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CYCLE-PANCYCLISM IN BIPARTITE TOURNAMENTS II

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Abstract

Let T be a hamiltonian bipartite tournament with n vertices, γ a hamiltonian directed cycle of T, and k an even number. In this paper the following question is studied: What is the maximum intersection with γ of a directed cycle of length k contained in $T[V(\gamma)]$? It is proved that for an even k in the range $\frac{n+6}{2} \leq k \leq n-2$, there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 4$ and the result is best possible. In a previous paper a similar result for $4 \leq k \leq \frac{n+4}{2}$ was proved.

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1. Introduction

The subject of pancyclism has been studied by several authors (e.g. [1, 2, 3, 5, 13, 15]). Three types of pancyclism have been considered. A digraph D is: *pancyclic* if it has directed cycles of all the possible lengths; *vertex-pancyclic* if given any vertex v there are directed cycles of every length containing v; and *arc-pancyclic* if given any arc e there are directed cycles of every length containing e.

It is well known that a hamiltonian bipartite tournament is pancyclic, and vertex-pancyclic (with only very few exceptions) but not necessarily arc-pancyclic (see for example [3, 12, 14]). The concept of cycle-pancyclism studies the following question: Given a directed cycle γ of a digraph D, find the maximum number of arcs which a directed cycle of length k, (if such a directed cycle exists) contained in $D[V(\gamma)]$ (the subdigraph of D induced by $V(\gamma)$) has in common with γ. Cycle-pancyclism in tournaments has been studied in [6, 7, 8, 9]. In a previous paper [10] it was attempted to study cycle-pancyclism in bipartite tournaments; in fact it was proved that for an even k, $4 \leq k \leq \frac{n+4}{2}$ there exists a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$ with $|A(\mathcal{C}_{h(k)}) \cap A(\gamma)| \geq h(k) - 3$ and the result is best possible. In this paper, the study of cycle-pancyclism in bipartite tournaments is completed. To study this question it is sufficient to consider a hamiltonian bipartite tournament where γ is a hamiltonian directed cycle (because we are looking for directed cycles of length k contained in $D[V(\gamma)]$ whose arcs intersect the arcs of γ the must possible). We will assume (without saying it explicitly in Lemmas, Theorems or Corollaries) that we are working in a hamiltonian bipartite tournament with a vertex set $V = \{0, 1, \ldots, n-1\}$ and arc set A. Also we assume without loss of generality that $\gamma = (0, 1, \ldots, n-1, 0)$ is a hamiltonian directed cycle of T; k will be an even number; $\mathcal{C}_{h(k)}$ will denote a directed cycle of length $h(k)$ with $h(k) \in \{k, k-2\}$ and $\mathcal{I}\left(\mathcal{C}_{h(k)}\right) = |A(\mathcal{C}_{h(k)}) \cap A(\gamma)|$. This paper is the second part of the study on the existence of a directed cycle $\mathcal{C}_{h(k)}$ where $\mathcal{I}(\mathcal{C}_{h(k)})$ is the maximum. For general concepts we refer the reader to [4].

2. Preliminaries

A *chord* of a cycle C is an arc not in C with both terminal vertices in C. The *length* of a chord $g = (u, v)$ of C, denoted $\ell(g)$, is equal to the length of $\langle u, C, v\rangle$ where $\langle u, C, v\rangle$ denotes the uv-directed path contained in C. We say that g is a *c-chord* if $\ell(g) = c$ and $g = (u, v)$ is a *−c-chord* if $\ell\langle v, C, u\rangle = c$. Observe that if g is a c-chord then it is also a $-(n-c)$-chord. All the chords considered in this paper are chords of γ, also observe that since T is bipartite all the the chords of γ have odd lengths. We will denote by $\mathcal{C}_k$ a directed cycle of length k. In what follows all notation is taken modulo n. In what follows we assume $k \geq 10$ (In [10] it was proved that for $k = 4, 6, 8$ there exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$).

