Discussiones Mathematicae
Graph Theory 24 (2004) 509–527

DIFFERENCE LABELLING OF DIGRAPHS

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Abstract

A digraph G is a *difference digraph* iff there exists an $S \subset I\!N^+$ such that G is isomorphic to the digraph $DD(S) = (V, A)$, where $V = S$ and $A = \{(i, j) : i, j \in V \wedge i - j \in V\}$.

For some classes of digraphs, e.g. alternating trees, oriented cycles, tournaments etc., it is known, under which conditions these digraphs are difference digraphs (cf. [5]). We generalize the so-called *source-join* (a construction principle to obtain a new difference digraph from two given ones (cf. [5])) and construct a difference labelling for the source-join of an even number of difference digraphs.

As an application we obtain a sufficient condition guaranteeing that certain (non-alternating) trees are difference digraphs.

Keywords: graph labelling, difference digraph, oriented tree.

2000 Mathematics Subject Classification: 05C78, 05C20.

1. Introduction and Basic Definitions

Harary [11] introduced the notion of sum graphs and difference graphs in 1988. In recent years, a lot of authors published papers dealing with sum graphs, e.g. [1, 2, 6, 9, 10, 12] – [20], or sum hypergraphs, cf. [23] – [28].

Some classes of difference graphs (paths, trees, cycles, cacti, special wheels, complete graphs, complete bipartite graphs etc.) were investigated by Bloom, Burr, Eggleton, Gervacio, Hell, Sonntag and Taylor in the

undirected (cf. [3, 4, 7, 21]) as well as in the directed case (cf. [5]). In [3, 4, 7] undirected difference graphs were referred to as *autographs* or *monographs*.

In our paper we generalize the *source-join* (a construction principle to obtain a new difference digraph from two given ones (cf. [5])) for even number of digraphs. As an application difference labellings can be constructed for a class of trees.

All digraphs considered in this article are supposed to be *oriented graphs*, i.e., nonempty and finite without loops, multiple arcs and 2-cycles.

As usually, a vertex v of a digraph $G = (V, A)$ is called a *source* [*sink*] iff v has in-degree [out-degree] 0.

Let $G = (V, A)$ be a digraph. G is a *difference digraph* iff there exist a finite $S \subset I\!N^+$ and a bijection $r : V \longrightarrow S$ such that $A = \{(u, v) : u, v \in V \wedge r(u) - r(v) \in S\}$. We call the bijection r a *difference labelling* of the difference digraph $G = (V, A)$.

Most of the time we will refer to vertices of difference digraphs by their labels. With this in mind, for finite $S \subset I\!N^+$ we denote $DD(S) = (V, A)$ as the *difference digraph of* S iff $V = S$ and $A = \{(i, j) : i, j \in V \wedge i - j \in V\}$.

Obviously, if $G = (V, A)$ is a difference digraph with difference labelling r, then G is isomorphic to $DD(S)$, where $S = \{r(v) : v \in V\}$ (and the isomorphism is defined by $V \ni v \mapsto r(v) \in S$).

Note whenever $i - j \in V$, the difference digraph $G = (V, A)$ must include the arc (i, j).

As an example of a difference digraph, consider the oriented wheel in Figure 1.

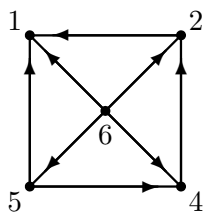

Figure 1

In difference digraphs there are only two different types of arcs: the first one is an arc of the form $(2x, x)$, the second one is an arc (z, x) with $z = x + y$, where $y \in V \setminus \{x, z\}$ and $(z, y) \in A$, i.e., arcs of the second type always appear in pairs (cf. Figure 2).

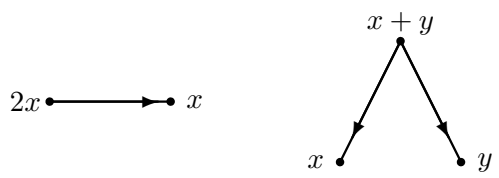

Figure 2

In [5] a pair of adjacent arcs is called an *inpair* [*outpair*] iff the arcs have the same terminal [initial] vertex. An inpair and an outpair having one arc in common is an *intersecting inpair and outpair* (cf. Figure 3).

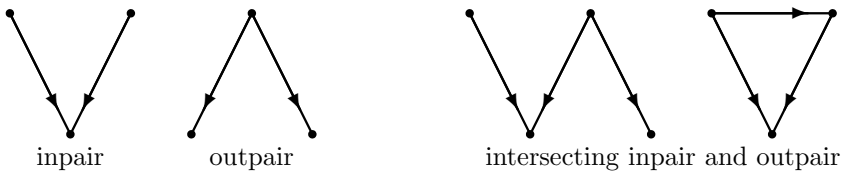

Figure 3

The following Theorem of Eggleton and Gervacio has been very useful for our investigations.

Theorem 1.1 [5]**.** *In a difference digraph, every inpair intersects an outpair.*

We say that a given digraph $G = (V, A)$ fulfills the *Inpair-Outpair-Condition* (IOC) iff in G every inpair intersects an outpair.

In Figure 4 there are examples to demonstrate that the IOC is not sufficient for a digraph to be a difference digraph. To see this, start the labelling procedure at the marked vertices and try to avoid pairs of vertices having the same label. For G_1 and G_3 this is impossible (for G_3 some modifications of the given labelling are possible but result in the same problem). The labelling of G_2 would involve the existence of the arcs $(4x, 3x)$, $(4x, x) \notin A(G_2)$.

