities involving these invariants:

<sup>\*</sup>Supported by a Discovery Project Grant (DP0344803) from the Australian Research Council.

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(I) 
$$\alpha(G) + \beta(G) = p$$

(II) 
$$\alpha'(G) + \beta'(G) = p$$
,

here in (II) G is assumed to have no isolated vertices. Now there are a number of similar Gallai-type equalities for a variety of graphical invariants. The reader is referred to [2] for a comprehensive survey on this topic. The purpose of this paper is to prove a Gallai-type equality for the total domination number of G.

A subset D of V(G) is said to be a dominating set of G if each vertex in V(G) - D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is the domination number of G, denoted by  $\gamma(G)$ . A dominating set D is a total dominating set of G if the subgraph G[D] induced by D has no isolated vertex. Note that G admits total dominating sets if and only if it contains no isolated vertex. In such a case, the total domination number of G, denoted by  $\gamma_t(G)$ , is defined to be the minimum cardinality of a total dominating set of G. A dominating set D of G is a connected domination number  $\gamma_c(G)$  is the minimum cardinality of a connected domination number  $\gamma_c(G)$  is the minimum cardinality of a connected dominating set of G. A degree-one vertex of a graph is said to be a pendant vertex; and an edge incident with a pendant vertex is a pendant edge of the graph. Denote by  $\varepsilon(G)$  the maximum number of pendant edges in a spanning forest of G. In [6] Nieminen gave the following Gallai-type equality for domination number  $\gamma(G)$ .

Theorem 1 ([6]). 
$$\gamma(G) + \varepsilon(G) = p$$
.

A similar equality holds for connected domination number. Denote by  $\varepsilon_T(G)$  the maximum number of pendant edges in a spanning tree of a connected graph G. Hedetniemi and Laskar [5] proved

(1) 
$$\gamma_c(G) + \varepsilon_T(G) = p$$

for any connected graph G. To the best knowledge of the author, there has been no similar Gallai-type equality so far for total domination number in the literature. In this paper we will provide such an equality, which has the same spirit as above.

For a spanning forest F of G, we denote by s(F) the number of pendant edges of F minus the number of star components of F. (A star is a complete bipartite graph  $K_{1,n}$  for some  $n \geq 1$ .) Denote by  $\varepsilon_t(G)$  the maximum s(F)

taken over all spanning forests F of G. Our main result is the following theorem.

**Theorem 2.** Let G be a graph with no isolated vertex. Then

$$\gamma_t(G) + \varepsilon_t(G) = p.$$

### 2. Proof of Theorem 2

In order to prove Theorem 2, let us first review some basic ideas (see [1, 2, 4]) involved in the derivation of a lot of known Gallai-type equalities.

Let S be a finite set and Q a property associated with the subsets of S. If a subset X of S possesses Q, then we call X a Q-set; otherwise a  $\overline{Q}$ -set. In the following we suppose that Q is cohereditary (or expanding as used in [2]) in the sense that whenever X is a Q-set and  $X \subseteq Y \subseteq S$  then Y is a Q-set. We say that  $Y \subseteq S$  is a  $Q^*$ -set if  $X \cup Y \neq S$  holds for each  $\overline{Q}$ -set X. Let  $\beta_Q(S)$  be the minimum cardinality of a Q-set of S, and  $\alpha_Q(S)$  the maximum cardinality of a  $Q^*$ -set of S. It is not difficult to see [2, Theorem 2'] that  $X \subseteq S$  is a Q-set if and only if  $\overline{X} = S - X$  is a  $Q^*$ -set. This implies the following basic Gallai-type equality:

(2) 
$$\alpha_Q(S) + \beta_Q(S) = |S|.$$

**Proof of Theorem 2.** Let V = V(G) be the vertex set of G. Let Q be the property defined on the subsets of V such that  $X \subseteq V$  is a Q-set if and only if it is a total dominating set of G. Then obviously Q is cohereditary and  $\beta_Q(V) = \gamma_t(G)$ . We have the following claim.

**Claim.** A subset Y of V is a  $Q^*$ -set if and only if Y is a set of pendant vertices of a spanning forest F of G such that

- (a) F contains no isolated vertex;
- (b) each edge of F is incident with at most one vertex in Y; and
- (c) the removal of Y from F results in a forest with no isolated vertices.