Observation 2.1. *If $n = 2k-6$, then there exists a directed cycle $\mathcal{C}_{k-2}$ with $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$.*

Proof. Consider the arc between 0 and $k-3$; when $(0,k-3) \in A$ we have $\mathcal{C}_{k-2} = (0,k-3) \cup \langle k-3,\gamma,0\rangle$ a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$; when $(k-3,0) \in A$ we obtain $\mathcal{C}_{k-2} = \langle 0,\gamma,k-3\rangle \cup (k-3,0)$ a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$. ■

In view of Observation 2.1 we will assume in what follows that $k+2 \le n \le 2k-8$.

Lemma 2.2. *At least one of the following properties holds:*

(i) *There exists a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \ge h(k)-1$ ($h(k) \in \{k,k-2\}$).*

(ii) *All the following arcs are in A:* (a) *Every $(k-1)$-chord;* (b) *Every $(k-3)$-chord.*

Proof. Suppose that (i) is not true. (a) If $(k-1,0)$ is a $-(k-1)$-chord, then $\mathcal{C}_k = \langle 0,\gamma,k-1\rangle \cup (k-1,0)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_k) = k-1$. (b) If $(k-3,0)$ is a $-(k-3)$-chord, then $\mathcal{C}_{k-2} = \langle 0,\gamma,k-3\rangle \cup (k-3,0)$ is a directed cycle with $\mathcal{I}(\mathcal{C}_{k-2}) = k-3$ ■

Lemma 2.3. *Let $P = (x,x+1,\dots,\ell)$, x,ℓ even, $\ell \ge x+4$, be a directed path contained in γ, let z be odd, $x \in V-V(P)$, and $\{(x,z),(x+2,z),(z,\ell),(z,\ell-2),\dots,(z,\ell-(a-1))\} \subseteq A$ with a odd, $1 \le a \le \ell-x-3$. Then there exists an index i, $x+2 \le i \le \ell-(a+1)$, such that $\{(i,z),(z,i+a+1)\} \subseteq A$.*

Proof. Let $i \in V(P)$ be the maximum vertex in P such that $(i,z) \in A$ clearly $x+2 \le i \le \ell-(a+1)$ and $\{(i,z),(z,i+a+1)\} \subseteq A$. ■

Lemma 2.4. *If all the $(k-3),(k-1),\dots,p$-chords, p is odd, $k-1 \le p < n-3$ are in T, then at least one of the two following properties holds.*

(i) *There exists a directed cycle $\mathcal{C}_k$ with $\mathcal{I}(\mathcal{C}_k) \ge k-3$.*

(ii) *Every $(p+2)$-chord is in T.*

Proof. We show that if (ii) is false, then (i) holds. Let (s_1,s_2) be a $-(p+2)$-chord and let z be odd in $\langle s_1,\gamma,s_2\rangle - \{s_1,s_2\}$. Assume w.l.o.g. that $s_2 = 0$. Let $x = z+n-p \pmod n$. Observe that

$$\{(x,z),(x+2,z),\dots,(x+p-(k-3),z)\} \subseteq A\,. \tag{1}$$

since these are the $p, p-2, \dots, (k-3)$-chords of γ ending in z. Similarly

$$\{(z, z+p), (z, z+p-2), \dots, (z, z+k-3)\} \subseteq A\,. \tag{2}$$

Observe that the start points of the arcs in set (1) are consecutive in-neighbors of z in γ and less than the endpoints of the arcs in set (2), which are consecutive out-neighbors of z in γ. This is because the largest start point of an arc in (1) is $z+n-(k-3)$ and the last endpoint of an arc in (2) is $z+(k-3)$ and $z+(k-3) > z+n-(k-3)$ (as $n \leq 2k-8$).