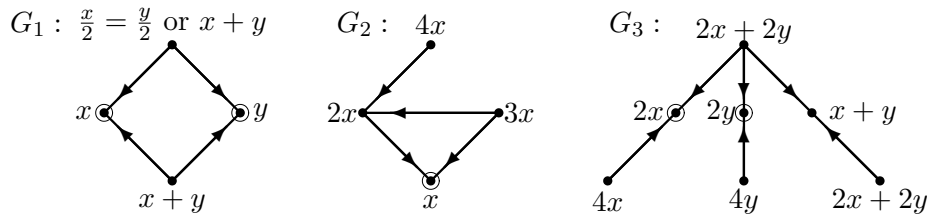

Figure 4. Three non-difference digraphs fulfilling the IOC

2. Generalized Source-Join

In [5] the *source-join* $G_1 \otimes G_2 = (V, A)$ of two disjoint difference digraphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ is defined as follows: let $v_1 \in V_1$ and $v_2 \in V_2$ be two vertices and $s \notin V_1 \cup V_2$ a new vertex. Then $G_1 \otimes G_2$ has the vertex set $V = V_1 \cup V_2 \cup \{s\}$ and the arc set $A = A_1 \cup A_2 \cup \{(s, v_1), (s, v_2)\}$. Hence the new vertex s is a source in $G_1 \otimes G_2$ which is referred to as *the source of* $G_1 \otimes G_2$.

Eggleton and Gervacio [5] proved the source-join $G_1 \otimes G_2 = (V, A)$ to be a difference digraph if $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$ are difference digraphs. To construct a difference labelling for $G_1 \otimes G_2$ they started with difference labellings of G_1 and G_2 and used the following labelling method

(LM): Choose primes $q_1 \neq q_2$ with $q_2 > \max V_1$ and $q_1 > \max V_2$.
Label the source $s \notin V_1 \cup V_2$ of $G_1 \otimes G_2$ by $s := q_1 v_1 + q_2 v_2$.
Relabel vertices $v \in V_1$ by $v := q_1 v$ and vertices $v \in V_2$ by $v := q_2 v$.

We generalize the source-join to an even number d of disjoint difference digraphs $G_1 = (V_1, A_1), G_2 = (V_2, A_2), \ldots, G_d = (V_d, A_d)$. To this end we choose $v_1 \in V_1, \ldots, v_d \in V_d$, a new vertex $s \notin \bigcup_{i=1}^{d} V_i$ and define the (*generalized*) *source-join* $G = \bigotimes_{i=1}^{d} G_i = (V, A)$ by $V = \bigcup_{i=1}^{d} V_i \cup \{s\}$ and $A = \bigcup_{i=1}^{d} A_i \cup \{(s, v_1), (s, v_2), \ldots, (s, v_d)\}$.

We construct the following labelling of $V(G_1 \otimes G_2 \otimes \ldots \otimes G_d)$:

Let the difference digraphs G_i be difference labelled and m be the maximum label of the vertices of $\bigcup_{i=1}^{d} V_i$. Choose primes $p_1, \ldots, p_d$ such that

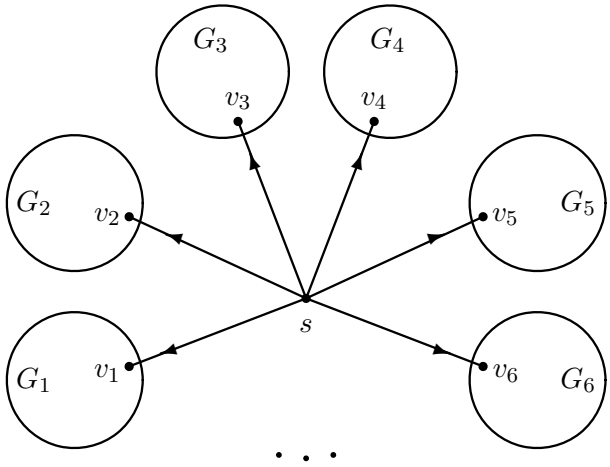

Figure 5. Generalized source-join $G_1 \otimes G_2 \otimes \ldots \otimes G_d$

$$p_1 > 2^{\sqrt{2}-1} m^{2\sqrt{2}-1} \tag{1}$$

and

$$\forall i \in \{1, \ldots, d-1\} : p_{i+1} > 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} p_i^{\sqrt{2}} \tag{2}$$

holds. For odd $i = 1, 3, \ldots, d-1$, let

$$P_i := \prod_{\text{odd } k=1 (k \neq i)}^{d-1} (p_k v_k + p_{k+1} v_{k+1})$$

and relabel the vertices $v_i \in V_i$ by $v_i := P_i \cdot p_i \cdot v_i$ as well as $v_{i+1} \in V_{i+1}$ by $v_{i+1} := P_i \cdot p_{i+1} \cdot v_{i+1}$. Finally, we label the source s by

$$s := \prod_{\text{odd } k=1}^{d-1} (p_k v_k + p_{k+1} v_{k+1}).$$

To demonstrate that this labelling is a difference labelling, in the proof of the corresponding theorem we will construct the same labelling in a slightly modified way: we apply (LM) to $G_i \otimes G_{i+1}$, for all odd $i \in \{1, 3, \ldots, d-1\}$,

then we relabel the vertices of $V(G_1 \otimes G_2 \otimes \ldots \otimes G_d)$ using the numbers P_i. In the second step we verify that all vertices of the source-join have obtained different labels and only the arcs of $G_1 \otimes G_2 \otimes \ldots \otimes G_d$ have been generated by this labelling.

In order to prove that the labelling induces no "additional" arcs, we need a technical lemma.