In fact, if  $Y \subseteq V$  is a  $Q^*$ -set, then V - Y is a total dominating set according to the discussion above. Thus, for each  $y \in Y$  there exists an edge, say  $e_y$ ,

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joining y and a vertex in V-Y. Also, the subgraph G[V-Y] of G induced by V-Y has no isolated vertex. Let  $E_Y$  be a minimal subset of the edge set of G[V-Y] such that it induces a spanning subgraph of G[V-Y] with no isolated vertex. By the minimality,  $E_Y$  induces a spanning forest of G[V-Y]. Thus, the graph induced by the edges  $E_Y \cup \{e_y : y \in Y\}$  is a spanning forest F of G satisfying (a), (b) and (c) above, and Y is a set of pendant vertices of F. Conversely, if  $Y \subseteq V$  is a set of pendant vertices of a spanning forest F of G such that (a), (b) and (c) are satisfied, then any  $X \subseteq V(G)$  with  $X \cup Y = V(G)$  is a total dominating set of G. In other words, in such a case Y is a  $Q^*$ -set and hence the Claim is proved.

Now by the Claim above  $\alpha_Q(V)$  is equal to the maximum cardinality of a subset Y of V such that Y is a set of pendant vertices of a spanning forest F of G satisfying (a), (b) and (c). Note that for a fixed spanning forest F with no isolated vertices, a set Y of pendant vertices of F satisfying (b) and (c) has the maximum cardinality if and only if Y contains all the pendant vertices of each non-star component and n-1 pendant vertices of each star component  $K_{1,n}$  of F. In other words, the maximum cardinality of a set Y of pendant vertices of F satisfying (b) and (c) is precisely s(F). Thus,  $\alpha_O(V)$  is the maximum s(F) taken over all spanning forests F with no isolated vertex. For a spanning forest F of G with isolated vertices, say  $x_1, x_2, \ldots, x_n$   $(1 \le n \le p)$ , since G contains no isolated vertex, each  $x_i$  is either adjacent to another  $x_i$  or adjacent to a vertex in a nontrivial component of F. (A nontrivial component is a connected component with at least two vertices.) Hence we can add some edges of G to F such that each  $x_i$  is incident with exactly one of the added edges. In this way we get a new spanning forest F' of G containing no isolated vertex. It is not difficult to check that  $s(F) \leq s(F')$ . Thus,  $\alpha_Q(V)$  is actually the maximum s(F)taken over all spanning forests F. That is,  $\alpha_Q(V) = \varepsilon_t(G)$ . Now from (2) we get  $\gamma_t(G) + \varepsilon_t(G) = p$  and the proof of Theorem 2 is complete.

We notice that Theorem 1 can be derived from (2) in a similar way. In fact, let Q be the property associated with the subsets of V = V(G) such that  $X \subseteq V$  is a Q-set if and only if X is a dominating set of G. Then Q is cohereditary and  $\beta_Q(V) = \gamma(G)$ . By an argument similar to the proof of Theorem 2 we get  $\alpha_Q(V) = \varepsilon(G)$  and hence Theorem 1 follows from (2). Note that (1) cannot be derived from (2) in a similar way since the property of being a connected dominating set is not a cohereditary property.

## References

- [1] B. Bollobás, E.J. Cockayne and C.M. Mynhardt, On Generalized Minimal Domination Parameters for Paths, Discrete Math. 86 (1990) 89–97.
- [2] E.J. Cockayne, S.T. Hedetniemi and R. Laskar, Gallai Theorems for Graphs, Hypergraphs and Set Systems, Discrete Math. 72 (1988) 35–47.
- [3] T. Gallai, Über Extreme Punkt-und Kantenmengen, Ann. Univ. Sci. Budapest Eötvös Sect. Math. 2 (1959) 133–138.
- [4] S.T. Hedetniemi, *Hereditary Properties of Graphs*, J. Combin. Theory **14** (1973) 16–27.
- [5] S.T. Hedetniemi and R. Laskar, *Connected Domination in Graphs*, in: B. Bollobás ed., Graph Theory and Combinatorics (Academic Press, 1984) 209–218.
- [6] J. Nieminen, Two Bounds for the Domination Number of a Graph, J. Inst. Math. Appl. 14 (1974) 183–187.

Received 9 October 2003 Revised 14 April 2004