Now, consider the directed path $\langle x, \gamma, z+p\rangle$. Since $x = z+n-p (\operatorname{mod} n)$ and $2p > n$ it is obvious that $z \notin V(\langle x, \gamma, z+p\rangle)$. Note that the cardinality of (1) is at least 2 and the cardinality of (2) is $\frac{p-k+5}{2}$. Thus letting $a = p-k+4$ and $\ell = z+p$ it follows from Lemma 2.3 that there exists j, $x \leq j < z+(k-3)$ such that $\{(j,z), (z, j+a+1)\} \subseteq A$. And then $C = (s_1, s_2) \cup \langle s_2, \gamma, j\rangle \cup (j, z, j+a+1) \cup \langle j+a+1, \gamma, s_1\rangle$ is a directed cycle. In order to see that $\ell(C) = k$ note that $\ell\langle s_1, \gamma, s_2\rangle = n-(p+2)$, and thus $\ell\langle s_2, \gamma, s_1\rangle = p+2$. Clearly, $\ell\langle j, \gamma, j+a+1\rangle = a+1$, therefore $\ell(C) = p+2-(a+1)+3 = k$ and $\mathcal{I}(C) = k-3$. ∎

It follows directly from Lemmas 2.2 and 2.4 the following

Theorem 2.5. *At least one of the following conditions holds.*

(i) *There exists a directed cycle* $\mathcal{C}_{h(k)}$ *with* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-3$.

(ii) *For each* p *odd,* $k-3 \leq p \leq n-3$, *every* p*-chord of* γ *is in* T.

3. The Main Result

In this section we prove the following

Theorem 3.1. *For every* k *such that* $\frac{n+6}{2} \leq k \leq n-2$, *there exists a directed cycle* $\mathcal{C}_{h(k)}$ *with* $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k)-4$.

Proof. In view of Observation 2.1 we will assume $k \geq \frac{n+8}{2}$. It follows from Theorem 2.5 that we can assume that for each odd p, $k-3 \leq p \leq n-3$, every p-chord is in T; i.e., for each odd q $3 \leq q \leq n-(k-3)$, every $(-q)$-chord is in T. This assumption will be maintained in the whole proof. Let s be the minimum integer such that γ has an s-chord. Note that $s \geq n-k+5$ (s is odd), and thus $-s \leq k-5$. This is because for each odd p, $k-3 \leq p \leq n-3$, every p-chord is in T, and therefore for each $3 \leq n-p \leq n-k+3$, every $(n-p)$-chord is not in T. Let $g=(u,v)$ be an s-chord of γ.

Denote by w the last vertex of $\langle v+1,\gamma,u-1\rangle$ such that there exists an arc (w,z) with $z \in \langle u+1,\gamma,v-1\rangle$. Notice that $\ell\langle u,\gamma,v\rangle \geq 7$ (because $\ell\langle u,\gamma,v\rangle = s \geq n-k+5$ and $n \geq k+2$), and thus $\langle u+1,\gamma,v-1\rangle$ has at least one vertex. Also, the vertex w is well defined because $(v+2,v-1)$ is a (-3)-chord and hence it is in A. Hence for every vertex $x \in \langle w+1,\gamma,u-1\rangle$, every (x',x) arc with $x' \in \langle u+1,\gamma,v-1\rangle$ and $x \not\equiv x' \pmod 2$, is in A; by Definition of w. Also for any x, x' such that: $x \not\equiv x' \pmod 2$, $2 \leq \ell\langle x,\gamma,x'\rangle < s$, and $\{x,x'\} \subseteq V(\langle u,\gamma,v-1\rangle)$ the arc (x',x) is in A because of the definition of s. Therefore we have the following Claims:

Claim 1. (a) For every $z_1 \in \langle z,\gamma,v-1\rangle$ and $u_1 \in \langle u,\gamma,z-1\rangle$ such that $u_1 \not\equiv z_1 \pmod 2$ and $\ell\langle u_1,\gamma,z_1\rangle \geq 2$, it holds $(z_1,u_1) \in A$.

(b) For every $u_2 \in \langle u+1,\gamma,v-1\rangle$ and $w_1 \in \langle w+1,\gamma,u\rangle$ such that $w_1 \not\equiv u_2 \pmod 2$ and $\ell\langle w_1,\gamma,u_2\rangle \geq 2$, it holds $(u_2,w_1) \in A$.

As a direct consequence we have the following Claim.