Lemma 2.1.

$$\forall i \in \{1, \ldots, d-1\} : p_{i+1} > 2m^2 p_i \,. \tag{3}$$

$$\forall i \in \{1, \ldots, d-2\} : p_{i+2} > 4m^3 p_i^2 \,. \tag{4}$$

Proof. Let $i \in \{1, \ldots, d-1\}$. Using (1) and (2) we get

$$\begin{aligned} p_{i+1} &> 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} p_i^{\sqrt{2}} = 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} p_i^{\sqrt{2}-1} p_i \\ &> 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} \left(2^{\sqrt{2}-1} m^{2\sqrt{2}-1}\right)^{\sqrt{2}-1} p_i \\ &= 2^{\frac{2}{\sqrt{2}+1}+(\sqrt{2}-1)(\sqrt{2}-1)} m^{\frac{3}{\sqrt{2}+1}+(2\sqrt{2}-1)(\sqrt{2}-1)} p_i \\ &= 2^{\frac{2+(2-1)(\sqrt{2}-1)}{\sqrt{2}+1}} m^{\frac{3+(2-1)(2\sqrt{2}-1)}{\sqrt{2}+1}} p_i = 2m^2 p_i \end{aligned}$$

as well as

$$\begin{aligned} p_{i+2} &> 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} p_{i+1}^{\sqrt{2}} > 2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} \left(2^{\frac{2}{\sqrt{2}+1}} m^{\frac{3}{\sqrt{2}+1}} p_i^{\sqrt{2}}\right)^{\sqrt{2}} \\ &= 2^{\left(\frac{2}{\sqrt{2}+1}+\frac{2\sqrt{2}}{\sqrt{2}+1}\right)} m^{\left(\frac{3}{\sqrt{2}+1}+\frac{3\sqrt{2}}{\sqrt{2}+1}\right)} p_i^2 = 2^2 m^3 p_i^2. \end{aligned}$$

■

Theorem 2.1. *The labelling described above is a difference labelling of the generalized source-join $\bigotimes_{i=1}^{d} G_i$ of the difference digraphs $G_1, G_2, \ldots, G_d$, for even d.*

Proof. Unless otherwise agreed, in the following $u_i, v_i, \ldots$ denote the (labels of the) vertices $u_i, v_i, \ldots \in V_i$, for $i = 1, 2, \ldots, d$, where the notations $v_1, v_2, \ldots, v_d$ are reserved for (the original labels of) the successors of the

source $s \notin \bigcup_{i=1}^{d} V_i$ in $\bigotimes_{i=1}^{d} G_i$ (cf. Figure 5). In detail, by *original label* we mean the label of a vertex in the difference digraphs $G_1, G_2, \ldots, G_d$ before the relabelling procedure.

We begin with difference labellings of $G_1, \ldots, G_d$. For every odd $i \in \{1, 3, \ldots, d-1\}$, we apply the labelling method (LM) from [5] to the source-join $G_i \otimes G_{i+1}$ with the primes p_i and p_{i+1}, respectively, i.e., we label the source s of $G_i \otimes G_{i+1}$ by $s := p_i v_i + p_{i+1} v_{i+1}$ and relabel vertices $v \in V_i$ by $v := p_i v$ and vertices $v \in V_{i+1}$ by $v := p_{i+1} v$. Note that (1) and Lemma 2.1 guarantee $p_{i+1} > \max V_i$ and $p_i > \max V_{i+1}$ (cf. (LM)).

Then we multiply the labels of all vertices of $G_i \otimes G_{i+1}$ by P_i and obtain a new difference labelling of $G_i \otimes G_{i+1}$, for all odd $i \in \{1, 3, \ldots, d-1\}$, with the property that s has the same label in $G_1 \otimes G_2, G_3 \otimes G_4, \ldots, G_{d-1} \otimes G_d$. Consequently, every arc of $\bigotimes_{i=1}^{d} G_i$ is generated by our vertex labelling.

Now we demonstrate

(a) different vertices have different labels and

(b) the labelling does not induce "new" arcs, i.e., arcs which are not contained in $A(\bigotimes_{i=1}^{d} G_i)$.

Obviously, no problems occur if we consider vertices $v, v' \in V(G_i \otimes G_{i+1})$ and arcs between such vertices, for odd $i \in \{1, 3, \ldots, d-1\}$.

To (a): Assume, we have labels $u_{i'} = u_{j'}$ with $i' \in \{i, i+1\}$ and $j' \in \{j, j+1\}$, where $i \neq j$ are odd elements of $\{1, 3, \ldots, d-1\}$. Moreover, let $x_{i'}$ and $x_{j'}$ be the original labels of $u_{i'}$ and $u_{j'}$ in $G_{i'}$ and $G_{j'}$, respectively, i.e., $P_i p_{i'} x_{i'} = u_{i'} = u_{j'} = P_j p_{j'} x_{j'}$. We divide this equation by

$$\prod_{\text{odd } k=1 (k \neq i,j)}^{d-1} (p_k v_k + p_{k+1} v_{k+1})$$

and obtain $p_{i'} x_{i'} (p_j v_j + p_{j+1} v_{j+1}) = p_{j'} x_{j'} (p_i v_i + p_{i+1} v_{i+1})$.

First, consider $i' = i \wedge j' = j$. It follows $p_i(x_i(p_j v_j + p_{j+1} v_{j+1}) - p_j x_j v_i) = p_j p_{i+1} x_j v_{i+1}$ $(\neq 0)$. Consequently, p_i divides one of p_j, p_{i+1}, x_j or v_{i+1}. This is incompatible with the fact that p_i, p_{i+1}, p_j are pairwise distinct primes and $p_i > m \geq \max\{x_j, v_{i+1}\}$.

The remaining cases $i' = i+1 \wedge j' = j$, $i' = i \wedge j' = j+1$ and $i' = i+1 \wedge j' = j+1$ can be considered analogously.

To (b): First we exclude $u_{i'} = 2u_{j'}$ with $i' \in \{i, i+1\}$ and $j' \in \{j, j+1\}$, where $i \neq j$ are odd elements of $\{1, 3, \ldots, d-1\}$. We see this in the same way like in (a), when we begin with $P_i p_{i'} x_{i'} = u_{i'} = 2u_{j'} = 2P_j p_{j'} x_{j'}$.

Now we have to show the non-existence of a set $\{i', j', k'\} \not\subseteq \{l, l+1\}$ with $u_{k'} - u_{j'} = u_{i'}$, where $u_{l'} \in V(G_{l'} \otimes G_{l'+1})$ holds for $l' \in \{i', j', k'\}$, and all odd $l \in \{1, 3, \ldots, d-1\}$.

At first, consider $u_{i'}, u_{j'}, u_{k'}$ with $u_{k'} - u_{j'} = u_{i'}$ and $s \notin \{u_{i'}, u_{j'}, u_{k'}\}$. Without loss of generality, we can assume $i' \leq j' \leq k'$. (Because of $u_{k'} - u_{j'} = u_{i'}$ we obtain $u_{k'} > u_{j'}, u_{i'}$, i.e., $k' \geq j', i'$. Since $u_{k'} - u_{j'} = u_{i'}$ is equivalent to $u_{k'} - u_{i'} = u_{j'}$, we can assume $u_{i'} < u_{j'}$, i.e. $i' \leq j'$.)

We distinguish three cases:

Case A. $u_{i'}, u_{j'} \in V(G_i \otimes G_{i+1}) \wedge u_{k'} \in V(G_k \otimes G_{k+1}) \wedge i < k \wedge i, k$ odd.

Case B. $u_{i'} \in V(G_i \otimes G_{i+1}) \wedge u_{j'}, u_{k'} \in V(G_j \otimes G_{j+1}) \wedge i < j \wedge i, j$ odd.

Case C. $u_{i'} \in V(G_i \otimes G_{i+1}) \wedge u_{j'} \in V(G_j \otimes G_{j+1}) \wedge u_{k'} \in V(G_k \otimes G_{k+1})$
$\wedge\, i < j < k \wedge i, j, k$ odd.

In each case we have to distinguish a lot of subcases, i.e., whether the vertices $u_{i'}, u_{j'}, u_{k'}$ are in $V(G_l)$ or in $V(G_{l+1})$ for certain $l \in \{i, j, k\}$. All these subcases can be treated similarly as done in (a) and at the beginning of (b), respectively, where in some situations Lemma 2.1 is needed to obtain a contradiction. Finally, $s \in \{u_{i'}, u_{j'}, u_{k'}\}$ must be investigated.

For details, see [22]. ■

3. Trees

In [5], the alternating trees which are difference digraphs are characterized. A tree is referred to as *alternating* iff every path of length of at least 2 in the tree is *alternating*, i.e., two consecutive arcs have always opposite orientation. An *odd source* in an alternating tree is a source having an odd out-degree. A sink is *ordinary* iff it is not adjacent to both an odd source u and an *end-source* (i.e., a source with out-degree 1) $v \neq u$.

Theorem 3.1 [5]**.** *An alternating tree is a difference digraph iff every odd source is adjacent to an ordinary sink.*

The generalized source-join enables us to verify a sufficient condition for the existence of a difference labelling of trees, which are not necessarily alternating. Let $N^-(v)$ and $N^+(v)$ denote the set of all predecessors and the set of all successors of the vertex $v \in V$, respectively.

Definition 3.1. A tree $T = (V, A)$ is called a *d-tree* iff T fulfills the IOC and for every $v \in V$ the following conditions hold:

(i) $d^+(v) \in \{0, 1\}$ or $d^+(v)$ even;

(ii) if there exists a $v' \in N^-(v)$ with $d^+(v') = 1$, then in $N^-(N^+(v))$ there are at most $\frac{d^+(v)}{2}$ vertices v'' with $d^+(v'') = 1$.

d-trees will be proved to be difference digraphs. Condition (i) results from the fact that we will need the generalized source-join of an even number of difference digraphs in the proof of the following Theorem 3.2. As to condition (ii) we note that there exist trees without difference labellings which violate (ii) but fulfill the IOC and (i). To see this, consider the tree T in Figure 6 and assume it has a difference labelling.

The vertex z has a predecessor a with out-degree 1 as well as more than $\frac{d^+(z)}{2} = 4$ successors with the property that each of them has a predecessor of out-degree 1. Because of the even out-degree of z and the different labels of all successors of z, at least two of these successors (with predecessors of out-degree 1), say x and y, have the property $x + y = z$. Their predecessors of out-degree 1 must have the labels $2x$ and $2y$. Since a has the label $2z$, the equation $a = 2z = 2x + 2y$ would imply the existence of two arcs $(a, 2x)$ and $(a, 2y)$ in contradiction to the definition of T.

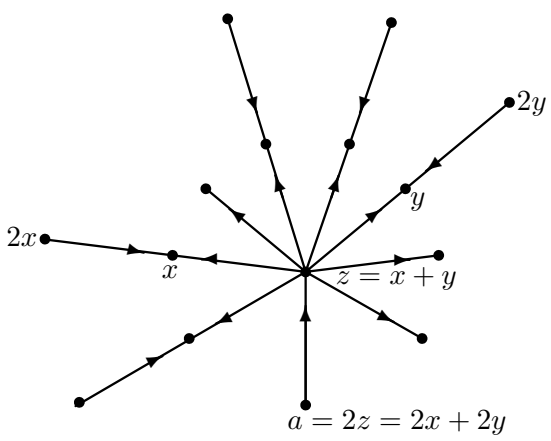

Figure 6. A nonalternating tree violating Definition 3.1(ii) that is not a difference digraph

To verify Theorem 3.2 (see below), we need some information on the structure of d-trees.

Lemma 3.1. *If $T = (V, A)$ is a d-tree then T has one of the following structures:*

(S1) $\forall\, v \in V :\ d^+(v) \leq 1$.
In this case T is a path.

(S2) $\exists\, s \in V : d^+(s) \geq 2 \wedge s$ *is a source.*
Since $d = d^+(s)$ is even, T is the generalized source-join of d trees $T_1, T_2, \ldots, T_d$ (cf. Figure 5 with $G_i = T_i$).