Claim 2. (a) If $z_1 \in \langle z,\gamma,v-1\rangle$ and $w_1 \in \langle w+1,\gamma,u\rangle$ such that $z_1 \not\equiv w_1 \pmod 2$ and $\ell\langle w_1,\gamma,z_1\rangle \geq 2$, then

$$\mathcal{C}^1 = (z_1,w_1) \cup \langle w_1,\gamma,u\rangle \cup (u,v) \cup \langle v,\gamma,w\rangle \cup (w,z) \cup \langle z,\gamma,z_1\rangle$$

is a directed cycle of T of length m with $\mathcal{I}(\mathcal{C}^1) = m-3$.

(b) If $z_1 \in \langle z,\gamma,v-1\rangle$, $w_1 \in \langle w+1,\gamma,u\rangle$, $u_1 \in \langle u+1,\gamma,z-1\rangle$ and $u_2 \in \langle u_1,\gamma,z-1\rangle$ such that: $u_1 \not\equiv z_1 \pmod 2$; $\ell\langle u_1,\gamma,z_1\rangle \geq 2$, $w_1 \not\equiv u_2 \pmod 2$ and $\ell\langle w_1,\gamma,u_2\rangle \geq 2$, then

$$\begin{aligned}\mathcal{C}^2 = (z_1,u_1) &\cup \langle u_1,\gamma,u_2\rangle \cup (u_2,w_1) \cup \langle w_1,\gamma,u\rangle \\ &\cup (u,v) \cup \langle v,\gamma,w\rangle \cup (w,z) \cup \langle z,\gamma,z_1\rangle\end{aligned}$$

is a directed cycle of length q with $\mathcal{I}(\mathcal{C}^2) = q-4$.

Observe that $\mathcal{C}^2$ is a directed cycle because $\langle u+1,\gamma,v-1\rangle$ is non-empty, and because $z \neq u_2$, $u \neq u_1$, $w \neq w_1$ and $v \neq z_1$. A similar observation holds for $\mathcal{C}^1$.

We proceed to prove the existence of a directed cycle of length k intersecting γ in at least $k-4$ arcs. We split the problem into several cases according to the position of z in $\langle u+1, \gamma, v-1\rangle$ and according to $\ell\langle u+1, \gamma, v-1\rangle$. We are able to use constructions equal or similar to $\mathcal{C}^1$ or $\mathcal{C}^2$. Consider the path $\alpha = (u, v) \cup \langle v, \gamma, w\rangle \cup (w, z)$, and let $r = k - \ell(\alpha)$. We now extend α to a directed cycle $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 4$. Observe that since $\ell\langle v, \gamma, u\rangle = n - s \leq k - 5$ (because $s \geq n - k + 5$) it follows that $\ell\langle v, \gamma, u - 1\rangle \leq k - 6$ and $\ell(\alpha) \leq k - 4$; hence $r \geq 4$.

Case 1. $\ell\langle w, \gamma, u\rangle - 1 + \ell\langle z, \gamma, v\rangle - 1 \geq r - 1 > 0$.
Let r_1 and r_2 be such that $r_1 + r_2 = r - 1$, $0 \leq r_1 \leq \ell\langle w, \gamma, u\rangle - 1$, $0 \leq r_2 \leq \ell\langle z, \gamma, v\rangle - 1$, $w_1 = u - r_1$ and $z_1 = z + r_2$.

The proof that $(z_1, w_1) \in A$ is as follows:
First, we prove that $\ell\langle w_1, \gamma, z_1\rangle \geq 2$. We have that $\ell\langle w_1, \gamma, z_1\rangle = z + r_2 - (u - r_1) = z - u + r - 1$, since $r_1 + r_2 = r - 1$; the definition of z implies $z - u \geq 1$, and therefore $z - u + r - 1 \geq r \geq 4$.