(S3) $\exists\, s \in V : d^+(s) \geq 2\ \wedge\ d^-(s) = 1 \wedge$ *the component of $T - s$, which contains the predecessor s' of s, is a directed path with terminal vertex s'. Again, $d = d^+(s)$ is even, and T has the structure shown in Figure 7.*

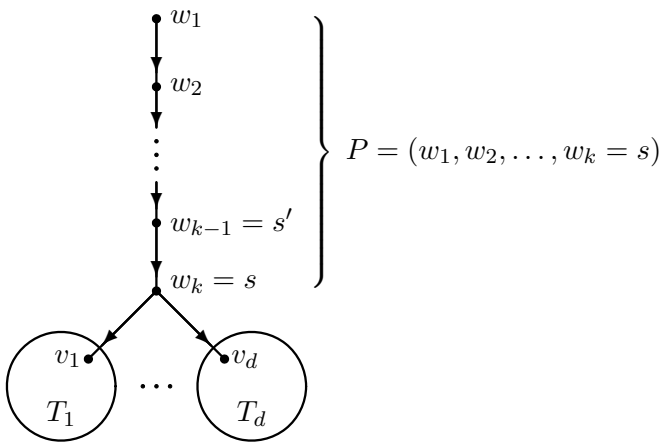

Figure 7

Proof. Assume, neither (S1) nor (S2) nor (S3) is valid. Then there exists a vertex $v^0 \in V$ with $d = d^+(v^0) \geq 2$ (because of not (S1)) and $d^-(v^0) \geq 1$ (because of not (S2)).

The IOC implies that there is at most one predecessor of v^0 with out-degree 1. Consequently, if v^0 has a second predecessor or v^0 has no predecessor of out-degree 1, there exists a predecessor v^1 of v^0 with even out-degree (≥ 2).

Let us delete the outgoing arcs $\{e_1, e_2, \ldots, e_d\}$ of v^0 and consider the component T' of $T - \{e_1, e_2, \ldots, e_d\}$ which contains v^0. Because of the IOC and since (S3) is forbidden, also in the case that v^0 has exactly one predecessor and this predecessor has out-degree one, we obtain the existence of a vertex $v^1 \in V(T')$ with even out-degree $d^+(v^1) \geq 2$.

Since (S2) cannot occur, $d^-(v^1) \geq 1$ holds. We delete the outgoing arcs $\{e'_1, e'_2, \ldots, e'_f\}$ of v^1 in T' and consider the component T'' of $T' - \{e'_1, e'_2, \ldots, e'_f\}$ which contains v^1. Because (S3) is forbidden, in T'' we obtain the existence of a vertex $v^2 \in V(T'')$ with even out-degree $d^+(v^2) \geq 2$ and so on.

Since the tree T is finite, this construction must stop in contradiction to our assumption. ■

Theorem 3.2. *If $T = (V, A)$ is a d-tree then there is a difference labelling of T with*

$$(*) \qquad \begin{aligned} &\forall\, v \in V \forall i \in I\!N^+ : (\exists\, v^* \in V : v^* = 2^i v) \\ &\Longrightarrow \exists\, v^- \in N^-(v) : d^+(v^-) = 1. \end{aligned}$$

Proof. The proof will be done by induction on the number t of the vertices $v \in V(T)$ with $d^+(v) \geq 2$.

In case $t = 0$ the tree T has structure (S1), i.e., T is an oriented path. Hence we can label its vertices by $2^0, 2^1, 2^2, \ldots, 2^{n-1}$.

Now let T contain $t+1$ vertices v with $d^+(v) \geq 2$. In the following Case A we use the same notation as in the proof of Theorem 2.1; in Case B we need additionally the notation $w_1, w_2, \ldots, w_k$ of the vertices of the path P (cf. Figure 7).

Case A. $\exists\, s \in V(T) : d^+(s) \geq 2 \wedge s$ is a source.
Now T has structure (S2) and we delete the vertex s and all of its outgoing arcs. Thus we obtain an even number of trees $T_1, T_2, \ldots, T_d$ and these trees fulfill the premise of the theorem. The induction hypothesis guarantees that we can construct a difference labelling with property $(*)$ for each of these trees. The generalized source-join provides a difference labelling of T.

Now we have to show the property $(*)$ for this labelling.

A1. $\exists\, i, j \in \{1, 2, \ldots, d\}\, \exists\, u_i \in V(T_i)\, \exists\, u_j \in V(T_j) : i \neq j \wedge u_i = 2^h u_j$.
With $u_i = P_i p_i x_i$ and $u_j = P_j p_j x_j$ we have $P_i p_i x_i = 2^h P_j p_j x_j$.

We construct a contradiction for odd i and j. If i or j is even, only slight modifications are necessary.

$$P_i p_i x_i = 2^h P_j p_j x_j \Leftrightarrow x_i p_i (p_j v_j + p_{j+1} v_{j+1}) = 2^h x_j p_j (p_i v_i + p_{i+1} v_{i+1})$$
$$\Leftrightarrow p_i (x_i (p_j v_j + p_{j+1} v_{j+1}) - 2^h x_j p_j v_i) = 2^h x_j p_j p_{i+1} v_{i+1}.$$

Then, we obtain the contradiction that p_i divides $2^h x_j p_j p_{i+1} v_{i+1}$.

A2. $\exists\, i \in \{1, 2, \ldots, d\}\, \exists\, u_i \in V(T_i) :\ s = 2^h u_i \vee 2^h s = u_i$.

$$s = 2^h u_i \Leftrightarrow P_i (p_i v_i + p_{i+1} v_{i+1}) = 2^h P_i p_i x_i \Leftrightarrow p_i v_i + p_{i+1} v_{i+1} = 2^h p_i x_i$$
$$\Leftrightarrow p_{i+1} v_{i+1} = p_i (2^h x_i - v_i).$$

Consequently, p_i has to divide $p_{i+1} v_{i+1}$, but this is impossible.