Now we prove that $z_1 \not\equiv w_1 \pmod 2$ by considering two possible cases: When $\ell(\alpha)$ is odd we have: r is odd (because $r = k - \ell(\alpha)$, and k even), $r - 1$ is even, $r_1 = r_2 \pmod 2$ (because $r_1 + r_2 = r - 1$), and consecuently $r_2 = -r_1 \pmod 2$ (Notice $r_1 \equiv -r_1 \pmod 2$); moreover, $\ell\langle v, \gamma, w\rangle$ is odd which implies $\ell\langle z, \gamma, v\rangle$ is even (Notice that since (w, z) is a chord we have $\ell\langle w, \gamma, z\rangle$ is odd and $\ell\langle z, \gamma, w\rangle$ is odd and therefore $\ell\langle u, \gamma, z\rangle$ is odd (because $\ell\langle u, \gamma, v\rangle$ is odd); so we have $u \not\equiv z \pmod 2$, we conclude $u - r_1 \not\equiv z + r_2 \pmod 2$ (because $u - r_1 \equiv z + r_2 \pmod 2$ implies $u \equiv z \pmod 2$ as $r_2 \equiv -r_1 \pmod 2$)). When $\ell(\alpha)$ is even we have: r is even, $r - 1$ is odd, $r_1 \not\equiv r_2 \pmod 2$, $r_2 \not\equiv -r_1 \pmod 2$; moreover, $\ell\langle v, \gamma, w\rangle$ is even, $\ell\langle z, \gamma, v\rangle$ is odd, $\ell\langle u, \gamma, z\rangle$ is even; so we have $u \equiv z \pmod 2$ and since $r_2 \not\equiv -r_1$ we conclude $u - r_1 \not\equiv z + r_2 \pmod 2$. So in any case we have $u - r_1 \not\equiv z + r_2 \pmod 2$.

It follows from Claim 2 that $\mathcal{C}^1$ is a directed cycle of length $\ell(\alpha) + r_1 + r_2 + 1 = \ell(\alpha) + r = k$ with $\mathcal{I}(\mathcal{C}^1) = k - 3$.

Case 2. $\ell\langle w, \gamma, u\rangle - 1 + \ell\langle z, \gamma, v\rangle - 1 < r - 1$.
Observe that $\ell\langle w, \gamma, u\rangle - 1 + \ell\langle z, \gamma, v\rangle - 1 + \ell\langle u, \gamma, z\rangle - 1 = n - \ell(\alpha) - 1 > k - \ell(\alpha) = r$. Thus $\ell\langle u, \gamma, z\rangle - 1 \geq r - (\ell\langle w, \gamma, u\rangle - 1 + \ell\langle z, \gamma, v\rangle - 1) + 1$

From the hyphotesis of Case 2, $\ell\langle u, \gamma, z\rangle \geq 4$.
Let $r_3 = r - (\ell\langle w, \gamma, u\rangle - 1 + \ell\langle z, \gamma, v\rangle - 1) - 2$. It follows from the hyphotesis of Case 2 that $0 \leq r_3 < \ell\langle u, \gamma, z\rangle - 1$.

Denote: $w_1 = w + 1$, $u_1 = u + 1$, $u_2 = u + r_3 + 1 = u_1 + r_3$ and $z_1 = v - 1$.

We have that $\ell\langle u_1, \gamma, z_1\rangle = v - 1 - u - 1 = v - u - 2 \geq 2$. The last equality follows because $\ell\langle u, \gamma, v\rangle = s \geq n - k + 5$ and $k \leq n - 2$, and hence $\ell\langle u, \gamma, v\rangle \geq 7$. So $\ell\langle u_1, \gamma, z_1\rangle \geq 2$ and the fact $u_1 \not\equiv z_1 \pmod 2$ (because $(u, v) \in A$ implies $u + 1 \not\equiv v - 1 \pmod 2$ (as $1 \equiv -1 \pmod 2$)).

Hence $(z_1, u_1) \in A$.

Now $\ell\langle w_1, \gamma, u_2\rangle = u + r_3 + 1 - w - 1 = u - w + r_3$. Since by Definition $u - w \geq 1$ and $r_3 \geq 0$; it is sufficient to consider two possibilities: $u_2 - w_1 \geq 2$ and $u_2 - w_1 = 1$.