A similar contradiction is found, if we assume $2^h s = u_i$.

Case B. $\forall\, v \in V(T) : d^+(v) \geq 2 \Rightarrow d^-(v) \geq 1$.
Obviously, T has structure (S3) (cf. Figure 7; note that $d = d^+(s)$). Because of the IOC no vertex of V has more than one predecessor with out-degree of exactly one.

We apply assumption (ii) of Definition 3.1 to the vertex s (cf. Figure 7). Of course, in $N^-(N^+(s))$ there are at most $\frac{d}{2}$ vertices $v' \in N^-(N^+(s))$ with $d^+(v') = 1$ and these vertices v' are predecessors of pairwise distinct vertices from $N^+(s)$. Hence at most $\frac{d}{2}$ of the trees $T_1, T_2, \ldots, T_d$ have such a vertex v' and every such tree contains at most one of these vertices.

Therefore, without loss of generality we can subscript the trees T_1, $T_2, \ldots, T_d$ in such a way that only trees T_i with odd i can contain such a vertex. Hence, in every pair (T_i, T_{i+1}), for odd i, we find at most one vertex $v' \in N^-(N^+(s))$ with $d^+(v') = 1$.

We apply the following

Labelling Algorithm

1. Delete all outgoing arcs of s, i.e., the arcs $(s, v_1), \ldots, (s, v_d)$.
2. $T_1, \ldots, T_d$ fulfill the induction hypothesis, so for $i = 1, \ldots, d$ construct a difference labelling of T_i with property $(*)$.

3. Construct a difference labelling (which has property $(*)$) of the source-join $\bigotimes_{i=1}^{d} T_i$ with source s, under consideration of the following conditions:

   (a) Let k be the length of the path P (cf. Figure 7). In addition to (1) and (2) (see Section 2), the primes $p_1, \ldots, p_d$ must have the properties $p_1 > 2^k m \ \wedge \ p_{i+1} > 2^{k+1} m^2 p_i$, for all $i \in \{1, \ldots, d-1\}$.

   (b) To construct the difference labelling of $\bigotimes_{i=1}^{d} T_i$ (with property $(*)$) we proceed as in Case A and use the algorithm described in Section 2.

   (c) Note that because of the special subscription of the trees $T_1, T_2, \ldots, T_d$ in every source-join $T_i \otimes T_{i+1}$, for odd i, there is at most one vertex $v' \in N^-(N^+(s))$ with $d^+(v') = 1$.

   Of course, the vertex $s = w_k$ obtains the label $s = \prod_{\text{odd } l=1}^{d-1}(p_l v_l + p_{l+1} v_{l+1}) = P_i(p_i v_i + p_{i+1} v_{i+1})$, for odd $i \in \{1, \ldots, n-1\}$.
4. For $i = 1, \ldots, k-1$ label the vertices of the path P by $w_{k-i} = 2\, w_{k-i+1} = 2^i s$.

It is easy to see that every arc (x, y) is generated by the labels of its end vertices x and y, i.e., for every arc (x, y) there exists a vertex z such that the label of the arc is the same as the label of the vertex: $x - y = z$.

Now we verify that the labelling constructed above does not induce "new" arcs, i.e.:

$$\forall u, u', u'' \in V : u - u' = u'' \Rightarrow (u, u') \in A. \tag{5}$$

(5) is obvious in case $u, u', u'' \in \bigcup_{i=1}^{d} V(T_i) \cup \{s\}$, since this is a triple of vertices in the source-join $\bigotimes_{i=1}^{d} T_i$. The situation $u, u', u'' \in V(P)$ is trivial, too, and the same holds for $u' = u''$.

Therefore assume there are three vertices $u, u', u'' \in V$ with $u - u' = u''$ and $|\{u, u', u''\} \cap V(P)| \in \{1, 2\}$; without loss of generality, we can suppose $u > u' > u''$.

B1. $|\{u, u', u''\} \cap V(P)| = 2$.
It suffices to investigate $u, u' \in V(P)$. Changing some signs the remaining cases can be considered analogously. With $u = 2^g s, u' = 2^h s$ and $u'' = P_i p_i x_i \in V(T_i)$ we obtain

$u - u' = u'' \Leftrightarrow (2^g - 2^h)s = P_i p_i x_i \Leftrightarrow (2^g - 2^h)\prod_{\text{odd } l=1}^{d-1}(p_l v_l + p_{l+1}v_{l+1}) = P_i p_i x_i \Leftrightarrow (2^g - 2^h)(p_i v_i + p_{i+1}v_{i+1}) = p_i x_i \Leftrightarrow (2^g - 2^h)p_{i+1}v_{i+1} = p_i(x_i - (2^g - 2^h)v_i)$.

Since p_i cannot divide the left hand side of the equation, we have got a contradiction. For $u'' = P_i p_{i+1} x_{i+1} \in V(T_{i+1})$ we similarly obtain a contradiction.

B2. $|\{u, u', u''\} \cap V(P)| = 1$.
We suppose $s \notin \{u, u', u''\}$ and distinguish whether or not two of the vertices u, u', u'' are in $V(T_i)$, $V(T_{i+1})$, $V(T_i) \cup V(T_{i+1})$ or one is in $V(T_i) \cup V(T_{i+1})$ and the other one in $V(T_j) \cup V(T_{j+1})$, for $i \neq j$, where $i, j \in \{1, 3, \ldots, d-1\}$ are odd. As done above, we will discuss only the most important cases. The rest can be obtained by slight modifications.

In the following, i and j are odd numbers from $\{1, 3, \ldots, d-1\}$.

B2.1. $u = 2^h s \in V(P) \ \wedge \ u' = P_i p_i x_i \in V(T_i) \ \wedge \ u'' = P_i p_i x_i' \in V(T_i)$.
$2^h s - P_i p_i x_i = P_i p_i x_i' \Leftrightarrow 2^h(p_i v_i + p_{i+1}v_{i+1}) - p_i x_i = p_i x_i' \Leftrightarrow 2^h p_{i+1} v_{i+1} = p_i(x_i + x_i' - 2^h v_i)$, but $2^h p_{i+1} v_{i+1}$ is not a multiple of p_i.

B2.2. $u = 2^h s \in V(P) \wedge u' = P_i p_{i+1} x_{i+1} \in V(T_{i+1}) \wedge u'' = P_i p_i x_i \in V(T_i)$.
$2^h s - P_i p_{i+1} x_{i+1} = P_i p_i x_i \Leftrightarrow 2^h(p_i v_i + p_{i+1}v_{i+1}) - p_{i+1}x_{i+1} = p_i x_i \Leftrightarrow p_{i+1}(2^h v_{i+1} - x_{i+1}) = p_i(x_i - 2^h v_i)$.

This is only possible if p_i divides $2^h v_{i+1} - x_{i+1}$, i.e., for $2^h v_{i+1} = x_{i+1}$, and if p_{i+1} divides $2^h v_i - x_i$, i.e., for $2^h v_i = x_i$. Because of step 2 of our Algorithm property $(*)$ holds and we have $\exists v_i^- \in N^-(v_i) : d^+(v_i^-) = 1$ and $\exists v_{i+1}^- \in N^-(v_{i+1}) : d^+(v_{i+1}^-) = 1$. This is incompatible with (c), since v_i and v_{i+1} are distinct successors of s.

B2.3. $u = 2^h s \in V(P) \wedge u' = P_i p_i x_i \in V(T_i) \wedge u'' = P_j p_j x_j \in V(T_j) \wedge i > j$.
$2^h s - P_i p_i x_i = P_j p_j x_j \Leftrightarrow 2^h(p_i v_i + p_{i+1}v_{i+1})(p_j v_j + p_{j+1}v_{j+1}) - p_i x_i(p_j v_j + p_{j+1}v_{j+1}) = p_j x_j(p_i v_i + p_{i+1}v_{i+1}) \Leftrightarrow p_i((2^h v_i - x_i)(p_j v_j + p_{j+1}v_{j+1}) - p_j x_j v_i) = p_{i+1}(p_j x_j v_{i+1} - 2^h v_{i+1}(p_j v_j + p_{j+1}v_{j+1}))$.

Define $\sigma = p_j x_j v_{i+1} - 2^h v_{i+1}(p_j v_j + p_{j+1}v_{j+1})$. Obviously, p_i must divide σ, moreover $\sigma < 0$ can be deduced from (3): $p_j x_j v_{i+1} < 2m^2 p_j < p_{j+1} < 2^h v_{i+1}(p_j v_j + p_{j+1}v_{j+1})$.

In the case $i > j$ immediately $i > j + 1$ follows and from (a) we get the contradiction $p_i > 2^{k+1}m^2 p_{j+1} \geq 2 \cdot 2^h m^2 p_{j+1} > 2^h m^2(p_i + p_{j+1}) \geq 2^h v_{i+1}(p_j v_j + p_{j+1}v_{j+1}) > |p_j x_j v_{i+1} - 2^h v_{i+1}(p_j v_j + p_{j+1}v_{j+1})| = |\sigma|$.

At the end of Case B we have to demonstrate the property $(*)$, i.e., for vertices v and v^* with $v^* = 2^i v$ we must show the existence of a vertex $v^- \in N^-(v)$ with $d^+(v^-) = 1$. Because of Case A we can suppose that exactly one of v, v^* is an element of $V(P)$.

Assume $v^* = 2^i v$ with $v^* = 2^h s \in V(P) - \{s\}$ and $v = P_k p_k x_k \in V(T_k) \subset V(\bigotimes_{i=1}^d T_i) - \{s\}$. It follows: $2^h s = 2^i P_k p_k x_k \Leftrightarrow 2^{h-i}(p_k v_k + p_{k+1} v_{k+1}) = p_k x_k \Leftrightarrow 2^{h-i} p_{k+1} v_{k+1} = p_k(x_k - 2^{h-i} v_k)$, but p_k cannot divide the left hand side of this equation.

The cases $v^* = 2^h s \in V(P) - \{s\} \wedge v = P_k p_{k+1} x_{k+1} \in V(T_{k+1})$ and $v^* \in V(\bigotimes_{i=1}^d T_i) - \{s\} \wedge v \in V(P) - \{s\}$ can be treated analogously.

This completes the proof. ■

Corollary 3.1. *If $T = (V, A)$ is a d-tree then T is a difference digraph.*

4. Remarks on Digraphs with Cycles

Among many other results, in [5] some basic properties of difference digraphs were given, e.g.:

Remark 4.1 ([5])**.**

(a) A difference digraph contains no directed cycles.

(b) A difference digraph is an oriented graph.

(c) A difference digraph has at least one source and at least one sink.

(d) A digraph with a total sink is a difference digraph iff it is a transitive tournament.
(A *total sink* is a vertex $v \in V$ with $N^-(v) = V - \{v\}$.)

In [5] the authors cite Gervacio [8] (unfortunately, the paper [8] is not available to me) and mention that he proved that transitive tournaments are the only difference digraphs in the class of tournaments. Moreover, in [8] the oriented cycles were characterized, which are difference digraphs. Because in [5] in this context a more general notion of *difference digraph* is used (they allow integers as labels as well as the use of the same label for different vertices), it is not clear, whether or not Gervacio used difference labellings of oriented cycles in the sense of our definition in his paper [8].

Thus we sketch the proof of the following theorem here, i.e., we describe a possible labelling procedure and make some remarks on its verification.