Case 2.a. $\ell\langle w_1, \gamma, u_2\rangle \geq 2$.
In this case we only need to prove that $w_1 \not\equiv u_2 \pmod 2$ in order to have $(u_2, w_1) \in A$. Since $w \not\equiv z \pmod 2$ (because $(w, z) \in A$), and $w_1 = w + 1$ it suffices to prove $\ell\langle u_2, \gamma, z\rangle$ is odd. We proceed by contradiction; suppose $\ell\langle u_2, \gamma, z\rangle$ is even and consider the two possible cases: When $\ell(\alpha)$ is odd we have r is odd, $r - 2$ is odd, and $\ell\langle v, \gamma, w\rangle$ is odd. Moreover, since $\ell\langle v, \gamma, w\rangle$ is odd we have $\ell\langle z, \gamma, z_1\rangle$ is odd (because $z_1 = v - 1$ and $\ell\langle z, \gamma, w\rangle$ is odd as $(w, z) \in A$); now $\ell\langle u_1, \gamma, u_2\rangle$ is even (Notice $\ell\langle u_1, \gamma, z_1\rangle$ is odd because $(z_1, u_1) \in A$) and $\ell\langle w_1, \gamma, u\rangle$ is odd (Notice $\ell\langle w, \gamma, z\rangle$ is odd and $\ell\langle u_1, \gamma, z\rangle$ is even), so we obtain $\ell\langle w_1, \gamma, u\rangle + \ell\langle u_1, \gamma, u_2\rangle + \ell\langle z, \gamma, z_1\rangle$ is even. Also we have $\ell\langle w_1, \gamma, u\rangle + \ell\langle u_1, \gamma, u_2\rangle + \ell\langle z, \gamma, z_1\rangle = \ell\langle w, \gamma, u\rangle - 1 + r_3 + \ell\langle z, \gamma, v\rangle - 1 = r - 2$ is odd; a contradiction. The case when $\ell(\alpha)$ is even is completely analogous (by interchanging even with odd).

We conclude $(u_2, w_1) \in A$ and by Claim 2 $\mathcal{C}_q = \mathcal{C}^2$ is a directed cycle with $\mathcal{I}(\mathcal{C}^2) = q - 4$. Furthermore, $q = k$ since by construction $\ell(\mathcal{C}^2) = \ell(\alpha) + r = k$.

We now divide the remaining case of $u_2 - w_1 = 1$ into two subcases. In both the following holds: $u - w + r_3 = 1$ (because $u_2 - w_1 = 1$), $w = u - 1$, $r_3 = 0$, $u = w_1$ and $u_2 = u_1 = u + 1$. Also, by Definition of r_3, $r = \ell\langle w, \gamma, u\rangle + \ell\langle z, \gamma, v\rangle$.

Case 2.b.1. $u_2 - w_1 = 1$ and $z \leq z_1 - 1$.
Notice that $\ell\langle u, \gamma, z_1 - 1\rangle = \ell\langle u, \gamma, v\rangle - 2 \geq 3$. (because $s \geq 5$). Moreover, since $u \not\equiv v \pmod 2$ (as $(u, v) \in A$) we have $u \not\equiv z_1 - 1 = v - 2 \pmod 2$. Thus $(z_1 - 1, u) \in A$, and the fact $z \leq z_1 - 1$ implies $C_q = \alpha \cup \langle z, \gamma, z_1 - 1\rangle \cup (z_1 - 1, u)$ is a directed cycle of length q with $\mathcal{I}(C_q) = q - 3$.

Now we prove $q = k - 2$; observe that $r = u - w + v - z$, and since $w = u - 1$ and $v = z_1 + 1$ we obtain $r = z_1 - z + 2$ and $z_1 - 1 - z = r - 3$. Thus $\ell(C_q) = \ell(\alpha) + r - 3 + 1 = \ell(\alpha) + r - 2 = k - 2$.

We conclude the proof of Theorem with the next subcase.