Theorem 4.1 ([5], [8]). *An oriented cycle $C = (V, A)$ is a difference digraph iff C fulfills the IOC, C is not a directed cycle and C is not isomorphic to C_4^* or C_5^* (cf. Figure 8).*

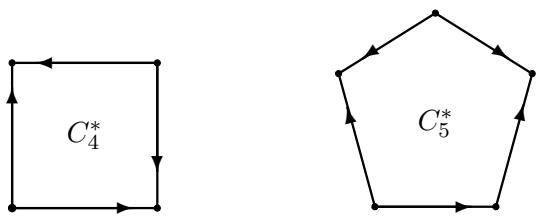

Figure 8. Two oriented cycles, which are not difference digraphs

Proof. Of course, we have only to verify the sufficiency of the given conditions; their necessity is obvious. So let $C = (v_0, v_1, \ldots, v_n, v_{n+1} = v_0)$ be an oriented cycle with IOC which is not directed and not isomorphic to C_4^* or C_5^*. Without loss of generality, let v_n be a source. In order to decompose C into directed paths $P_1, P_2, \ldots, P_r$, we delete all sources $v_{j_1}, v_{j_2}, \ldots, v_{j_r} = v_n$ of C and obtain $C = (P_1, v_{j_1}, P_2, v_{j_2}, \ldots, v_{j_{r-1}}, P_r, v_{j_r} = v_n, v_{n+1} = v_0)$. Now, step by step, we label the vertices of the paths $P_1, P_2, \ldots, P_r$ and the sources $v_{j_1}, v_{j_2}, \ldots, v_{j_r} = v_n$.

Labelling algorithm

1. $i := 1$, $L_0 := 0$.
2. **If** $i \geq 2$, **then** let L_{i-1} be the maximum of the labels of the vertices of $P_1, P_2, \ldots, P_{i-1}$ and of the sources $v_{j_1}, v_{j_2}, \ldots, v_{j_{i-2}}$.
3. Start at the terminal vertex of P_i and label the vertices of P_i along the path by $2L_{i-1}+1, 2(2L_{i-1}+1), 2^2(2L_{i-1}+1), \ldots, 2^{l(P_i)}(2L_{i-1}+1)$, where $l(P_i)$ is the length of P_i.
4. **If** $i \geq 2$, **then** label the source $v_{j_{i-1}}$ by the sum of the labels of its successors. (Note that these successors obtained their labels when P_{i-1} and P_i were labelled.)
5. **If** $i < r$, **then** $i := i + 1$ **and** go to 2.
6. Label $v_{j_r} = v_n$ by the sum of the labels of its successors.

As an example, see Figure 9.

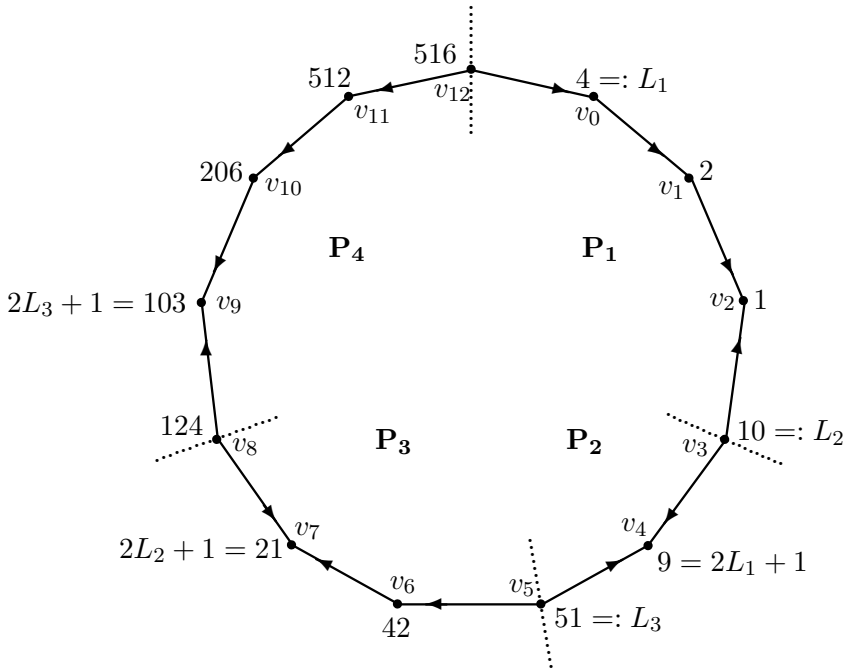

Figure 9

It is clear that this labelling generates all arcs of the cycle C but no additional arcs inside one of the paths P_i ($i \in \{1, 2, \ldots, r\}$). Because of the definition of $L_1, L_2, \ldots, L_{r-1}$ and the labelling of the vertices of the paths $P_1, P_2, \ldots, P_r$ (cf. step 3) we have "sufficiently large" differences between the vertex labels of different paths, i.e., there cannot be vertices $u_i \in V(P_i)$ and $u_j \in V(P_j)$ with $i \neq j$ and $u_i - u_j \in V(P_k)$ for a $k \in \{1, 2, \ldots, r\}$. For the same reason this fact holds true, if we involve also the sources $v_{j_1}, v_{j_2}, \ldots, v_{j_{r-1}}, v_{j_r} = v_n$ beside the vertices of $P_1, P_2, \ldots, P_r$. To demonstrate this, a detailed, but simple distinction of cases is necessary. ■

The next step to investigate difference digraphs with cycles would be to consider cycles with additional hanging arcs ("prickles") as preparatory work to combine cycles and paths (or trees) to oriented cacti (cf. [21] for undirected cacti). Of course partial results are possible, but even adding hanging arcs to cycles causes a lot of problems and can result in a difference digraph or not. Many different cases must be considered, e.g. whether or not there

are ingoing/outgoing arcs at adjacent vertices of the cycles, where the direction of the arcs along the cycle is important, too. In the undirected case the composition of difference labellings of cycles with prickles ("hedgehogs") and paths with prickles ("caterpillars") causes no problems, but in the directed case we have only the source-join as a tool (if we desist from very special structures).

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Received 21 July 2003
Revised 24 February 2004