Case 2.b.2. $u_2 - w_1 = 1$ and $z = z_1 = v - 1$.
From $r_3 = 0$, $w = u - 1$, $z = v - 1$ and $r = \ell\langle w, \gamma, u\rangle + \ell\langle z, \gamma, v\rangle$ it follows that $r = 2$, thus $\ell(\alpha) = k - r = k - 2$. Since $\ell\langle v, \gamma, w\rangle = \ell(\alpha) - 2 = k - 4$ and $w = u - 1$ we have $\ell\langle v, \gamma, u\rangle = k - 3$ and hence $\mathcal{C}_{k-2} = \langle v, \gamma, u\rangle \cup (u, v)$ is a directed cycle of length $k - 2$ with $\mathcal{I}(\mathcal{C}_{k-2}) = k - 3$. ■

4. Remarks

In this section it is proved that the hyphotesis of Theorem 3.1 are tight, and the result is best possible.

Definition 4.1 [10]**.** A digraph D with vertex set V is called cyclically p-partite complete ($p \geq 3$) provided one can partition $V = V_0 \cup V_1 \cup \cdots \cup V_{p-1}$ so that (u, v) is an arc of D if and only if $u \in V_i$, $v \in V_{i+1}$ (notation modulo p).

Remark 4.2 [10]**.** The cyclically 4-partite complete digraph T_4 is a bipartite tournament and clearly, every directed cycle of T_4 has length$\equiv 0 \pmod 4$. So for $k = 4m + 2$, T_4 has no directed cycles of length k and for $k = 4m$, T_4 has no directed cycles of length $k - 2$.

Remark 4.3. For $n \geq 6$, $k \geq 6$, such that $n \leq 2k - 8$, there exits a bipartite hamiltonian tournament T_n with no directed cycles $\mathcal{C}_{h(k)}$ with $\mathcal{I}(\mathcal{C}_{h(k)}) \geq h(k) - 3$. (recall $h(k) \in \{k, k-2\}$).

Proof. Define T_n as follows:

$$A(T_n) = \{(i, i+1) \mid i \in \{0, 1, \ldots, n-1\}\}$$
$$\cup \left\{(i, i+j) \mid j \in \left\{\frac{n}{2} + 1, \frac{n}{2} + 3, \dot{s}, n - 3\right\}\right\}.$$

Notice that $n \equiv 0 \pmod 4$, otherwise the arc $(i, i+j)$ is not defined.

Consider a directed cycle $\mathcal{C}_{h(k)}$ of length $h(k)$, $h(k) \in \{k, k-2\}$. Observe that the definition of T_n and the fact $n \leq 2k - 8$ imply $\mathcal{I}(\mathcal{C}_{h(k)}) < h(k) - 2$. We prove that $\mathcal{I}(\mathcal{C}_{h(k)}) < h(k) - 3$ by showing that for any directed cycle $\mathcal{C}$ with $\mathcal{I}(\mathcal{C}) = k - 3$, it holds $\ell(\mathcal{C}) \leq k - 4$.

Let $f_1 = (x_1, x_2)$, $f_2 = (x_3, x_4)$, and $f_3 = (x_5, x_6)$ be the three arcs of $\mathcal{C}$ not in γ. Hence without loss of generality

$$\mathcal{C} = (x_1, x_2) \cup \langle x_2, \gamma, x_3\rangle \cup (x_3, x_4) \cup \langle x_4, \gamma, x_5\rangle \cup (x_5, x_6) \cup \langle x_6, \gamma, x_1\rangle .$$

By the definition of T_n it follows that $\ell(f_i) \geq \frac{n}{2} + 1$, for each $i \in \{1, 2, 3\}$. On the other hand,

$$\begin{aligned} \ell(\mathcal{C}) &= \ell\langle x_2, \gamma, x_1\rangle + \ell\langle x_6, \gamma, x_5\rangle - \ell\langle x_3, \gamma, x_4\rangle + 3 \\ &= n - \ell(f_1) + n - \ell(f_3) - \ell(f_2) + 3 \\ &\leq \frac{n}{2} - 1 + \frac{n}{2} - 1 - \frac{n}{2} - 1 + 3 \\ &= \frac{n}{2} \end{aligned}$$

Therefore $\ell(\mathcal{C}) \leq k - 4$, since $n \leq 2k - 8$. ∎

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